Unification and Projectivity in De Morgan and Kleene Algebras

Simone Bova
Institut für Informationssysteme
Technische Universität Wien (Wien, Austria)
simone.bova@tuwien.ac.at

Leonardo Cabrer
Mathematics Institute
University of Bern (Bern, Switzerland)

Abstract

We provide a complete classification of solvable instances of the equational unification problem over De Morgan and Kleene algebras with respect to unification type. The key tool is a combinatorial characterization of finitely generated projective De Morgan and Kleene algebras.

1 Introduction

A *De Morgan* algebra $\mathbf{A} = (A, \wedge, \vee, ', 0, 1)$ is a bounded distributive lattice with an involution satisfying De Morgan laws, that is a unary operation $'$ satisfying $x = x''$ and $x \wedge y = (x' \vee y')'$. A *Kleene* algebra is a De Morgan algebra satisfying $x \wedge x' \leq y \vee y'$. A *Boolean* algebra is a Kleene algebra satisfying $x \wedge x' = 0$. In [10], Kalman shows that the lattice of (nontrivial) varieties of De Morgan algebras is a three-element chain formed by Boolean algebras (Kleene algebras satisfying $x \wedge x' = 0$), Kleene algebras, and De Morgan algebras, and these varieties are locally finite; that is, finite, finitely presented, and finitely generated algebras coincide.

In a variety of algebras, the symbolic (equational) unification problem is the problem of solving finite systems of equations over free algebras. An instance of the symbolic unification problem is a finite system of equations, and a solution (a unifier) is an assignment of the variables to terms such that the system holds identically in the variety. The set of unifiers of a solvable instance supports a natural order, and the instances are classified depending on the properties of their maximal unifiers. In this paper, we provide a complete (first-order, decidable) classification of solvable instances of the unification problem over De Morgan and Kleene algebras with respect to their unification type.

The key tool towards the classification is a combinatorial (first-order, decidable) characterization of finitely generated projective De Morgan and Kleene algebras, motivated by the nice theory of algebraic (equational) unification introduced by Ghilardi [9]. In the algebraic unification setting, an instance of the unification problem is a finitely presented algebra in a certain variety, a unifier is a homomorphism to a finitely presented projective algebra in the variety, and unifiers support a natural order that determines the unification type of the instance, in such a way that it coincides with the unification type of its finite presentation, viewed as an instance of the symbolic unification problem.

Even if projective Boolean algebras have been characterized in [3, 14], a complete characterization of projective De Morgan and Kleene algebras lacks in the literature. In this note, also motivated by an effective application of the algebraic unification framework, we initiate the study of projective De Morgan and Kleene algebras, and relying on finite duality theorems [7], we provide a combinatorial characterization of finitely generated projective algebras, and we exploit it to classify all solvable instances of the equational unification problem over De Morgan and Kleene algebras with respect to their unification type; in particular, we establish that De Morgan and Kleene algebras have nullary equational unification type (and avoid the infinitary type).

The paper is organized as follows. In Section 2, we collect from the literature the background on projective algebras, duality theory, and unification theory necessary for the rest of the paper. For standard undefined notions and facts in order theory, universal algebra, category theory, and unification theory, we refer the reader to [8], [12], [11], and [2] respectively. In Section 3, we introduce the characterization of finite projective De Morgan and Kleene algebras (respectively, Theorem 11 and Theorem 12). In Section 4, we provide a complete classification with respect to unification type of all solvable instances of the equational unification problem over bounded distributive lattices, Kleene algebras, and De Morgan algebras (respectively, Theorem 15, Theorem 22, and Theorem 31). The distributive lattices case tightens previous results by Ghilardi [9], and outlines the key ideas involved in the study of the more demanding cases of Kleene and De Morgan algebras.

2 Preliminaries

Let $\mathbf{P} = (P, \leq)$ be a preorder, that is, $\leq$ is reflexive and transitive. If x and y are incomparable in $\mathbf{P}$, we write $x \parallel y$. Given $X, Y \subseteq P$, we write $X \leq Y$ iff $x \leq y$ for all $x \in X$ and $y \in Y$; we freely omit brackets, writing for instance $x \leq y, z$ instead of $\{x\} \leq \{y, z\}$. If $X \subseteq P$, we denote by $(X]$ and $[X)$ respectively the *downset* and *upset* in P generated by X, namely $(X] = \{y \in P \mid y \leq x \text{ for some } x \in X\}$ and $[X) = \{y \in P \mid y \geq x \text{ for some } x \in X\}$; if $X = \{x\}$ we freely write $(x]$ and $[x)$. If $x, y \in P$, we write $[x, y] = \{z \in P \mid x \leq z \leq y\}$. A set $X \subseteq P$ is *directed* if for all $x, y \in X$ there exists $z \in X$ such that $x, y \leq z$. We denote minimal elements in $\mathbf{P}$ by $\min(\mathbf{P}) = \{x \in P \mid y \leq x \text{ implies } x \leq y \text{ for all } y \in P\}$. Similarly we denote maximal elements in $\mathbf{P}$ by $\max(\mathbf{P})$. Let $\mathbf{P} = (P, \leq)$ and $\mathbf{Q} = (Q, \leq)$ be preorders. A map $f \colon P \to Q$ is *monotone* if $x \leq y$ implies $f(x) \leq f(y)$.

2.1 Projective Algebras

Let $\mathcal{V}$ be a variety of algebras and κ be an arbitrary cardinal. An algebra $\mathbf{B} \in \mathcal{V}$ is said to have the *universal mapping* property for κ if, there exists $X \subseteq B$ such that $|X| = \kappa$ and for every $\mathbf{A} \in V$ and every map $f \colon X \to \mathbf{A}$ there exists a (unique) homomorphism $g\colon \mathbf{B} \to \mathbf{A}$ extending f (any $x \in X$ is said a *free generator*, and $\mathbf{B}$ is said *freely generated* by X). For every cardinal κ, there exists a unique algebra (up to isomorphism) with the universal mapping property freely generated by a set of cardinality κ, called the *free* κ-generated algebra in $\mathcal{V}$, and denoted by $\mathbf{F}_{\mathcal{V}}(\kappa)$.

Since the varieties of De Morgan and Kleene algebras, in symbols $\mathcal{M}$ and $\mathcal{K}$ respectively, are generated by single finite algebras [10], they are locally finite, that is, finitely generated and finite algebras coincide.

Example 1. *By direct computation,* $\mathbf{F}_{\mathcal{M}}(1)$ *is the bounded distributive lattice over* $\{0, x \wedge x', x, x', x \vee x', 1\}$ *shown in Fig. 1.*

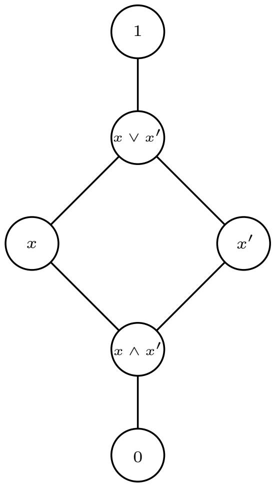

Figure 1: $\mathbf{F}_{\mathcal{M}}(1)$.

Let $\mathcal{V}$ be a variety of algebras. An algebra $\mathbf{A} \in \mathcal{V}$ is said to be *projective* if for every pair of algebras $\mathbf{B}, \mathbf{C} \in \mathcal{V}$, every surjective homomorphism $f\colon \mathbf{B} \to \mathbf{C}$, and every homomorphism $h\colon \mathbf{A} \to \mathbf{C}$, there exists a homomorphism $g\colon \mathbf{A} \to \mathbf{B}$ such that $f \circ g = h$.

We exploit the following well-known characterization of projective algebras (see [9, Lemma 3.2]).

Theorem 2. *Let* $\mathcal{V}$ *be a variety, and let* $\mathbf{A} \in \mathcal{V}$*. Then,* $\mathbf{A}$ *is projective in* $\mathcal{V}$ *iff* $\mathbf{A}$ *is a* retract *of a free algebra* $\mathbf{F}_{\mathcal{V}}(\kappa)$ *in* $\mathcal{V}$ *for some cardinal* κ*, that is, there exist homomorphisms* $r\colon \mathbf{F}_{\mathcal{V}}(\kappa) \to \mathbf{A}$ *and* $f\colon \mathbf{A} \to \mathbf{F}_{\mathcal{V}}(\kappa)$ *such that* $r \circ f = \mathrm{id}_{\mathbf{A}}$.

Since in this paper we are only working with finitely generated projective algebras, the cardinal κ of the theorem above can be chosen to be finite. Moreover, each variety considered in this paper is locally finite, that is finitely generated algebras coincide with finite algebras. So we concentrate on the finite algebras of each class and the duality for them.

2.2 Finite Duality

We recall duality theorems for the categories of finite bounded distributive lattices, $\mathcal{D}_f$, finite De Morgan algebras, $\mathcal{M}_f$, and finite Kleene algebras, $\mathcal{K}_f$.

First, we present Birkhoff duality between finite bounded distributive lattices and finite posets [6]. The category $\mathcal{P}_f$ of finite posets has finite posets $(P, \leq)$ as objects, and monotone maps as morphisms.

Define the map $\mathsf{J}\colon \mathcal{D}_f \to \mathcal{P}_f$ as follows: For every $\mathbf{A}$ in $\mathcal{D}_f$, let

$$\mathsf{J}(\mathbf{A}) = (\{x \mid x \text{ join irreducible in } \mathbf{A}\}, \leq),$$

where $\leq$ is the order inherited from the order in $\mathbf{A}$. For every $h\colon \mathbf{A} \to \mathbf{B}$ in $\mathcal{D}_f$, let $\mathsf{J}(h)\colon \mathsf{J}(\mathbf{B}) \to \mathsf{J}(\mathbf{A})$, be the map defined by

$$\mathsf{J}(h)(x) = \bigwedge\{y \mid h(y) \geq x\},$$

for all $x \in \mathsf{J}(\mathbf{B})$.

Define the map $\mathsf{D}\colon \mathcal{P}_f \to \mathcal{D}_f$ as follows: For every $\mathbf{P} = (P, \leq) \in \mathcal{P}_f$, let

$$\mathsf{D}(\mathbf{P}) = (\{X \subseteq P \mid (X] = X\}, \cap, \cup, \emptyset, P).$$

For every $f\colon \mathbf{P} \to \mathbf{Q}$ in $\mathcal{P}_f$, let $\mathsf{D}(f)\colon \mathsf{D}(\mathbf{Q}) \to \mathsf{D}(\mathbf{P})$ be the map defined by

$$\mathsf{D}(f)(X) = f^{-1}(X)$$

for all $X \in \mathsf{D}(\mathbf{Q})$.

Theorem 3 (Birkhoff, [6])**.** J *and* D *are well defined contravariant functors. Moreover, they determine a dual equivalence between* $\mathcal{D}_f$ *and* $\mathcal{P}_f$*.*

Building on the duality for bounded distributive lattices developed by Priestley [13], in [7, Theorem 2.3 and Theorem 3.2] Cornish and Fowler present a duality for De Morgan and Kleene algebras. We rely on Theorem 3, to restrict such duality to finite objects.

Definition 4 (Finite Involutive Posets, $\mathcal{PM}_f$ and $\mathcal{PK}_f$)**.** The category $\mathcal{PM}_f$ of *finite involutive posets* is defined as follows:

Objects : Structures $(P, \leq, i)$, where $(P, \leq)$ is a finite poset, and $i\colon P \to P$ is such that $x \leq y$ implies $i(y) \leq i(x)$ and $i(i(x)) = x$.

Morphisms : Maps $f\colon (P, \leq, i) \to (P', \leq', i')$ such that $x \leq y$ implies $f(x) \leq' f(y)$ and $f(i(x)) = i'(f(x))$.

The category $\mathcal{PK}_f$ is the full subcategory of $\mathcal{PM}_f$ whose objects $(P, \leq, i)$ are such that $i(x)$ is comparable to x for all $x \in P$.

The map $\mathsf{J}_{\mathcal{M}}\colon \mathcal{M}_f \to \mathcal{PM}_f$ is defined by: For every $\mathbf{A} = (A, \wedge, \vee, ', 0, 1)$ in $\mathcal{M}_f$, let

$$\mathsf{J}_{\mathcal{M}}(\mathbf{A}) = (\mathsf{J}(A, \wedge, \vee, 0, 1), i),$$

where $i(x) = \bigwedge(A \setminus \{a' \mid a \in [x)\})$ for each $x \in \mathsf{J}(A, \wedge, \vee, 0, 1)$. Moreover, $\mathsf{J}_{\mathcal{M}}(h) = \mathsf{J}(h)$ for every $h\colon \mathbf{A} \to \mathbf{B}$ in $\mathcal{FM}$.

The map $\mathsf{D}_{\mathcal{M}}\colon \mathcal{PM}_f \to \mathcal{M}_f$ is defined by: For every $\mathbf{P} = (P, \leq, i) \in \mathcal{PM}_f$,

$$\mathsf{D}_{\mathcal{M}}(\mathbf{P}) = \mathbf{A} = (A, \wedge, \vee, ', 0, 1)$$

where $(A, \wedge, \vee, 0, 1) = \mathsf{D}(P, \leq)$, and $X' = P \setminus i(X)$. Moreover, $\mathsf{D}_{\mathcal{M}}(f) = \mathsf{D}(f)$ for every $f\colon \mathbf{P} \to \mathbf{Q}$ in $\mathcal{PM}_f$.

We will denote by $\mathsf{J}_{\mathcal{K}}\colon \mathcal{K}_f \to \mathcal{PK}_f$ and $\mathsf{D}_{\mathcal{K}}\colon \mathcal{PK}_f \to \mathcal{K}_f$ the restrictions of the functors $\mathsf{J}_{\mathcal{M}}$ and $\mathsf{D}_{\mathcal{M}}$ to the categories $\mathcal{K}_f$ and $\mathcal{PK}_f$ respectively.

Theorem 5 (Cornish and Fowler)**.** $\mathsf{J}_{\mathcal{M}}$ *and* $\mathsf{D}_{\mathcal{M}}$ *(respectively,* $\mathsf{J}_{\mathcal{K}}$ *and* $\mathsf{D}_{\mathcal{K}}$*) are well defined contravariant functors. Moreover, they determine a dual equivalence between the categories* $\mathcal{M}_f$ *and* $\mathcal{PM}_f$ *(respectively,* $\mathcal{K}_f$ *and* $\mathcal{PK}_f$*) .*

Let $\mathbf{D} = (D, \leq, i) \in \mathcal{PM}_f$ be as in Fig. 2. In light of Example 1, $\mathsf{J}_{\mathcal{M}}(\mathbf{F}_{\mathcal{M}}(1))$ and $\mathbf{D}$ are isomorphic via the map $x \wedge x' \mapsto 2$, $x \mapsto 0$, $x' \mapsto 1$, $1 \mapsto 3$.

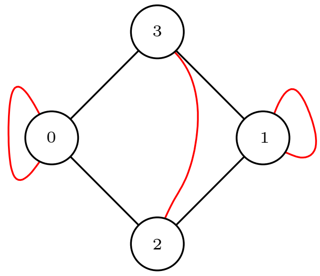

Figure 2: $\mathsf{J}_{\mathcal{M}}(\mathbf{F}_{\mathcal{M}}(1)) \simeq \mathbf{D}$. Curved edges depict the map $i\colon D \to D$.

Let $\mathbf{P} = (P, \leq, i) \in \mathcal{PM}_f$. By [7, Theorem 2.4], the *product* of n copies of $\mathbf{P}$ in the category $\mathcal{PM}_f$, denoted by

$$\mathbf{P}^n = (P^n, \leq^n, i^n),$$

is the finite poset over P^n with the order and the involution defined coordinatewise, that is for all $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in P^n$, $x \leq^n y$ iff $x_i \leq y_i$ for all $i = 1, \ldots, n$, and $i^n(x) = (i(x_1), \ldots, i(x_n))$.

Proposition 6. $\mathsf{J}_{\mathcal{M}}(\mathbf{F}_{\mathcal{M}}(n)) \simeq \mathbf{D}^n$*.*

Proof. $\mathbf{F}_{\mathcal{M}}(n)$ is the coproduct of n copies of $\mathbf{F}_{\mathcal{M}}(1)$. Therefore, by Theorem 5, $\mathsf{J}_{\mathcal{M}}(\mathbf{F}_{\mathcal{M}}(n)) \simeq \mathsf{J}_{\mathcal{M}}(\mathbf{F}_{\mathcal{M}}(1))^n \simeq \mathbf{D}^n$, the product of n copies of $\mathsf{J}_{\mathcal{M}}(\mathbf{F}_{\mathcal{M}}(1))$. The statement follows. □

The diagram in Fig. 3 represents the poset of $\mathsf{J}_{\mathcal{M}}(\mathbf{F}_{\mathcal{M}}(2))$. The action of the involution is represented by curved lines. To simplify the notation we write strings of two characters instead of oredered pairs, for example 01 stands for $(0, 1) \in \mathbf{D}^2$.

Let $\mathbf{P} = (P, \leq, i) \in \mathcal{PM}_f$. By [1], subobjects of $\mathbf{P}$ are subsets $X \subseteq P$ with the inherited order such that $X = i(X)$. By Theorem 5, subobjects of $\mathbf{P}$ correspond exactly to quotients on $\mathsf{D}_{\mathcal{M}}(\mathbf{P})$. For each $\mathbf{P}$ in $\mathcal{PM}_f$, let $\mathbf{P}_k$ be the largest subobject of $\mathbf{P}$ lying in the subcategory $\mathcal{PK}_f$, that is, $\mathbf{P}_k$ is the subobject of $\mathbf{P}$ (possibly empty) such that each element x of $\mathbf{P}_k$ is comparable with $i(x)$. Therefore, $\mathsf{D}_{\mathcal{M}}(\mathbf{P}_k)$ is the largest quotient of $\mathsf{D}_{\mathcal{M}}(\mathbf{P})$ lying in $\mathcal{K}_f$.

Proposition 7. $\mathsf{J}_{\mathcal{K}}(\mathbf{F}_{\mathcal{K}}(n)) \simeq (\mathbf{D}^n)_k$*.*

Proof. $\mathbf{F}_{\mathcal{K}}(n)$ is the largest quotient of $\mathbf{F}_{\mathcal{M}}(n)$ that is a Kleene algebra. By Proposition 6, $\mathsf{J}_{\mathcal{M}}(\mathbf{F}_{\mathcal{M}}(n)) \simeq \mathbf{D}^n$. Then by the mentioned correspondence between quotients and subobjects under the duality [1], $\mathsf{J}_{\mathcal{K}}(\mathbf{F}_{\mathcal{K}}(n)) = \mathsf{J}_{\mathcal{M}}(\mathbf{F}_{\mathcal{K}}(n))$ arises as the largest subobject of $\mathbf{D}^n$ lying in $\mathcal{PK}_f$. □

Following Proposition 7 in Fig. 4 we represent the poset of $\mathsf{J}_{\mathcal{K}}(\mathbf{F}_{\mathcal{K}}(2))$ by selecting the elements of $\mathsf{J}_{\mathcal{M}}(\mathbf{F}_{\mathcal{M}}(2))$ (see Fig. 3) that are comparable with their involution. In this case we only need to drop the elements 23 and 32 of $\mathbf{D}^2$.

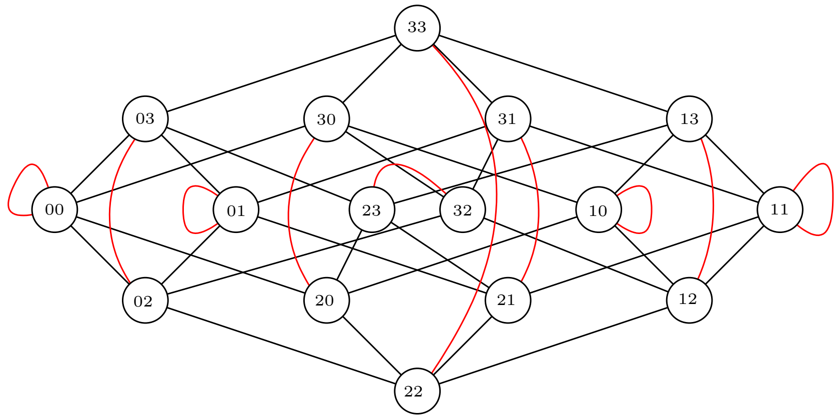

Figure 3: $\mathsf{J}_{\mathcal{M}}(\mathbf{F}_{\mathcal{M}}(2)) \simeq \mathbf{D}^2$.

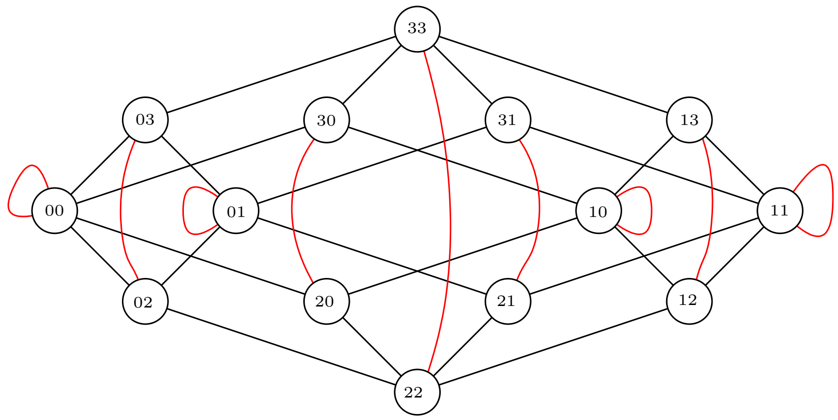

Figure 4: $\mathsf{J}_{\mathcal{K}}(\mathbf{F}_{\mathcal{K}}(2)) \simeq (\mathbf{D}^2)_k$.

2.3 Unification Theory

Let $\mathbf{P} = (P, \leq)$ be a preorder. A *μ-set* for $\mathbf{P}$ is a subset $M \subseteq P$ such that $x \parallel y$ for all $x, y \in M$ such that $x \neq y$, and for every $x \in P$ there exists $y \in M$ such that $x \leq y$. It is easy to check that if $\mathbf{P}$ has a μ-set, then every μ-set of $\mathbf{P}$ has the same cardinality.

We say that $\mathbf{P}$ has type:

nullary if $\mathbf{P}$ has no μ-sets (in symbols, $\text{type}(\mathbf{P}) = 0$);

infinitary if $\mathbf{P}$ has a μ-set of infinite cardinality ($\text{type}(\mathbf{P}) = \infty$);

finitary if $\mathbf{P}$ has a finite μ-set of cardinality greater than 1 ($\text{type}(\mathbf{P}) = \omega$);

unitary if $\mathbf{P}$ has a μ-set of cardinality 1 ($\text{type}(\mathbf{P}) = 1$).

Observe that if $\mathbf{P}$ and $\mathbf{Q}$ are isomorphic preordered sets then they have the same type. Moreover, if $\mathbf{P}$ and $\mathbf{Q}$ are equivalent as categories (see [11, pp. 11]) then they have the same type (see [9, Lemma 2.1]). We prepare for later use some easy consequences of the definitions.

Lemma 8. *The set* $\{1, \omega, \infty, 0\}$ *carries a natural total order* $1 \leq \omega \leq \infty \leq 0$. *If* $\mathbf{P}$ *is a preorder and* $Q \subseteq P$ *is an upset of* $\mathbf{P}$ *and* $\mathbf{Q}$ *denotes the preorder with universe* Q *and relation inherited from* $\mathbf{P}$*, then* $\text{type}(\mathbf{Q}) \leq \text{type}(\mathbf{P})$.

Lemma 9. *Let* $\mathbf{P} = (P, \leq)$ *be a directed preorder. Then,* $\text{type}(\mathbf{P}) = 0$ *or* $\text{type}(\mathbf{P}) = 1$.

The algebraic unification theory by Ghilardi [9] reduces the traditional symbolic unification problem over an equational theory to the following:

Problem UNIF($\mathcal{V}$).

Instance A finitely presented algebra $\mathbf{A} \in \mathcal{V}$.

Solution A homomorphism $u \colon \mathbf{A} \to \mathbf{P}$, where $\mathbf{P}$ is a finitely presented projective algebra in $\mathcal{V}$.

A solution to an instance $\mathbf{A}$ is called an *(algebraic) unifier* for $\mathbf{A}$, and $\mathbf{A}$ is called *solvable in* $\mathcal{V}$ if $\mathbf{A}$ has a solution.

Let $\mathbf{A} \in \mathcal{V}$ be finitely presented, and for $i = 1, 2$ let $u_i \colon \mathbf{A} \to \mathbf{P}_i$ be a unifier for $\mathbf{A}$. Then, u_1 is more general than u_2, in symbols, $u_2 \leq u_1$, if there exists a homomorphism $f \colon \mathbf{P}_1 \to \mathbf{P}_2$ such that $f \circ u_1 = u_2$. For $\mathbf{A}$ solvable in $\mathcal{V}$, let $U_{\mathcal{V}}(\mathbf{A})$ be the preorder induced by the generality relation over the unifiers for $\mathbf{A}$. We define the type of $\mathbf{A}$ as the type of the preordered set $U_{\mathcal{V}}(\mathbf{A})$, in symbols $\text{type}_{\mathcal{V}}(\mathbf{A}) = \text{type}(U_{\mathcal{V}}(\mathbf{A}))$. It follows from this definition that $\mathbf{A} \in \mathcal{V}$ has type 1 if and only if there exists a unifier u for $\mathbf{A}$ that is more general than each unifier of $\mathbf{A}$. Such a unifier when it exists is called a *most general unifier* for $\mathbf{A}$.

We say that the variety $\mathcal{V}$ has type:

nullary if $\{\text{type}_{\mathcal{V}}(\mathbf{A}) \mid \mathbf{A} \text{ solvable in } \mathcal{V}\} \cap \{0\} \neq \emptyset$;

infinitary if $\infty \in \{\text{type}_{\mathcal{V}}(\mathbf{A}) \mid \mathbf{A} \text{ solvable in } \mathcal{V}\} \subseteq \{\infty, \omega, 1\}$;

finitary if $\omega \in \{\text{type}_{\mathcal{V}}(\mathbf{A}) \mid \mathbf{A} \text{ solvable in } \mathcal{V}\} \subseteq \{\omega, 1\}$;

unitary if $\{\text{type}_{\mathcal{V}}(\mathbf{A}) \mid \mathbf{A} \text{ solvable in } \mathcal{V}\} = \{1\}$.

3 Finite Projective Algebras

We provide first-order decidable characterizations of the finite involutive posets corresponding to finite projective De Morgan (Theorem 11) and Kleene (Theorem 12) algebras.

Definition 10 ([5])**.** Let κ be a cardinal. A poset $(P, \leq)$ is κ-complete if, whenever $X \subseteq P$ is such that all $Y \subseteq X$ with $|Y| < \kappa$ have an upper bound, then $\bigvee X$ exists in $(P, \leq)$.

Theorem 11 (De Morgan Projective)**.** *Let* $\mathbf{A} \in \mathcal{M}_f$*. Then* $\mathbf{A}$ *is projective in* $\mathcal{M}$ *iff* $\mathsf{J}_{\mathcal{M}}(\mathbf{A}) = (P, \leq, i) \in \mathcal{PM}_f$ *satisfies the following:*

(M_1) $(P, \leq)$ *is a nonempty lattice;*

(M_2) *for all* $x \in P$*, if* $x \leq i(x)$*, then there exists* $y \in P$ *such that* $x \leq y = i(y)$*;*

(M_3) $\{x \in P \mid x \le i(x)\}$ *with inherited order is* 3*-complete.*

Proof. There exists $n \in \mathbb{N}$ such that $\mathsf{J}_{\mathcal{M}}(\mathbf{A}) = (P, \le, i) = \mathbf{P}$ is a subobject of $\mathsf{J}_{\mathcal{M}}(\mathbf{F}_{\mathcal{M}}(n)) = \mathbf{D}^n = (D^n, \le, i) \in \mathcal{PM}_f$ by Proposition 6. That is, it is possible to display $\mathbf{P}$ as a subset of $\mathbf{D}^n$ with inherited order and involution. Combining this together with Theorem 2 and Theorem 5, $\mathbf{A}$ is projective iff there exists an onto morphism $r\colon \mathbf{D}^n \to \mathbf{P}$ in $\mathcal{PM}_f$ such that $r \circ r = r$, that is $r|_P = \mathrm{id}_{\mathbf{P}}$, where $r|_P$ denotes the restriction of r to P. Therefore, it is sufficient to show that conditions (M_1)-(M_3) are necessary and sufficient for the existence of such a map r.

Below, $Z = \{z \in D^n \mid z = i(z)\} = \{0,1\}^n$ and $Y = Z \cap P$.

($\Rightarrow$) Let $r\colon \mathbf{D}^n \to \mathbf{P}$ be a morphism in $\mathcal{PM}_f$ such that $r|_P = \mathrm{id}_{\mathbf{P}}$. We show that $\mathbf{P}$ satisfies (M_1), (M_2), and (M_3).

For (M_1): In particular, r is a poset retraction of $\mathbf{D}^n$ onto $\mathbf{P}$. Since $\mathbf{D}^n$ is a nonempty lattice, it follows straightforwardly that $\mathbf{P}$ is a nonempty lattice.

For (M_2): Let $y \in P$ be such that $y \le i(y)$. Then $y \in \{2,0,1\}^n$. Let $z \in Z$ be such that for $i = 1, \ldots, n$, if $y_i \in \{0,1\}$ then $z_i = y_i$ and $z_i = 0$ otherwise. Then $y \le z \le i(y)$, and $y = r(y) \le r(z) = r(i(z)) = i(r(z))$.

For (M_3): Observe that $r((Z]) = (Y]_P$, because if $x \le z$ for $x \in D^n$ and $z \in Z$, then $r(x) \le r(z) \in Y$. Then the restriction of r to $(Z]$ is a poset retraction of $(Z]$ onto $(Y]_P$. Since $(Z] = \{2,0,1\}^n$ is 3-complete, by [5, Corollary 2.6] $r((Z]) = (Y]_P$ is 3-complete.

($\Leftarrow$) Assume that $\mathbf{P}$ satisfies (M_1), (M_2), and (M_3). We show that there exists a morphism $r\colon \mathbf{D}^n \to \mathbf{P}$ in $\mathcal{PM}_f$ such that $r|_P = \mathrm{id}_{\mathbf{P}}$.

To define the retraction $r\colon D^n \to P$, we first introduce the following notation. For all $x \in D^n$, let $L_x = \{z \in P \mid z \le x\}$ and $U_x = \{z \in P \mid x \le z\}$. Since i is an order reversing involution,

$$i(L_x) = U_{i(x)} \tag{1}$$

for each $x \in D^n$.

If $x \in Z$, then there exists $y \in Y$ is such that $\bigvee_P L_x \le y \le \bigwedge_P U_x$. In fact, if $z_1, z_2 \in L_x$, then $z_1, z_2 \le x \le i(z_1), i(z_2)$. By ($M_1$), $z_1 \vee_P z_2 \le i(z_1) \wedge_P i(z_2) = i(z_1 \vee_P z_2)$. Combining this with ($M_3$), we have $\bigvee_P L_x \le i(\bigvee_P L_x)$. Now, by ($M_2$), there exists $y \in Y$ such that $\bigvee_P L_x \le y$. Finally by (1) and the fact that $x = i(x)$, $\bigvee_P L_x \le y = i(y) \le i(\bigvee_P L_x) = \bigwedge_P i(L_x) = \bigwedge_P U_{i(x)} = \bigwedge_P U_x$, as desired.

For each $x \in Z$, we fix $r(x) \in Y$ such that

$$\textstyle\bigvee_P L_x \le r(x) \le \bigwedge_P U_x. \tag{2}$$

If $x \in D^n \setminus Z$, then let m be the smallest number in $\{1, \ldots, n\}$ such that $x_m \in \{2,3\}$. We define

$$r(x) = \begin{cases} \bigvee_P L_x, & \text{if } x_m = 2; \\ \bigwedge_P U_x, & \text{if } x_m = 3. \end{cases} \tag{3}$$

The map $r\colon D^n \to P$ is well defined. Also, $r(x) = x$ for all $x \in P$ because $x \in U_x \cap L_x$.

Claim 1: For each $x \in D^n$, $r(i(x)) = i(r(x))$.

If $x \in Z$, then $r(i(x)) = i(r(x))$ holds because $r(x) \in Y$. Let $x \in D^n \setminus Z$, and m be the smallest number in $\{1, \ldots, n\}$ such that $x_m \in \{2, 3\}$. If $x_m = 2$ and $(i(x))_m = 3$, then $r(x) = \bigvee_P L_x$ and $r(i(x)) = \bigwedge_P U_{i(x)}$. Then, by (1),

$$\begin{aligned} r(i(x)) &= \textstyle\bigwedge_P U_{i(x)} = \bigwedge_P i(L_x) \\ &= i\left(\textstyle\bigvee_P L_x\right) \\ &= i(r(x)). \end{aligned}$$

The case $x_m = 3$ and $(i(x))_m = 2$ reduces to the previous case, which concludes the proof of the claim.

Claim 2: r is monotone.

Let $x, y \in D^n$ such that $x < y$. Then $L_x \subseteq L_y$, $U_y \subseteq U_x$. Therefore

$$\textstyle\bigvee_P L_x \leq \bigvee_P L_y \leq \bigwedge_P U_y, \tag{4}$$

and

$$\textstyle\bigwedge_P U_x \leq \bigwedge_P U_y. \tag{5}$$

If $x \in Z$, observe that $y \in \{0, 1, 3\}^n \setminus \{0, 1\}^n$. By (2), (3) and (5), $r(x) \leq \bigwedge_P U_x \leq \bigwedge_P U_y = r(y)$. A similar argument proves that $r(x) \leq r(y)$ if $y \in Z$.

If $x, y \in D^n \setminus Z$ and $r(x) = \bigvee_P L_x$, then $r(x) \leq r(y)$ by (4) and (3). If $r(x) = \bigwedge_P U_x$, then by (3), letting m be the smallest number in $\{1, \ldots, n\}$ such that $x_m = 3$, since $x \leq y$ it follows that $y_k \in \{0, 1, 3\}$ for every $k < m$, and $y_m = 3$. Again by (3), $r(y) = \bigwedge_P U_y$. Therefore, $r(x) \leq r(y)$ by (5), which concludes the proof of the claim.

By Claim 1 and Claim 2, $r \colon \mathbf{D}^n \to \mathbf{P}$ is the required retraction, so that $\mathbf{A}$ is a retract of $\mathbf{F}_{\mathcal{M}}(n)$ in $\mathcal{M}_f$, that is, it is projective in $\mathcal{M}_f$. □

Since $(P, \leq)$ is a finite lattice by (M_1), condition (M_3) reduces to the following first-order statement: $x \vee y \vee z \leq i(x \vee y \vee z)$, for all x, y, z such that $x \vee y \leq i(x \vee y)$, $x \vee z \leq i(x \vee z)$, and $y \vee z \leq i(y \vee z)$.

Theorem 12 (Kleene Projective). *Let $\mathbf{A} \in \mathcal{K}_f$. Then $\mathbf{A}$ is projective in $\mathcal{K}$ iff $\mathsf{J}_{\mathcal{K}}(\mathbf{A}) = (P, \leq, i) \in \mathcal{PK}_f$ satisfies conditions (M_2), (M_3) in Theorem 11 and the conditions:*

(K_1) *$\{x \in P \mid x \leq i(x)\}$ with inherited order is a nonempty meet semilattice;*

(K_2) *every $x, y \in P$ such that $x, y \leq i(y), i(x)$ have a common upper bound $z \in P$ such that $z \leq i(z)$.*

Proof. There exists $n \in \mathbb{N}$ such that $\mathsf{J}_{\mathcal{K}}(\mathbf{A}) = (P, \leq, i) = \mathbf{P}$ is a subobject of $\mathsf{J}_{\mathcal{K}}(\mathbf{F}_{\mathcal{K}}(n)) = (\mathbf{D}^n)_k = ((D^n)_k, \leq, i) \in \mathcal{PK}_f$ by Proposition 7. Combining Theorem 2 and Theorem 5, $\mathbf{A}$ is projective iff there exists a morphism $r \colon (\mathbf{D}^n)_k \to \mathbf{P}$ in $\mathcal{PK}_f$ such that $r|_P = \mathrm{id}_{\mathbf{P}}$. Therefore, it is sufficient to show that conditions (K_1), (K_2), (M_2), and (M_3) are necessary and sufficient for the existence of such a map r.

Below, $Z = \{z \in (D^n)_k \mid z = i(z)\} = \{0, 1\}^n$ and $Y = Z \cap P$.

$(\Rightarrow)$ Let $r \colon (\mathbf{D}^n)_k \to \mathbf{P}$ be a morphism in $\mathcal{PK}_f$ such that $r|_P = \mathrm{id}_{\mathbf{P}}$.

The proof that $\mathbf{P}$ satisfies (M_2) and (M_3) follows by the same argument used in the proof of Theorem 11.

For (K_1): First observe that $r(Z) = Y$ and $r((Z]) = (Y]_P$. Then the restriction of r to $(Z]$ is a poset retraction of $(Z]$ onto $(Y]_P$. Since $(Z]$ is a nonempty meet semilattice, $(Y]_P$ is a nonempty meet semilattice [5, Lemma 2.4].

For (K_2): Let $x, y \in P$ be such that $x, y \leq_P i(x), i(y)$. Then there does not exist $i \in \{1, \ldots, n\}$ such that $x_i = 0$ and $y_i = 1$ (otherwise, $x \parallel_P i(y)$), which proves that $x \vee_{D^n} y \in \{2, 0, 1\}^n = (Z] \subseteq D^n_k$. Then $z = r(x \vee_{D^n} y) \in (Y]_P$. Therefore, $x, y \leq z \leq i(z)$, as desired.

($\Leftarrow$) Let $\mathbf{P}$ be a subset of $(\mathbf{D}^n)_k$ with inherited order and involution satisfying (K_1), (K_2), (M_2) and (M_3). We define a morphism $r\colon (\mathbf{D}^n)_k \to \mathbf{P}$ in $\mathcal{PK}_f$ such that $r|_P = \mathrm{id}_{\mathbf{P}}$.

Since $\mathbf{P} \in \mathcal{PK}_f$, by (M_2) we have that $P = (Z] \cup [Z) = (Z] \cup i((Z])$. Moreover, since Z is an antichain, $(Z] \cap [Z) = Z$. For all $x \in (Z]$, let $L_x = \{z \in P \mid z \leq x\}$. Observe that $L_x \subseteq (Y]_P$ by (M_2). Also, if $v, w \in L_x$, then $v, w \leq x \leq i(v), i(w)$, and combining (K_1) and (K_2), we obtain that $v \vee_P w \in (Y]_P$ for all $v, w \in L_x$. Therefore, $\bigvee_P L_x \in (Y]_P$ by (M_3).

For all $x \in Z$, we define $r(x) \in Y$ such that

$$\bigvee\nolimits_P L_x \leq r(x), \tag{6}$$

whose existence is ensured by condition (M_2). And for all $x \in (Z] \setminus Z$ we define

$$r(x) = \bigvee\nolimits_P L_x, \tag{7}$$
$$r(i(x)) = i(r(x)). \tag{8}$$

The map $r\colon (D^n)_k \to P$ is well defined. We prove that r is the desired retraction. Clearly, if $x \in P$, then $r(x) = x$. By definition, r commutes with i. We check monotonicity. Let $x, y \in (D^n)_k$ be such that $x < y$. If $x, y \in (Z]$, then $L_x \subseteq L_y$, then

$$r(x) = \bigvee\nolimits_P L_x \leq \bigvee\nolimits_P L_y \leq r(y),$$

where the last inequality always holds by (6) and (7). If $x \in (Z]$ and $y \in [Z)$, then there exists $z \in Z$ such that $x \leq z \leq y$, so that $i(y) \leq z$. Then $r(x), r(i(y)) \leq r(z)$ by the previous case, but $r(i(y)) = i(r(y))$ by commutativity of r, so that $r(z) = r(i(z)) = i(r(z)) \leq r(y)$ by the properties of i and commutativity of r. If $x, y \in [Z)$, then $i(x), i(y) \in (Z]$ and $i(y) \leq i(x)$, then $r(i(y)) \leq r(i(x))$, then $i(r(y)) \leq i(r(x))$, and so $r(x) \leq r(y)$. $\square$

Observe that the conditions (K_1), (K_2) are first-order conditions on the set $\{x \in P \mid x \leq i(x)\}$.

4 Classification of Unification Problems

We obtain a complete, decidable, first-order classification of unification problems over bounded distributive lattices (Theorem 15), Kleene algebras (Theorem 22) and De Morgan algebras (Theorem 31) algebras with respect to unification type. In particular, we establish that unification over the varieties of De Morgan and Kleene algebras is nullary.

For the sake of presentation, we introduce the following notion. An *alphabet* Σ is a set of letters. A *word* over Σ is a finite sequence of letters in Σ. A *formal language* over Σ is a subset of words over Σ.

4.1 Distributive Lattices

We classify all solvable instances of the unification problem over bounded distributive lattices with respect to their unification type (Theorem 15), thus tightening the nullarity result by Ghilardi in [9]. This case study prepares the technically more involved cases of Kleene and De Morgan algebras.

In [4], Balbes and Horn characterize projective bounded distributive lattices. In the finite case, the characterization states that a finite bounded distributive lattice $\mathbf{L} \in \mathcal{D}_f$ is projective iff the finite poset $\mathsf{J}(\mathbf{L}) \in \mathcal{P}_f$ is a nonempty lattice. Thus, combining the algebraic unification theory developed by Ghilardi [9] and the finite duality by Birkhoff (Theorem 3), a unification problem over bounded distributive lattices reduces to the following combinatorial question:

Problem $\textsc{Unif}(\mathcal{DL})$.

Instance A finite poset $\mathbf{P}$.

Solution A monotone map $u\colon \mathbf{L} \to \mathbf{P}$, where $\mathbf{L}$ is a finite nonempty lattice.

We call a solution for a finite poset $\mathbf{P}$ a *unifier* for $\mathbf{P}$. The reason for this is that by the finite duality in Theorem 3 unifiers for $\mathbf{P}$ correspond to (algebraic) unifiers for $\mathsf{D}(\mathbf{P})$. Accordingly, we define the preorder of the of unifiers of $\mathbf{P}$ to agree with this identification. For $i = 1, 2$ let $u_i\colon \mathbf{L}_i \to \mathbf{P}$ be unifiers for $\mathbf{P}$. Then u_1 is more general than u_2, in symbols, $u_2 \leq u_1$, iff there exists a monotone map $f\colon \mathbf{L}_2 \to \mathbf{L}_1$ such that $u_1 \circ f = u_2$. Let $U_{\mathcal{DL}}(\mathbf{P})$ denote the preordered set of unifiers of $\mathbf{P}$. Then, the unification type of $\mathbf{P}$ is defined as usual, $\mathrm{type}_{\mathcal{DL}}(\mathbf{P}) = \mathrm{type}(U_{\mathcal{DL}}(\mathbf{P}))$. By the duality in Theorem 5, $U_{\mathcal{DL}}(\mathbf{P})$ and $U_{\mathcal{DL}}(\mathsf{D}(\mathbf{P}))$ are equivalent as categories. Therefore, [9, Lemma 2.1] implies that $\mathrm{type}_{\mathcal{DL}}(\mathbf{P}) = \mathrm{type}_{\mathcal{DL}}(\mathsf{D}(\mathbf{P}))$.

Remark 13. An instance $\mathbf{P} = (P, \leq)$ of $\textsc{Unif}(\mathcal{DL})$ is solvable iff $P \neq \emptyset$.

We now embark in the proof of the main result of this section. The structure of the proof is the following: using a slight modification of [9, Theorem 5.7], we identify a sufficient condition for an instance of the unification problem to have nullary type (Lemma 14), and then we prove that the identified condition is indeed necessary for nullarity (Theorem 15).

Lemma 14. *Let $\mathbf{Q} = (Q, \leq) \in \mathcal{P}_f$ be an instance of* $\textsc{Unif}(\mathcal{DL})$*. If there exist $x, a, b, c, d, y \in Q$ such that:*

(i) $x \leq a, b \leq c, d \leq y$;

(ii) there does not exist $e \in Q$ such that $a, b \leq e \leq c, d$;

then $\mathrm{type}_{\mathcal{DL}}(\mathbf{Q}) = 0$ *(see Fig. 5).*

Proof. Since $\mathbf{Q}$ is a finite poset, we assume without loss of generality $x \in \min(\mathbf{Q})$ and $y \in \max(\mathbf{Q})$. By (ii), we have $a \neq b$ and $c \neq d$. Let

$$V = \{u\colon \mathbf{P} \to \mathbf{Q} \in U_{\mathcal{DL}}(\mathbf{Q}) \mid x, y \in u(P)\}.$$

Since V is an upset of $U_{\mathcal{DL}}(\mathbf{Q})$, by Lemma 8, it is enough to prove that $\mathrm{type}(V) = 0$ to conclude that $\mathrm{type}(U_{\mathcal{DL}}(\mathbf{Q})) = \mathrm{type}_{\mathcal{DL}}(\mathbf{Q}) = 0$. We first

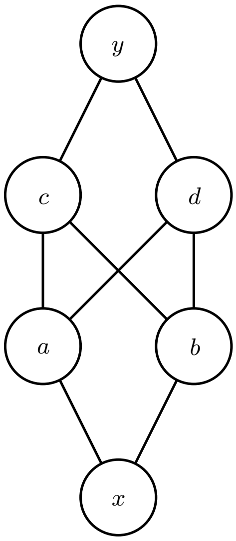

Figure 5: Subposet of $\mathbf{Q}$ in Lemma 14.

observe that V is directed. Indeed, if $u_1 \colon \mathbf{R}_1 \to \mathbf{Q}$ and $u_2 \colon \mathbf{R}_2 \to \mathbf{Q}$ in V, define $\mathbf{P} = (P, \leq)$ by adjoining a fresh bottom $\bot$ and a fresh top $\top$ to the disjoint union of $\mathbf{R}_1$ and $\mathbf{R}_2$. It is easy to check that $\mathbf{P}$ is a lattice. Define $u(y) = u_j(y)$ iff $y \in R_j$ for $j = 1, 2$, $u(\bot) = x$ and $u(\top) = y$. Since x and y are minimal and maximal in $\mathbf{Q}$ respectively, u is a monotone map from $\mathbf{P}$ into $\mathbf{Q}$ and $u \in V$. For $j = 1, 2$, let $f_j \colon \mathbf{R}_j \to \mathbf{P}$ in $\mathcal{P}_f$ be the injection of $\mathbf{R}_j$ into $\mathbf{P}$. Then $u_j = u \circ f_j$ for $j = 1, 2$, which proves that V is directed.

Since V is a directed preordered set with the inherited order of $U_{\mathcal{DL}}(\mathbf{Q})$, by Lemma 9, type$(V) \in \{0, 1\}$. We show that type$(V) \neq 1$. For every $n \in \mathbb{N}$, we define a unifier $u_n \colon \mathbf{T}_n \to \mathbf{Q}$ in V as follows. For $\mathbf{T}_n = (T_n, \leq) \in \mathcal{P}_f$ we let

$$T_n = \{\bot, \top, 1, \dots, n, j \cdot k \mid j < k \text{ in } \{1, \dots, n\} \text{ and } j + k \text{ is odd}\};$$

here, T_n is a formal language over the alphabet $\{\bot, \top, \cdot, 1, \dots, n\}$. The partial order over T_n is defined by the following cover relation, where $j, k \in \{1, \dots, n\}$:

$\bot \prec j$;

$j, k \prec j \cdot k$ for all $j \cdot k \in T_n$;

$j \cdot k \prec \top$ for all $j \cdot k \in T_n$;

where $j, k \in \{1, \dots, n\}$.

Then $\mathbf{T}_n$ is a lattice. See Fig. 6 for the Hasse diagram of $\mathbf{T}_5$.

We define $u_n \colon \mathbf{T}_n \to \mathbf{Q}$ as follows, where $j, k \in \{1, \dots, n\}$:

$u_n(\bot) = x$ and $u_n(\top) = y$;

$u_n(j) = a$ and $u_n(j \cdot k) = c$, for all $j, j \cdot k \in T_n$ with j odd;

$u_n(j) = b$ and $u_n(j \cdot k) = d$, for all $j, j \cdot k \in T_n$ with j even.

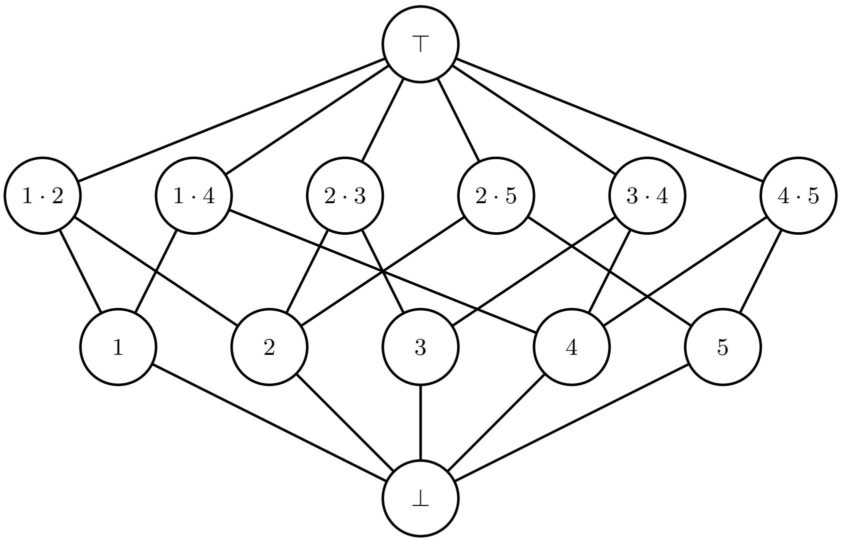

Figure 6: $\mathbf{T}_5$ in Lemma 14.

Since for each $n \in \{1, 2, \ldots\}$, $u_n\colon \mathbf{T}_n \to \mathbf{Q}$ is a monotone map, $\mathbf{T}_n$ is a lattice and $x, y \in u_n(\mathbf{T}_n)$, then u_n is a unifier for $\mathbf{Q}$ in V.

Let $u\colon \mathbf{P} \to \mathbf{Q}$ be a unifier in V. We show that $u_n \leq u$ implies $|P| \geq n$. Let $u_n = u \circ f$. We claim that $f(j) \neq f(k)$ for all $j < k$ with $j, k \in \{1, \ldots, n\}$. The claim is clear if j and k have different parity. If j and k have the same parity, without loss of generality assume j and k are both odd, then let l be even such that $j < l < k$. By construction $j, l \leq j \cdot l$, then we have $f(j), f(l) \leq f(j \cdot l)$. Since $\mathbf{P}$ is a lattice,

$$f(j), f(l) \leq f(j) \vee f(l) \leq f(j \cdot l).$$

Similarly

$$f(l), f(k) \leq f(l) \vee f(k) \leq f(l \cdot k).$$

Assume for a contradiction that $f(j) = f(k)$. Then

$$f(j) = f(k), f(l) \leq f(j) \vee f(l) = f(l) \vee f(k) \leq f(j \cdot l), f(l \cdot k),$$

and applying u through, since $u_n = u \circ f$,

$$a, b \leq u(f(l) \vee f(k)) \leq c, d,$$

contradicting (ii). Therefore, a most general unifier $u\colon \mathbf{P} \to \mathbf{Q}$ has $|P| \geq n$ for every $n \in \mathbb{N}$, impossible because $\mathbf{P}$ is finite. Thus, $\mathrm{type}(V) \neq 1$.

Then $\mathrm{type}(V) = 0$ and by Lemma 8, $\mathrm{type}_{\mathcal{DL}}(\mathbf{Q}) = 0$, as desired. □

Theorem 15. *Let $\mathbf{P} = (P, \leq) \in \mathcal{P}_f$ be a solvable instance of* $\textsc{Unif}(\mathcal{DL})$. *Then:*

$$\mathrm{type}_{\mathcal{DL}}(\mathbf{P}) = \begin{cases} 1, & \textit{iff } \mathbf{P} \textit{ is a lattice;} \\ \omega, & \textit{iff } \mathbf{P} \textit{ is not a lattice,} \\ & \textit{but } [x, y] \textit{ is a lattice for all } x \leq y \textit{ in } \mathbf{P}; \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. If $\mathbf{P}$ is a lattice, then $\text{type}_{\mathcal{DL}}(\mathbf{P}) = 1$ because $\text{id}_{\mathbf{P}}$ is a most general unifier for $\mathbf{P}$.

Suppose that $\mathbf{P}$ is not a lattice and $[x, y]$ is a lattice for all $x \leq y$ in $\mathbf{P}$. Define, for every $x, y \in P$ such that $x \leq y$, $x \in \min(\mathbf{P})$, and $y \in \max(\mathbf{P})$, the (inclusion) unifier $u_{x,y} : [x, y] \to \mathbf{P}$ by $u_{x,y}(z) = z$ for all $z \in [x, y]$. Clearly, there are finitely many unifiers of the form $u_{x,y}$ with $x \leq y$ in $\mathbf{P}$, because P is finite. We claim that they form a μ-set in $U_{\mathcal{DL}}(\mathbf{P})$. Since $x \in \min(\mathbf{P})$, and $y \in \max(\mathbf{P})$, any two unifiers of the type $u_{x,y}$ and $u_{x',y'}$ are comparable iff $x = x'$ and $y = y'$. Now let $u : \mathbf{L} \to \mathbf{P}$ be a unifier for $\mathbf{P}$. Now, $\mathbf{L}$ is bounded, with bottom $\perp$ and top $\top$. Let $x \in \min(\mathbf{P})$ and $y \in \max(\mathbf{P})$ be such that $x \leq u(\perp) \leq u(\top) \leq y$. Then $u(L) \subseteq [x, y]$ and $u_{x,y} \circ u = u$, so that $u_{x,y}$ is more general than u. Thus $\text{type}_{\mathcal{DL}}(\mathbf{P}) \in \{1, \omega\}$.

Since $\mathbf{P}$ is not a lattice but for each $x \leq y$ in $\mathbf{P}$, $[x, y]$ is a lattice, then $\mathbf{P}$ cannot be bounded. Assume that $x_1 \neq x_2$ are minimal points in $\mathbf{P}$ (the argument is similar for maximal points). Then let $\mathbf{L} = (\{p\}, \leq)$, for $i = 1, 2$ let $u_i : \mathbf{L} \to \mathbf{P}$ be the unifier such that $u_i(p) = x_i$. Suppose for a contradiction that there exists a unifier $u : \mathbf{M} \to \mathbf{P}$ such that $u_1, u_2 \leq u$. For $i = 1, 2$, let $u \circ f_i = u_i$ be a factorization of u_i where $u : \mathbf{M} \to \mathbf{P}$. Then by monotonicity $u(f_1(p) \wedge f_2(p)) \leq u(f_1(p)), u(f_2(p))$, and since $x_1 \parallel x_2$, we have $u(f_1(p) \wedge f_2(p)) < x_1, x_2$ which contradicts the minimality of x_1 and x_2. Thus u_1 and u_2 have no common upper bound in $U_{\mathcal{DL}}(\mathbf{P})$, and $\text{type}_{\mathcal{DL}}(\mathbf{P}) \neq 1$. This concludes the proof that $\text{type}_{\mathcal{DL}}(\mathbf{P}) = \omega$.

Finally, let $x \leq y$ in P be such that $[x, y]$ is not a lattice. Then there exist $a, b, c, d \in P$ such that $x \leq a, b \leq c, d \leq y$ and there does not exist $e \in P$ such that $a, b \leq e \leq c, d$. By Lemma 14, it follows that $\text{type}_{\mathcal{DL}}(\mathbf{P}) = 0$. □

4.2 Kleene Algebras

We provide a complete classification of solvable instances of the unification problem over Kleene algebras (Theorem 22). Combining the algebraic unification theory by Ghilardi [9], Theorem 12, and Theorem 5, the unification problem over Kleene algebras reduces to the following combinatorial question:

Problem Unif($\mathcal{K}$).

Instance $\mathbf{Q} = (Q, \leq, i) \in \mathcal{PK}_f$.

Solution A morphism $u : \mathbf{P} \to \mathbf{Q}$ in $\mathcal{PK}_f$, where $\mathbf{P}$ satisfies (K_1), (K_2), (M_2), and (M_3).

Remark 16. An instance $\mathbf{Q} = (Q, \leq, i)$ of Unif($\mathcal{K}$) is solvable iff $\{x \in Q \mid x = i(x)\} \neq \emptyset$. Indeed, if $\mathbf{P} \in \mathcal{PK}_f$ satisfies (K_1) and (M_2), and $\mathbf{Q}$ admits a morphism from $\mathbf{P}$, then $\{x \in Q \mid x = i(x)\} \neq \emptyset$. Conversely, if $\mathbf{Q}$ is such that $\{x \in Q \mid x = i(x)\} \neq \emptyset$, then $\mathbf{Q}$ admits a morphism from $\mathbf{P} = (P, \leq, i)$ where $P = \{x\}$, and $i(x) = x$; clearly, $\mathbf{P}$ satisfies (K_1), (K_2), (M_2), and (M_3).

Given a solvable instance $\mathbf{Q}$ of Unif($\mathcal{K}$), we let $U_{\mathcal{K}}(\mathbf{Q})$ denote the preordered set of unifiers of $\mathbf{Q}$, which is defined as in Section 4.1.

We now embark in the proof of the main result of this section. The structure of the proof is the following: we identify two sufficient conditions for an instance of the unification problem to have nullary type (Lemma 18 and Lemma 19), and then we prove that the identified conditions are indeed necessary for nullarity (Theorem 22).

We first establish the following fact for later use.

Lemma 17. *Let* $\mathbf{Q} = (Q, \leq, i) \in \mathcal{PK}_f$ *be an instance of* $\textsc{Unif}(\mathcal{K})$ *and* $x \in Q$ *be a minimal element of* $\mathbf{Q}$*. Then*

$$V = \{u \colon \mathbf{P} \to \mathbf{Q} \in U_{\mathcal{K}}(\mathbf{Q}) \mid x \in u(P)\} \tag{9}$$

is a directed upset in $U_{\mathcal{K}}(\mathbf{Q})$*.*

Proof. Clearly, V is an upset in $U_{\mathcal{K}}(\mathbf{Q})$. If V is empty, directedness is trivial. Otherwise, there exists $y \in Q$ such that $x \leq y = i(y)$, which proves that $x \leq i(x)$. Let $u_1 \colon \mathbf{R}_1 \to \mathbf{Q}$ and $u_2 \colon \mathbf{R}_2 \to \mathbf{Q}$ in V, with $\mathbf{R}_j = (R_j, \leq_j, i_j)$ for $j = 1, 2$. Define $\mathbf{P} = (P, \leq, i)$ by adjoining a fresh bottom $\bot$ and a fresh top $\top$ to the disjoint union of $\mathbf{R}_1$ and $\mathbf{R}_2$, and by letting $i(\bot) = \top$, $i(\top) = \bot$, and $i(y) = i_j(y)$ iff $y \in R_j$ for $j = 1, 2$. Since $\mathbf{R}_1$ and $\mathbf{R}_2$ satisfy (K_1), (K_2), (M_2), and (M_3), so does $\mathbf{P}$. Let $u \colon P \to Q$ be the map defined by: $u(y) = u_j(y)$ iff $y \in R_j$ for $j = 1, 2$, $u(\bot) = x$ and $u(\top) = i(x)$. It follows that $u \in V$. For $j = 1, 2$, let $f_j \colon \mathbf{R}_j \to \mathbf{P}$ in $\mathcal{PK}_f$ be the injection of $\mathbf{R}_j$ into $\mathbf{P}$. Then $u = u_j \circ f_j$ for $j = 1, 2$, as desired. ☐

Lemma 18. *Let* $\mathbf{Q} = (Q, \leq, i) \in \mathcal{PK}_f$ *be an instance of* $\textsc{Unif}(\mathcal{K})$*. If there exist* $x, a, b, c, d, y, z \in Q$ *such that:*

(i) $x \leq a, b \leq c, d$*;*

(ii) $c \leq y = i(y)$*;* $d \leq z = i(z)$*;*

(iii) there does not exist $e \in Q$ *such that* $a, b \leq e \leq c, d$*;*

then $\mathrm{type}_{\mathcal{K}}(\mathbf{Q}) = 0$ *(see Fig. 7).*

Proof. Since $\mathbf{Q}$ is a finite poset, we assume without loss of generality $x \in \min(\mathbf{Q})$. By *(iii)*, we have $a \neq b$ and $c \neq d$. Let

$$V = \{u \colon \mathbf{P} \to \mathbf{Q} \in U_{\mathcal{K}}(\mathbf{Q}) \mid x \in u(P)\}.$$

By Lemma 17 V is a directed upset of $U_{\mathcal{K}}(\mathbf{Q})$. By Lemma 8, to prove that $\mathrm{type}(U_{\mathcal{K}}(\mathbf{Q})) = 0$ it is enough to prove that $\mathrm{type}(V) = 0$. Since V is directed, by Lemma 9, $\mathrm{type}(V) \in \{0, 1\}$. We show that $\mathrm{type}(V) \neq 1$. For every $n \in \mathbb{N}$, we define a unifier $u_n \colon \mathbf{T}_n \to \mathbf{Q}$ in V as follows. For $\mathbf{T}_n = (T_n, \leq, i) \in \mathcal{PM}_f$ we let

$$T_n = \{\bot, \overline{\bot}, j, \overline{j}, j \cdot k, \overline{j \cdot k}, j \diamond k \mid j < k \text{ in } \{1, \ldots, n\} \text{ and } j + k \text{ is odd}\};$$

here, T_n is a formal language over the alphabet $A \cup \{\overline{s} \mid s \in A\}$, with $A = \{\bot, \cdot, \diamond, 1, \ldots, n\}$. The map $i \colon T_n \to T_n$ is defined as follows, where $j, k \in \{1, \ldots, n\}$ and $y \in \{\bot, j, j \cdot k \mid j < k \text{ in } \{1, \ldots, n\} \text{ and } j + k \text{ is odd}\} \subseteq T_n$:

$i(j \diamond k) = j \diamond k$ for all $j \diamond k \in T_n$;

$i(y) = \overline{y}$ and $i(\overline{y}) = y$ for all $y, \overline{y} \in T_n$.

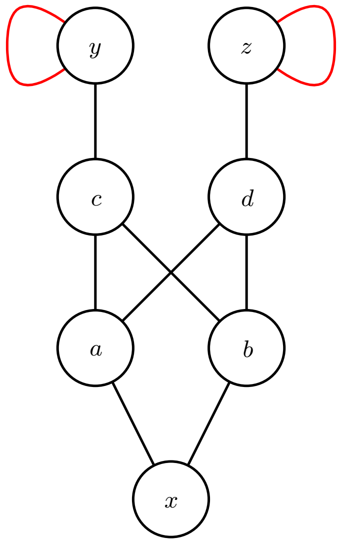

Figure 7: Subposet of $\mathbf{Q}$ in Lemma 18.

The partial order over T_n is defined by the following cover relation, for all $j, k \in \{1, \ldots, n\}$:

$\bot \prec j$ and $i(j) \prec i(\bot)$, for all $j \in T_n$

$j, k \prec j \cdot k$ and $i(j \cdot k) \prec i(j), i(k)$, for all $j, k, j \cdot k \in T_n$.

It is easy to check that $\mathbf{T}_n$ satisfies (K_1), (K_2), (M_2), and (M_3). Figure 8 provides the Hasse diagram of $\mathbf{T}_4$.

For $j, k \in \{1, \ldots, n\}$, we define $u_n \colon \mathbf{T}_n \to \mathbf{Q}$ by putting

$u_n(\bot) = x$;

$u_n(j) = a$, $u_n(j \cdot k) = c$, $u_n(j \diamond k) = y$, for all $j, j \cdot k, j \diamond k \in T_n$ with j odd;

$u_n(j) = b$, $u_n(j \cdot k) = d$, $u_n(j \diamond k) = z$, for all $j, j \cdot k, j \diamond k \in T_n$ with j even;

and, for all $y \in \{\bot, j, j \cdot k, j \diamond k \mid j < k \text{ in } \{1, \ldots, n\} \text{ and } j + k \text{ is odd}\} \subseteq T_n$,

$u_n(i(y)) = i(u_n(y))$.

It follows by a straightforward computation that $u_n \colon \mathbf{T}_n \to \mathbf{Q}$ is a morphism in $\mathcal{PK}_f$. Therefore u_n is a unifier for $\mathbf{Q}$ in V for each $n \in \mathbb{N}$.

Let $u \colon \mathbf{P} \to \mathbf{Q}$ be a unifier for $\mathbf{Q}$. We show that $u_n \leq u$ implies $|P| \geq n$. Let $u_n = u \circ f$. We claim that $f(j) \neq f(k)$ for all $j < k$ with $j, k \in T_n$. If $j + k$ is odd, it is straightforward. If $j + k$ is even, without loss of generality assume j, k both odd. Then let l be an even number such that $j < l < k$. By construction $j, l \leq j \cdot l \leq i(j \cdot l)$, then we have $f(j), f(l) \leq f(j \cdot l) \leq i(f(j \cdot l))$. By (K_1), $f(j) \vee f(l)$ exists in $\mathbf{P}$ and it satisfies:

$$f(j) \vee f(l) \leq i(f(j) \vee f(l)).$$

Then

$$f(j), f(l) \leq f(j) \vee f(l) \leq f(j \cdot l).$$

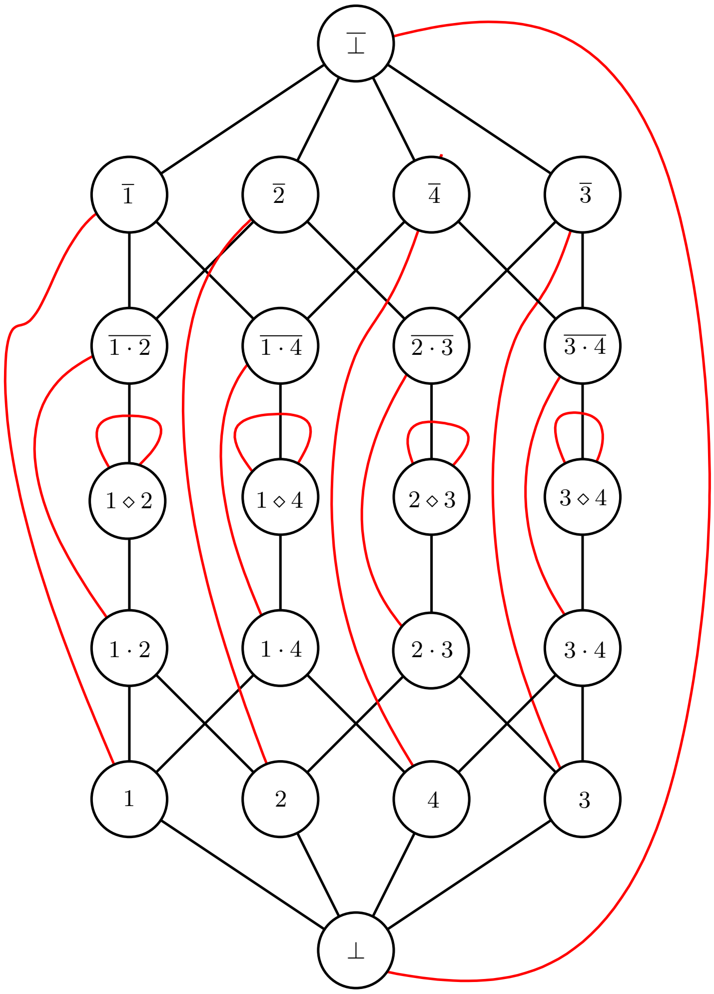

Figure 8: $\mathbf{T}_4$ in Lemma 18.

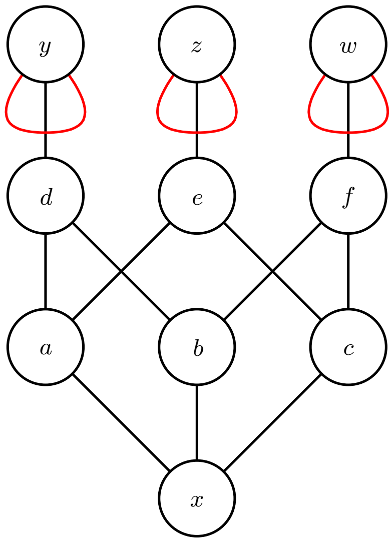

Figure 9: Subposet of **Q** in Lemma 19 and Lemma 28.

Similarly, $f(l) \vee f(k)$ exists in **P** and it satisfies:

$$f(l) \vee f(k) \leq i(f(l) \vee f(k)).$$

Then

$$f(l), f(k) \leq f(l) \vee f(k) \leq f(l \cdot k).$$

By way of contradiction assume $f(j) = f(k)$, then

$$f(j) = f(k), f(l) \leq f(j) \vee f(l) = f(l) \vee f(k) \leq f(j \cdot l), f(l \cdot k),$$

and applying u through, since $u_n = u \circ f$,

$$a, b \leq u(f(l) \vee f(k)) \leq c, d,$$

which contradicts (iii). Therefore, $|P| \geq n$.

This proves that $\text{type}(V) \neq 1$. Then $\text{type}(V) = 0$. By Lemma 8, $\text{type}_{\mathcal{K}}(\mathbf{Q}) = 0$, as desired. ☐

Lemma 19. *Let* $\mathbf{Q} = (Q, \leq, i)$ *be an instance of* $\textsc{Unif}(\mathcal{K})$. *If there exist* $x, a, b, c, d, e, f, y, z, w \in Q$ *such that:*

(i) $x \leq a, b, c$; $a \leq d, e$; $b \leq d, f$; $c \leq e, f$;

(ii) $d \leq y = i(y)$; $e \leq z = i(z)$; $f \leq w = i(w)$;

(iii) there does not exist $g \in Q$ *such that* $a, b, c \leq g \leq i(g)$,

then $\text{type}_{\mathcal{K}}(\mathbf{Q}) = 0$ *(see Fig. 9).*

Proof. Since **Q** is a finite poset, we assume without loss of generality that $x \in \min(\mathbf{Q})$. By (i) and (iii), we have $|\{a, b, c\}| = |\{d, e, f\}| = 3$. Let

$$V = \{u \colon \mathbf{P} \to \mathbf{Q} \in U_{\mathcal{K}}(\mathbf{Q}) \mid x \in u(P)\}.$$

By Lemma 17, V is an directed upset in $U_{\mathcal{K}}(\mathbf{Q})$. Then Lemma 9 proves $\mathrm{type}(V) \in \{0, 1\}$. We show that $\mathrm{type}(V) \neq 1$. For every $n \in \mathbb{N}$, we define a unifier $u_n \colon \mathbf{T}_n \to \mathbf{Q}$ in V as follows. Let $T_n = L \cup I \cup \overline{L}$ where

$$\begin{aligned} L =& \{\bot, j, j \cdot k, j \circ j \cdot k \mid j \neq k \text{ in } \{1, \ldots, n\}\} \cup \\ & \{j \cdot k \circ k \cdot j \mid j < k \text{ in } \{1, \ldots, n\}\}, \\ \overline{L} =& \{\overline{v} \mid v \in L\}, \\ I =& \{j \diamond j \cdot k \mid j \neq k \text{ in } \{1, \ldots, n\}\} \cup \{j \cdot k \diamond k \cdot j \mid j < k \text{ in } \{1, \ldots, n\}\}; \end{aligned}$$

here, T_n is a formal language over $A \cup \{\overline{s} \mid s \in A\}$ with $A = \{\bot, \circ, \diamond, \cdot, 1, \ldots, n\}$. The map $i \colon T_n \to T_n$ is defined by:

$$i(v) = v \text{ for all } v \in I;$$

$$i(v) = \overline{v} \text{ and } i(\overline{v}) = v \text{ for all } v \in L.$$

The partial order over T_n is defined by the cover relation containing the covers listed below, where $j, k \in \{1, \ldots, n\}$:

$$\bot \prec j, j \cdot k \text{ for all } j, j \cdot k \in T_n;$$

$$j, j \cdot k \prec j \circ j \cdot k \text{ for all } j, j \cdot k, j \circ j \cdot k \in T_n;$$

$$j \cdot k, k \cdot j \prec j \cdot k \circ k \cdot j \text{ for all } j \cdot k, k \cdot j \in T_n;$$

$$j \circ j \cdot k \prec j \diamond j \cdot k \text{ for all } j \circ j \cdot k, j \diamond j \cdot k \in T_n;$$

$$j \cdot k \circ k \cdot j \prec j \cdot k \diamond k \cdot j \text{ for all } j \cdot k \circ k \cdot j, j \cdot k \diamond k \cdot j \in T_n;$$

and, for each $x \prec y$ in the list, the cover

$$i(y) \prec i(x).$$

It is easy to check that $\mathbf{T}_n$ satisfies (M_1), (M_2), and (M_3). Notice that (M_1) implies (K_1) and (K_2). Figure 10 provides the Hasse diagram of $\mathbf{T}_2$.

We define $u_n \colon \mathbf{T}_n \to \mathbf{Q}$ as follows, where $j, k \in \{1, \ldots, n\}$:

$$u_n(\bot) = x;$$

$$u_n(j) = a \text{ for all } j \in T_n;$$

$$u_n(j \cdot k) = b \text{ for all } j \cdot k \in T_n \text{ with } j < k;$$

$$u_n(j \cdot k) = c \text{ for all } j \cdot k \in T_n \text{ with } k < j;$$

$$u_n(j \circ j \cdot k) = d \text{ for all } j \circ j \cdot k \in T_n \text{ with } j < k;$$

$$u_n(j \cdot k \circ k \cdot j) = e \text{ for all } j \cdot k \circ k \cdot j \in T_n \text{ with } j < k;$$

$$u_n(j \circ j \cdot k) = f \text{ for all } j \circ j \cdot k \in T_n \text{ with } k < j;$$

$$u_n(j \diamond j \cdot k) = y \text{ for all } j \diamond j \cdot k \in T_n \text{ with } j < k;$$

$$u_n(j \cdot k \diamond k \cdot j) = z \text{ for all } j \cdot k \diamond k \cdot j \in T_n \text{ with } j < k;$$

$$u_n(j \diamond j \cdot k) = w \text{ for all } j \diamond j \cdot k \in T_n \text{ with } k < j;$$

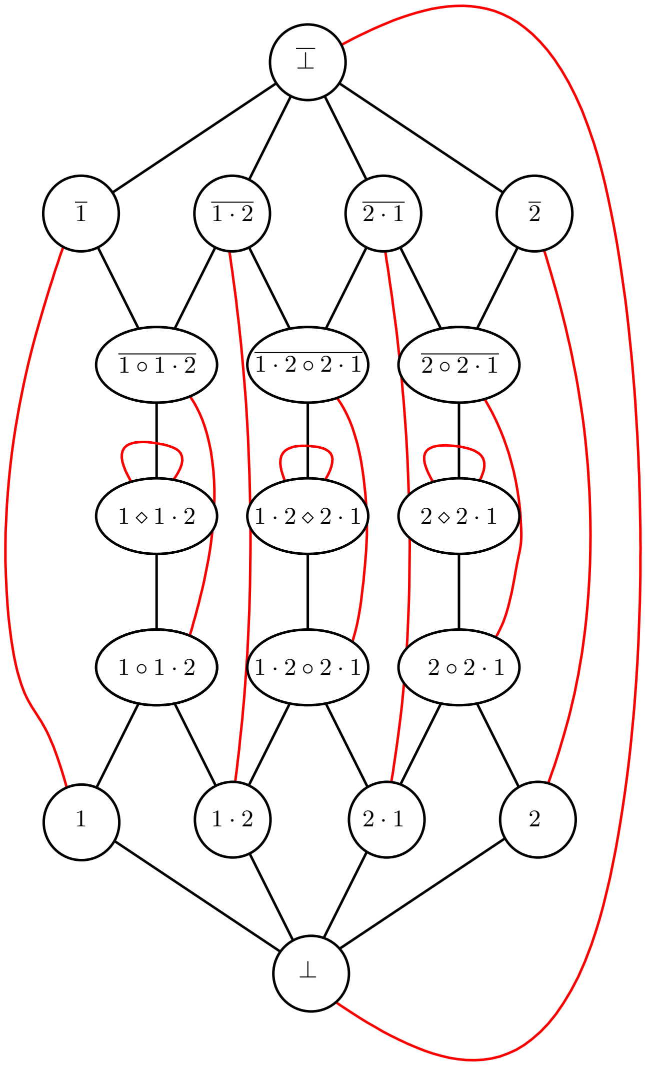

Figure 10: $\mathbf{T}_2$ in Lemma 19.

and, for all $y \in L \cup I \subseteq T_n$,

$$u_n(i(y)) = i(u_n(y)).$$

It is easy to check that $u_n\colon \mathbf{T}_n \to \mathbf{Q}$ is a unifier for $\mathbf{Q}$ in V.

Let $u\colon \mathbf{P} \to \mathbf{Q}$ be a unifier for $\mathbf{Q}$ in V, where $\mathbf{P} = (P, \leq, i)$. We show that $u_n \leq u$ implies $|P| \geq n$. Let $u_n = u \circ h$. We claim that $h(j) \neq h(k)$ for all $j < k$ in $\{1, \ldots, n\}$. Let $j < k$ in $\{1, \ldots, n\}$. By construction

$$j, j \cdot k \leq j \circ j \cdot k \leq i(j \circ j \cdot k) \leq i(j), i(j \cdot k),$$

then

$$h(j), h(j \cdot k) \leq h(j \circ j \cdot k) \leq i(h(j \circ j \cdot k)) \leq i(h(j)), i(h(j \cdot k)),$$

and by (K_1), $h(j) \vee h(j \cdot k)$ exists in $\mathbf{P}$ and

$$h(j) \vee h(j \cdot k) \leq i(h(j) \vee h(j \cdot k)),$$

so that

$$h(j), h(j \cdot k) \leq h(j) \vee h(j \cdot k) \leq h(j \circ j \cdot k).$$

Similarly

$$h(j \cdot k), h(k \cdot j) \leq h(j \cdot k) \vee h(k \cdot j) \leq h(j \cdot k \circ k \cdot j),$$

and

$$h(k \cdot j), h(k) \leq h(k \cdot j) \vee h(k) \leq h(k \circ k \cdot j).$$

If we assume the contrary, that is, $h(j) = h(k)$, then

$$h(j) \vee h(l) = h(l) \vee h(k);$$

and applying (M_3) to $h(j), h(j \cdot k), h(k \cdot j)$, we have

$$h(j), h(j \cdot k), h(k \cdot j) \leq h(j) \vee h(j \cdot k) \vee h(k \cdot j) \leq i(h(j) \vee h(j \cdot k) \vee h(k \cdot j)).$$

Applying u through, recalling that $u_n = u \circ h$, we have

$$a, b, c \leq u(h(j) \vee h(j \cdot k) \vee h(k \cdot j)) \leq i(u(h(j) \vee h(j \cdot k) \vee h(k \cdot j))),$$

which contradicts clause (iii) in the statement.

This proves that $\text{type}(V) \neq 1$. Then $\text{type}(V) = 0$. Now, by Lemma 8 $\text{type}_{\mathcal{K}}(\mathbf{Q}) = 0$, as desired. □

The proof of the main result in this section (Theorem 22) relies on the following notion.

Definition 20 (Kleene Unification Core)**.** Let $\mathbf{Q} = (Q, \leq, i) \in \mathcal{PK}_f$. The *Kleene unification core* of $\mathbf{Q}$ is the structure $\mathbf{Q'} = (Q', \leq', i') \in \mathcal{PK}_f$ where:

(i) $Q' = \{x, i(x) \in Q \mid x \leq z = i(z) \text{ for some } z \in Q\}$;

(ii) $x \leq' y$ iff $x \leq y$ and either of the following three cases occurs:

(a) $x \leq i(x)$ and $y \leq i(y)$;

(*b*) $i(x) \leq x$ and $i(y) \leq y$;

(*c*) $x \leq z = i(z) \leq y$ for some $z \in Q$;

(*iii*) $i'(x) = i(x)$ for all $x \in Q'$.

The following lemma justifies the terminology introduced.

Lemma 21. *Let $\mathbf{Q} = (Q, \leq, i) \in \mathcal{PK}_f$ and $\mathbf{Q}' \in \mathcal{PK}_f$ be its Kleene unification core.*

(*i*) *If $u\colon \mathbf{P} \to \mathbf{Q}$ is a unifier for $\mathbf{Q}$, then $u(P) \subseteq Q'$ and $u\colon \mathbf{P} \to \mathbf{Q}'$ is a unifier for $\mathbf{Q}'$.*

(*ii*) *$U_{\mathcal{K}}(\mathbf{Q}) \simeq U_{\mathcal{K}}(\mathbf{Q}')$.*

(*iii*) *$\mathbf{Q}' = (Q', \leq', i') \in \mathcal{PK}_f$ satisfies (M_2) and (K_2).*

Proof. (*i*) Let $u\colon \mathbf{P} \to \mathbf{Q}$ in $\mathcal{PK}_f$ be a unifier for $\mathbf{Q}$, with $\mathbf{P} = (P, \leq_P, i_P)$.

We show that $u(P) \subseteq Q'$. Let $x \in P$. Without loss of generality, we may assume $x \leq i_P(x)$. By (M_2) there exists $z \in P$ such that $x \leq z = i_P(z)$. Then $u(x) \leq u(z) = i(u(z))$, concluding that $u(x) \in Q'$.

We show that $u\colon \mathbf{P} \to \mathbf{Q}'$ is a unifier for $\mathbf{Q}'$. For all $x \in P$, we have $u(i_P(x)) = i(u(x)) = i'(u(x))$ by part (*i*) and Definition 20(*iii*). For monotonicity, let $x \leq_P y$. If $u(x) \leq i(u(x))$ and $u(y) \leq i(u(y))$, or $i(u(x)) \leq u(x)$ and $i(u(y)) \leq u(y)$, then $u(x) \leq' u(y)$ by Definition 20(*ii*). Otherwise, assume that $u(x) \leq i(u(x))$, $i(u(y)) \leq u(y)$, and there exists no $w \in Q'$ such that $u(x) \leq w = i(w) \leq u(y)$. Then, $u(x) < i(u(x)) = u(i_P(x))$ and $u(i_P(y)) = i(u(y)) < u(y)$, which implies $x <_P i_P(x)$ and $i_P(y) <_P y$. Since $x \leq_P y$ by hypothesis, we have $x, i_P(y) \leq_P i_P(x), y$. By (K_2), there exists $z \in P$ such that $x \leq_P z =_P i_P(z) \leq y$. But then, $u(x) \leq u(z) = i(u(z)) \leq u(y)$, a contradiction.

(*ii*) It follows from part (*i*) and the fact that the inclusion map from Q' into Q is a morphism in $\mathcal{PK}_f$.

(*iii*) The statement holds by Definition 20. In details, if $x \leq' i'(x) = i(x)$, either $x = i(x)$ or there exists $y \in Q$ such that $x \leq y = i(y) \leq i(x)$, so that (M_2) holds in $\mathbf{Q}'$. For (K_2), let $x, y \in Q'$ be such that $x, y \leq' i'(y), i'(x)$. We want to show that there exists $z \in Q'$ such that $x, y \leq' z \leq i'(z)$. Since i' is the restriction of i to Q', $x, y \leq' i(y), i(x)$. If $x = i(x)$ or $y = i(y)$ the result follows straightforwardly. If $x < i(x)$ and $y < i(y)$, since $x \leq' i(y)$ by (*c*) there exists $z \in Q$ such that $x \leq z = i(z) \leq i(y)$. Then $x, y \leq' z = i'(z)$, and the result follows. ☐

Theorem 22. *Let $\mathbf{Q} = (Q, \leq, i) \in \mathcal{PK}_f$ be a solvable instance of* Unif($\mathcal{K}$) *and $\mathbf{Q}' \in \mathcal{PK}_f$ be the Kleene unification core of $\mathbf{Q}$. Then:*

$$\mathrm{type}_{\mathcal{K}}(\mathbf{Q}) = \begin{cases} 1, & \textit{iff } \mathbf{Q}' \textit{ satisfies } (K_1) \textit{ and } (M_3) \\ \omega, & \textit{iff } \mathbf{Q}' \textit{ does not satisfy } (K_1) \\ & \textit{but } [x, i(x)]_{Q'} \textit{ satisfies } (K_1) \textit{ and } (M_3) \textit{ for each } x \in Q'; \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. Assume first that $\mathbf{Q}'$ satisfies (K_1) and (M_3). By Lemma 21(*iii*), $\mathbf{Q}'$ satisfies (M_2) and (K_2). Then $\mathsf{D}_{\mathcal{K}}(\mathbf{Q}')$ is projective by Theorem 12 and $\mathrm{type}_{\mathcal{K}}(\mathbf{Q}') = 1$. Now by Lemma 21(*ii*) $\mathrm{type}_{\mathcal{K}}(\mathbf{Q}) = \mathrm{type}_{\mathcal{K}}(\mathbf{Q}') = 1$.

Suppose that $\mathbf{Q'}$ does not satisfy (K_1) and $[x, i(x)]_{Q'}$ satisfies (K_1) and (M_3) for all $x \leq i(x)$ in $\mathbf{Q'}$. Since $\mathbf{Q'}$ satisfies (M_2) and (K_2), it follows that $[x, i(x)]_{Q'}$ satisfies (M_2) and (K_2) for all $x \leq i(x)$ in $\mathbf{Q'}$. Thus, define for every $x \in \min(\mathbf{Q'})$ the (inclusion) unifier $u_x\colon [x, i(x)]_{Q'} \to \mathbf{Q'}$ by $u_x(z) = z$ for all $z \in [x, i(x)]_{Q'}$. Clearly, there are finitely many unifiers of the form u_x in $\mathbf{Q'}$, because Q' is finite, and at least one such unifier because Q' is nonempty. We claim that the above unifiers form a μ-set in $U_{\mathcal{K}}(\mathbf{Q'})$. Clearly, if $x \neq y$ are minimal points in Q' then $u_x \parallel u_y$. Now, let $u\colon \mathbf{P} \to \mathbf{Q'}$ be a unifier for $\mathbf{Q'}$. Since $\mathbf{P}$ is bounded, let $\bot$ denote its lower bound. Since $\mathbf{Q'}$ is finite there exists $x \in \min(\mathbf{Q'})$ such that $x \leq u(\bot)$. Then $u(P) \subseteq [x, i(x)]_{Q'}$. Now let f be the inclusion (monotone) map, then $u_x \circ f = u$, that is, u_x is more general than u. Thus, $\text{type}_{\mathcal{K}}(\mathbf{Q'}) \in \{1, \omega\}$. We claim that $\text{type}_{\mathcal{K}}(\mathbf{Q'}) \neq 1$, that is, there exist two distinct unifiers for $\mathbf{Q'}$ with no common upper bound in $U_{\mathcal{K}}(\mathbf{Q'})$. In fact, $\mathbf{Q'}$ is not bounded, otherwise if $\mathbf{Q'}$ is bounded by $\bot$ and $\top$, then $\bot \leq i(\bot) = \top$ and then $[\bot, i(\bot)]_{Q'} = Q'$ satisfies (K_1). Hence, there are two distinct minimal points in $\mathbf{Q'}$, $x_1 \neq x_2$, so that $u_{x_1} \neq u_{x_2}$ are two distinct maximals in $U_{\mathcal{K}}(\mathbf{Q'})$. Finally by Lemma 21$(ii)$, $\text{type}_{\mathcal{K}}(\mathbf{Q}) = \text{type}_{\mathcal{K}}(\mathbf{Q'}) = \omega$.

Suppose now that there exists $x \leq i(x)$ in Q' such that $[x, i(x)]_{Q'}$ does not satisfy (K_1). Then $\{z \in Q' \mid x \leq' z \leq' i(z)\}$ with restricted order is not a meet semilattice, that is, there exist $x, a, b, c, d, y, z \in Q'$ such that $x \leq a, b \leq c, d$, $c \leq y = i(y)$, $d \leq z = i(z)$ but there does not exist $e \in Q'$ such that $a, b \leq e \leq c, d$. By Lemma 18, $\text{type}_{\mathcal{K}}(\mathbf{Q'}) = 0$. Thus, by Lemma 21$(ii)$, $\text{type}_{\mathcal{K}}(\mathbf{Q}) = 0$.

Finally suppose that for all $x \leq i(x)$ in $\mathbf{Q'}$ the interval $[x, i(x)]_{Q'}$ satisfies (K_1), but there exists $x \leq i(x)$ in $\mathbf{Q'}$ such that $[x, i(x)]_{Q'}$ does not satisfy (M_3); this case includes the case where $\mathbf{Q'}$ satisfies (K_1) and does not satisfy (M_3). Then there exist $x, a, b, c, d, e, f, y, z, w \in Q'$ such that: $x \leq a, b, c$; $a \leq d, e$; $b \leq d, f$; $c \leq e, f$; $d \leq y = i(y)$; $e \leq z = i(z)$; $f \leq w = i(w)$; there does not exist $g \in Q'$ such that $a, b, c \leq g \leq i(g)$. By Lemma 19, $\text{type}_{\mathcal{K}}(\mathbf{Q'}) = 0$. Finally by Lemma 21$(ii)$, $\text{type}_{\mathcal{K}}(\mathbf{Q}) = 0$. □

Using Lemma 18 and Lemma 19, it is easy to construct examples of Kleene unification problems having nullary type. These examples are witness of the fact stated in the following corollary.

Corollary 23. *The variety of Kleene algebras has nullary unification type.*

4.3 De Morgan Algebras

We provide a complete classification of solvable instances of the unification problem over De Morgan algebras (Theorem 31). Using [9], Theorem 5, and Theorem 11, the problem reduces to the following:

Problem $\textsc{Unif}(\mathcal{M})$.

Instance $\mathbf{Q} = (Q, \leq, i) \in \mathcal{PM}_f$.

Solution A morphism $u\colon \mathbf{P} \to \mathbf{Q}$ in $\mathcal{PM}_f$, where $\mathbf{P}$ satisfies (M_1)-(M_3).

This section follows a similar structure than the previous one. We first identify three sufficient conditions for an instance of the unification problem to have nullary type (Lemma 26, Lemma 27, and Lemma 28), and then we prove that the identified conditions are indeed necessary for nullarity (Theorem 31).

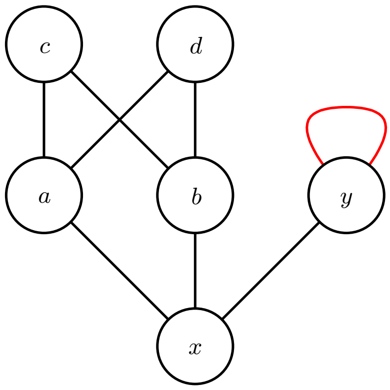

Figure 11: Subposet of $\mathbf{Q}$ in Lemma 26.

Remark 24. An instance $\mathbf{Q} = (Q, \leq, i)$ of $\textsc{Unif}(\mathcal{M})$ is solvable iff $\{x \in Q \mid x = i(x)\} \neq \emptyset$. The proof follows as in Remark 16.

Given a solvable instance $\mathbf{Q}$ of $\textsc{Unif}(\mathcal{M})$, we let $U_{\mathcal{M}}(\mathbf{Q})$ denote the preordered set of unifiers of $\mathbf{Q}$, which is defined as in Section 4.1.

Lemma 25. *Let $\mathbf{Q} = (Q, \leq, i) \in \mathcal{PM}_f$ be an instance of* $\textsc{Unif}(\mathcal{M})$ *and $x \in Q$ be a minimal element of $\mathbf{Q}$. Then*

$$V = \{u\colon \mathbf{P} \to \mathbf{Q} \in U_{\mathcal{M}}(\mathbf{Q}) \mid x \in u(P)\} \tag{10}$$

is a directed upset in $U_{\mathcal{M}}(\mathbf{Q})$.

Proof. Along the lines of Lemma 17. □

Lemma 26. *Let $\mathbf{Q} = (Q, \leq, i) \in \mathcal{PM}_f$ be an instance of* $\textsc{Unif}(\mathcal{M})$*. If there exist $x, a, b, c, d, y \in Q$ such that:*

(i) $x \leq a, b \leq c, d$;

(ii) $x \leq y = i(y)$;

(iii) there does not exist $e \in Q$ such that $a, b \leq e \leq c, d$;

then $\text{type}_{\mathcal{M}}(\mathbf{Q}) = 0$ (see Fig. 11).

Proof. Since $\mathbf{Q}$ is a finite poset, we assume without loss of generality $x \in \min(\mathbf{Q})$. Notice that by (iii), we have $a \neq b$ and $c \neq d$. Let

$$V = \{u\colon \mathbf{P} \to \mathbf{Q} \in U_{\mathcal{M}}(\mathbf{Q}) \mid x \in u(P)\}.$$

By Lemma 25, V is an directed upset in $U_{\mathcal{M}}(\mathbf{Q})$. By Lemma 8, to prove that $\text{type}(U_{\mathcal{M}}(\mathbf{Q})) = 0$ it is enough to prove that $\text{type}(V) = 0$. Since V is directed, by Lemma 9, $\text{type}(V) \in \{0, 1\}$. We show that $\text{type}(V) \neq 1$.

For every $n \in \mathbb{N}$, we define a unifier $u_n\colon \mathbf{T}_n \to \mathbf{Q}$ in V as follows. For $\mathbf{T}_n = (T_n, \leq, i) \in \mathcal{PK}_f$ we let

$$T_n = \{\bot, \overline{\bot}, 0, j, \overline{j}, j \cdot k, \overline{j \cdot k} \mid j < k \text{ in } \{1, \ldots, n\} \text{ and } j + k \text{ is odd}\}.$$

The map $i\colon T_n \to T_n$ is defined by:

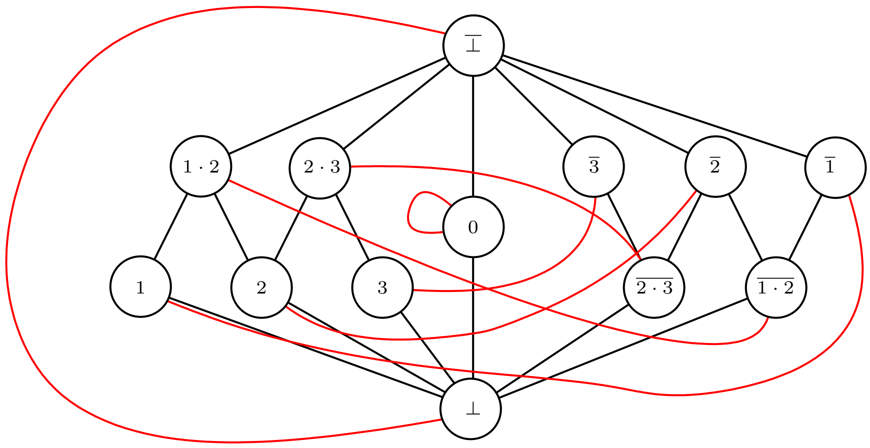

Figure 12: $\mathbf{T}_3$ in Lemma 26.

$i(0) = 0$;

$i(y) = \overline{y}$ and $i(\overline{y}) = y$ for all $y \in T_n \setminus \{0\}$.

The partial order over T_n is defined by the following cover relation, for all $j, k \in \{1, \ldots, n\}$:

$\bot \prec j$ and $i(j) \prec i(\bot)$;

$\bot \prec i(j \cdot k)$ and $j \cdot k \prec i(\bot)$ if $j < k$;

$j \prec j \cdot k$ and $i(j \cdot k) \prec i(j)$ if $j < k$;

$j \prec k \cdot j$ and $i(k \cdot j) \prec i(j)$ if $k < j$.

It is easy to check that $\mathbf{T}_n$ satisfies (M_1)-(M_3). Figure 12 provides the Hasse diagram of $\mathbf{T}_3$. We define $u_n \colon \mathbf{T}_n \to \mathbf{Q}$ as follows, where $j, k \in \{1, \ldots, n\}$:

$u_n(\bot) = x$;

$u_n(j) = a$ and $u_n(j \cdot k) = c$, for all $j, j \cdot k \in T_n$ with j odd;

$u_n(j) = b$ and $u_n(j \cdot k) = d$, for all $j, j \cdot k \in T_n$ with j even;

$u_n(0) = y$;

and, for all $y \in \{\bot, 0, j, j \cdot k \mid j < k \text{ in } \{1, \ldots, n\} \text{ and } j + k \text{ is odd}\} \subseteq T_n$,

$u_n(i(y)) = i(u_n(y))$.

It is easy to check that $u_n \colon \mathbf{T}_n \to \mathbf{Q}$ is a unifier for $\mathbf{Q}$ in V.

Let $u \colon \mathbf{P} \to \mathbf{Q}$ be a unifier for $\mathbf{Q}$ such that $u \in V$. We show that $u_n \leq u$ implies $|P| \geq n$. Let $u_n = u \circ f$. We claim that $f(j) \neq f(k)$ for all $j < k$ with $j, k \in \{1, \ldots, n\}$. If j and k have a different parity, then it is clear. If j and k have the same parity, without loss of generality assume that both are odd, then let l be even such that $j < l < k$. Since by construction $j, l \leq j \cdot l$, we have $f(j), f(l) \leq f(j \cdot l)$. By (M_1),

$$f(j), f(l) \leq f(j) \vee f(l) \leq f(j \cdot l).$$

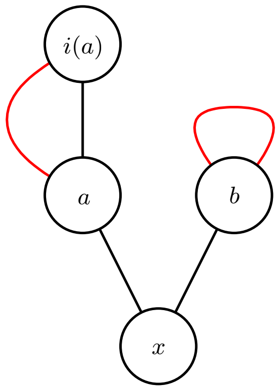

Figure 13: Subposet of $\mathbf{Q}$ in Lemma 27.

Similarly

$$f(l), f(k) \leq f(l) \vee f(k) \leq f(l \cdot k).$$

Assume $f(j) = f(k)$ for a contradiction. Then

$$f(j) = f(k), f(l) \leq f(j) \vee f(l) = f(l) \vee f(k) \leq f(j \cdot l), f(l \cdot k),$$

and applying u through recalling that $u_n = u \circ f$,

$$a, b \leq u(f(l) \vee f(k)) \leq c, d,$$

contradicting clause (iii) in the statement.

This proves that $\text{type}(V) \neq 1$. Then $\text{type}(V) = 0$. Therefore, $\text{type}_{\mathcal{M}}(\mathbf{Q}) = 0$, as desired. □

Lemma 27. *Let* $\mathbf{Q} = (Q, \leq, i) \in \mathcal{PM}_f$ *be an instance of* $\text{UNIF}(\mathcal{M})$*. If there exist* $x, a, b \in Q$ *such that:*

(i) $x \leq a, b$;

(ii) $a \leq i(a)$; $b = i(b)$;

(iii) there does not exist $c \in Q$ *such that* $a \leq c = i(c)$;

then $\text{type}_{\mathcal{M}}(\mathbf{Q}) = 0$ *(see Figure 13).*

Proof. Since $\mathbf{Q}$ is a finite poset, we assume without loss of generality $x \in \min(\mathbf{Q})$. Let

$$V = \{u \colon \mathbf{P} \to \mathbf{Q} \in U_{\mathcal{M}}(\mathbf{Q}) \mid x \in u(P)\}.$$

By Lemma 25, V is an directed upset in $U_{\mathcal{M}}(\mathbf{Q})$. By Lemma 8, to prove that $\text{type}(U_{\mathcal{K}}(\mathbf{Q})) = 0$ it is enough to prove that $\text{type}(V) = 0$. Since V is directed, by Lemma 9, $\text{type}(V) \in \{0, 1\}$. We show that $\text{type}(V) \neq 1$.

For every odd $n \in \mathbb{N}$, let

$$T_n = \{0, 1\}^n \cup \{d\}.$$

The map $i \colon T_n \to T_n$ is defined by:

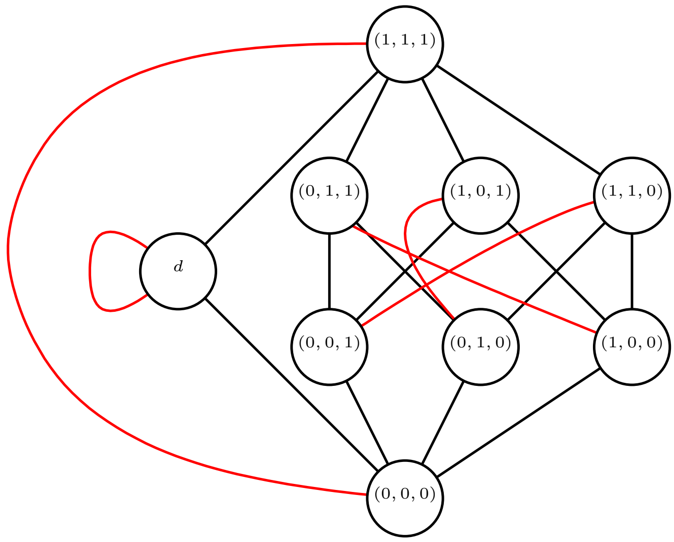

Figure 14: $\mathbf{T}_3$ in Lemma 27.

$i(d) = d$;

$i(e_1, \ldots, e_n) = (f_1, \ldots, f_n)$ where $f_j = 0$ iff $e_j = 1$ for $j = 1, 2, \ldots, n$.

The partial order over T_n is defined by:

$(e_1, \ldots, e_n) \leq (f_1, \ldots, f_n)$ if $e_j \leq f_j$ for $j = 1, 2, \ldots, n$;

$(0, \ldots, 0) \leq d \leq (1, \ldots, 1)$.

It is easy to check that $\mathbf{T}_n$ satisfies (M_1), (M_2), and (M_3). Figure 14 provides the Hasse diagram of $\mathbf{T}_3$.

For each n odd we define $u_n \colon \mathbf{T}_n \to \mathbf{Q}$ by:

$u_n(0, \ldots, 0) = x$ and $u_n(1, \ldots, 1) = i(x)$;

$u_n(d) = b$;

$u_n(e_1, \ldots, e_n) = a$ if $1 \leq e_1 + \cdots + e_n < n/2$;

$u_n(e_1, \ldots, e_n) = i(a)$ if $n/2 < e_1 + \ldots + e_n \leq n - 1$.

It follows straightforwardly that $u_n \colon \mathbf{T}_n \to \mathbf{Q}$ is a morphism in $\mathcal{PM}_f$, and therefore a unifier for $\mathbf{Q}$ in V.

Let $u \colon \mathbf{P} \to \mathbf{Q}$ be a unifier for $\mathbf{Q}$ such that $u \in V$. We show that $u_n \leq u$ implies $|P| \geq n$. Let $u_n = u \circ h$. We claim that $h(e) \neq h(f)$ for all $e \neq f$ in T_n such that $e_1 + \cdots + e_n = f_1 + \cdots + f_n = 1$. Suppose for a contradiction that $h(e) = h(f)$. By construction, $e \leq i(f)$, therefore

$$h(e) \leq h(i(f)) = i(h(f)) = i(h(e));$$

since $\mathbf{P}$ satisfies (M_2), there exists $z \in P$ such that

$$h(e) \leq z = i(z) \leq i(h(e));$$

but applying u:
$$a \leq u(z) = i(u(z)) \leq i(a),$$
in clear contradiction with clause (*iii*) in the statement.

This proves that $\text{type}(V) \neq 1$. Then $\text{type}(V) = 0$. Therefore, $\text{type}_{\mathcal{M}}(\mathbf{Q}) = 0$, as desired. □

Lemma 28. *Let $\mathbf{Q} = (Q, \leq, i) \in \mathcal{PM}_f$ be an instance of* $\textsc{Unif}(\mathcal{M})$*. If there exist $x, a, b, c, d, e, f, y, z, w \in Q$ such that:*

(*i*) *$x \leq a, b, c$; $a \leq d, e$; $b \leq d, f$; $c \leq e, f$;*

(*ii*) *$d \leq y = i(y)$; $e \leq z = i(z)$; $f \leq w = i(w)$;*

(*iii*) *there does not exist $g \in Q$ such that $a, b, c \leq g \leq i(g)$;*

then $\text{type}_{\mathcal{M}}(\mathbf{Q}) = 0$ *(see Fig. 9).*

Proof. Observe that conditions (*i*)-(*iii*) above are exactly conditions (*i*)-(*iii*) in Lemma 19. Moreover, as observed in Lemma 19, the structure $\mathbf{T}_n$ satisfies (M_1)-(M_3). Therefore, u_n is a unifier for $\mathbf{Q}$ in $\mathcal{PM}_f$ for all $n \in \mathbb{N}$, and the argument in Lemma 19 applies. □

Definition 29 (De Morgan Unification Core)**.** Let $\mathbf{Q} = (Q, \leq, i) \in \mathcal{PM}_f$. The *De Morgan unification core* of $\mathbf{Q}$ in $\mathcal{PM}_f$ is the structure $\mathbf{Q}' = (Q', \leq', i') \in \mathcal{PM}_f$ defined by:

(*i*) $Q' = \{x, i(x) \in Q \mid y \leq z, x, i(x) \text{ for some } y, z \in Q \text{ such that } z = i(z)\}$;

(*ii*) $x \leq' y$ iff $x \leq y$;

(*iii*) $i'(x) = i(x)$ for all $x \in Q'$.

Lemma 30. *Let $\mathbf{Q}' = (Q', \leq', i')$ be the De Morgan unification core of $\mathbf{Q} = (Q, \leq, i)$. Then:*

(*i*) *If $u\colon \mathbf{P} \to \mathbf{Q}$ is a unifier for $\mathbf{Q}$, then $u(P) \subseteq Q'$ and $u\colon \mathbf{P} \to \mathbf{Q}'$ is a unifier for $\mathbf{Q}'$.*

(*ii*) *$U_{\mathcal{M}}(\mathbf{Q}) \simeq U_{\mathcal{M}}(\mathbf{Q}')$.*

Proof. (*i*) We claim that $u(P) \subseteq Q'$. Indeed, let $x \in P$. If $x \leq_P i_P(x)$, then by (M_2) there exists $z \in P$ such that $z = i_P(z)$ and $x \leq_P z$. Then $u(x) \leq u(z) = i(u(z)) \leq i(u(x))$, so that $u(x) \in Q'$ by Definition 29(*i*). If $x \parallel_P i_P(x)$, then by (M_1), there exists $x \wedge i(x)$ and it satisfies $x \wedge i(x) \leq i(x \wedge i(x))$. By (M_2), there exists $z \in P$ such that $z = i_P(z)$ and $x \wedge i(x) \leq_P x, i_P(x), z$. Then $u(x \wedge i(x)) \leq u(x), i(u(x)), u(z)$, so that $u(x) \in Q'$ by Definition 29(*i*).

(*ii*) It follows from part (*i*) and the fact that the inclusion $Q' \subseteq Q$ is in $\mathcal{PM}_f$. □

Theorem 31. *Let $\mathbf{Q} = (Q, \leq, i) \in \mathcal{PM}_f$ be a solvable instance of* $\textsc{Unif}(\mathcal{M})$*, and $\mathbf{Q}' = (Q', \leq', i') \in \mathcal{PM}_f$ be the De Morgan unification core of $\mathbf{Q}$. Then:*

$$\text{type}_{\mathcal{M}}(\mathbf{Q}) = \begin{cases} 1, & \textit{iff } \mathbf{Q}' \textit{ satisfies } (M_1), (M_2), \textit{and } (M_3) \\ \omega, & \textit{iff } \mathbf{Q}' \textit{ does not satisfy } (M_1), \textit{but for every } x \in Q' \\ & \quad \textit{with } x \leq' i(x), [x, i(x)]_{Q'} \textit{ satisfies } (M_1), (M_2), \textit{and } (M_3); \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. If $\mathbf{Q}'$ satisfies (M_1)-(M_3), $\mathsf{D}_{\mathcal{M}}(\mathbf{Q}')$ is projective by Theorem 11, and $\text{type}_{\mathcal{M}}(\mathbf{Q}') = 1$. Therefore, $\text{type}_{\mathcal{M}}(\mathbf{Q}) = 1$ because $\text{type}_{\mathcal{M}}(\mathbf{Q}) = \text{type}_{\mathcal{M}}(\mathbf{Q}')$ by Lemma 30(ii).

Suppose that $\mathbf{Q}'$ does not satisfy (M_1) and $[x, i(x)]_{Q'}$ satisfies (M_1), (M_2), and (M_3) for all $x \leq' i(x)$ in $\mathbf{Q}'$. Along the lines of the second part of the proof of Theorem 22, it follows that $\text{type}_{\mathcal{M}}(\mathbf{Q}) = \omega$.

Now suppose that there exist $x \leq' i(x)$ in $\mathbf{Q}'$ such that $[x, i(x)]_{Q'}$ does not satisfy (M_1); without loss of generality, $x \in \min(\mathbf{Q}')$. Then $[x, i(x)]_{Q'}$ with restricted order is not a lattice, that is, there exist $a, b, c, d \in Q'$ such that $x \leq' a, b \leq' c, d \leq i(x)$ but there does not exist $e \in Q'$ such that $a, b \leq' e \leq' c, d$. Moreover, by minimality of x and Definition 29(i), there exists $y \in Q'$ such that $x \leq y = i(y)$. Therefore, by Lemma 26, $\text{type}_{\mathcal{M}}(\mathbf{Q}') = 0$. Thus, by Lemma 30($ii$), $\text{type}_{\mathcal{M}}(\mathbf{Q}) = 0$.

Next suppose that for all $x \leq' i(x)$ in $\mathbf{Q}'$ the interval $[x, i(x)]_{Q'}$ satisfies (M_1), but there exists $x \leq' i(x)$ in $\mathbf{Q}'$ such that $[x, i(x)]_{Q'}$ does not satisfy (M_2); without loss of generality, $x \in \min(\mathbf{Q}')$. This case includes the case where $\mathbf{Q}'$ satisfies (M_1) and not (M_2). Then there exists $a \leq i(a)$ in $[x, i(x)]_{Q'}$ such that there does not exist $c \in [x, i(x)]_{Q'}$ satisfying $a \leq c = i(c)$. By minimality of x and Definition 29(i), there exists $b = i(b)$ in Q' such that $x \leq b$. Therefore by Lemma 27, $\text{type}_{\mathcal{M}}(\mathbf{Q}') = 0$. Thus, by Lemma 30($ii$), $\text{type}_{\mathcal{M}}(\mathbf{Q}) = 0$.

Finally suppose that for all $x \leq' i(x)$ in $\mathbf{Q}'$ the interval $[x, i(x)]_{Q'}$ satisfies (M_1) and (M_2), but there exists $x \leq' i(x)$ in $\mathbf{Q}'$ such that $[x, i(x)]_{Q'}$ does not satisfy (M_3); this case includes the case where $\mathbf{Q}'$ satisfies (M_1) and (M_2) but not (M_3). Then there exist $x, a, b, c, d, e, f, y, z, w \in Q'$ such that: $x \leq' a, b, c$; $a \leq' d, e$; $b \leq' d, f$; $c \leq' e, f$; $d \leq' y = i(y)$; $e \leq' z = i(z)$; $f \leq' w = i(w)$; there does not exist $g \in Q'$ such that $a, b, c \leq' g \leq i(g)$. Therefore, by Lemma 28, $\text{type}_{\mathcal{M}}(\mathbf{Q}') = 0$. Thus, by Lemma 30($ii$), $\text{type}_{\mathcal{M}}(\mathbf{Q}) = 0$. □

Lemma 26, Lemma 27, and Lemma 28 yield examples of De Morgan unification instances having nullary type. This proves the following corollary.

Corollary 32. *The variety of De Morgan algebras has nullary unification type.*

Acknowledgments. The authors thank George Metcalfe for fruitful discussions about unification and admissibility. The first author was supported by ERC Starting Grant 239962. The second author was supported by Marie Curie Reintegration Grant PIRG06-GA-2009-256492.

References

[1] M. E. Adams. Principal Congruences in De Morgan Algebras. *Proc. Edinburgh Math. Soc.*, 30:415–421, 1987.

[2] F. Baader and W. Snyder. Unification theory. In A. Robinson and A. Voronkov, editors, *Handbook of Automated Reasoning*, volume 8, pages 445–533. Elsevier, 2001.

[3] R. Balbes and A. Horn. Injective and Projective Heyting Algebras. *Trans. Amer. Math. Soc.*, 148:549–559, 1970.

[4] R. Balbes and A. Horn. Projective distributive lattices. *Pacific J. Math.*, 33:273–279, 1970.

[5] J. Berman and W. J. Blok. Generalizations of Tarski's Fixed Point Theorem for Order Varieties of Complete Meet Semilattices. *Order*, 5:381–392, 1989.

[6] G. Birkhoff. Rings of Sets. *Duke Math. J.*, 3(3):443–454, 1937.

[7] W. H. Cornish and P. R. Fowler. Coproducts of De Morgan Algebras. *Bull. Austral. Math. Soc.*, 16:1–13, 1977.

[8] B.A Davey and H.A. Priestley. *Introduction to Lattices and Order*. Cambridge University Press, second edition, 2002.

[9] S. Ghilardi. Unification through Projectivity. *J. Logic Computat.*, 7(6):733–752, 1997.

[10] J. A. Kalman. Lattices with Involution. *Trans. Amer. Math. Soc.*, 87:485–491, 1958.

[11] S. MacLane. *Categories for the Working Mathematician*. Springer-Verlag, second edition, 1998.

[12] R. McKenzie, G.F. McNulty, and W. Taylor. *Algebras, Lattices, Varieties*. Wadsworth and Brooks, 1987.

[13] H. A. Priestley. Representation of distributive lattices by means of ordered Stone spaces. *Bul. Lon. Math. Soc.*, 2(2):186–190, 1970.

[14] R. Sikorski. Homomorphisms, Mappings and Retracts. *Colloquium Math.*, 2:202–211, 1951.