Quantified Conjunctive Queries on Partially Ordered Sets

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Abstract. We study the computational problem of checking whether a quantified conjunctive query (a first-order sentence built using only conjunction as Boolean connective) is true in a finite poset (a reflexive, antisymmetric, and transitive directed graph). We prove that the problem is already NP-hard on a certain fixed poset, and investigate structural properties of posets yielding fixed-parameter tractability when the problem is parameterized by the query. Our main algorithmic result is that model checking quantified conjunctive queries on posets of bounded width is fixed-parameter tractable (the width of a poset is the maximum size of a subset of pairwise incomparable elements). We complement our algorithmic result by complexity results with respect to classes of finite posets in a hierarchy of natural poset invariants, establishing its tightness in this sense.

1 Introduction

Motivation. The *model checking* problem for first-order logic is the problem of deciding whether a given first-order sentence is true in a given finite structure; it encompasses a wide range of fundamental combinatorial problems. The problem is trivially decidable in $O(n^k)$ time, where n is the size of the structure and k is the size of the sentence, but it is not polynomial-time decidable or even *fixed-parameter tractable* when parameterized by k (under complexity assumptions in classical and parameterized complexity, respectively).

Restrictions of the model checking problem to fixed classes of structures or sentences have been intensively investigated from the perspective of parameterized algorithms and complexity [5,10,11]. In particular, starting from seminal work by Courcelle [6] and Seese [16], structural properties of *graphs* sufficient for fixed-parameter tractability of model checking have been identified. An important outcome of this research is the understanding of the interplay between structural properties of graphs and the expressive power of first-order logic, most notably the interplay between sparsity and locality, culminating in the recent result by Grohe, Kreutzer, and Siebertz that model checking first-order logic on classes of *nowhere dense* graphs is fixed-parameter tractable [14,12]. On graph classes closed under subgraphs the result is known to be tight; at the same time, there are classes of *somewhere dense* graphs (not closed under subgraphs) with fixed parameter tractable first-order (and even monadic second-order) logic model checking; the prominent examples are graph classes of bounded clique-width solved by Courcelle, Makowsky, and Rotics [7].

In this paper, we investigate *posets* (short for *partially ordered sets*). Posets form a fundamental class of combinatorial objects [9] and may be viewed as reflexive, antisymmetric, and transitive directed graphs. Besides their naturality, our motivation towards posets is that they challenge our current model checking knowledge; indeed, posets are somewhere dense (but not closed under substructures) and have unbounded clique-width [1, Proposition 5]. Therefore, not only are they not covered by the aforementioned results [12,7], but most importantly, it seems likely that new structural ideas and algorithmic techniques are needed to understand and conquer first-order logic on posets.

In recent work, we started the investigation of first-order logic model checking on finite posets, and obtained a parameterized complexity classification of *existential* and *universal* logic (first-order sentences in prefix form built using only existential or only universal quantifiers) with respect to classes of posets in a hierarchy generated by basic poset invariants, including for instance width and depth [1].[1] In particular, as articulated more precisely in [1], a complete understanding of the first-order case reduces to understanding the parameterized complexity of model checking first-order logic on bounded width posets (the *width* of a poset is the maximum size of a subset of pairwise incomparable elements); these classes are hindered by the same obstructions as general posets, since already posets of width 2 have unbounded clique-width [1, Proposition 5].

Contribution. In this paper we push the tractability frontier traced in [1] closer towards full first-order logic, by proving that model checking *(quantified) conjunctive positive* logic (first-order sentences built using only conjunction as Boolean connective) is tractable on bounded width posets.[2] The problem of model checking conjunctive positive logic on finite structures, also known as the *quantified constraint satisfaction* problem, has been previously studied with various motivations in various settings [3,5]; somehow surprisingly, conjunctive logic is also capable of expressing rather interesting poset properties (as sampled in Proposition 2).

More precisely, our contribution is twofold. First, we identify conjunctive positive logic as a minimal syntactic fragment of first-order logic that allows for full quantification, and has computationally hard expression complexity on posets; namely, we prove that *there exists a finite poset where model checking (quantified) conjunctive positive logic is* NP*-hard* (Theorem 1). Next, as our main algorithmic result, we establish that *model checking conjunctive positive logic on finite posets, parameterized by the width of the poset and the size of the sentence, is fixed-parameter tractable* with an elementary parameter dependence (Theorem 2). The aforementioned fact that model checking conjunctive positive logic is already NP-hard on a fixed poset justifies the relaxation to fixed-parameter tractability by

[1] Existential and universal logic are maximal syntactic fragments properly contained in first-order logic.

[2] Conjunctive positive logic and existential (respectively, universal) logic are incomparable syntactic fragments of first-order logic.

showing that, if we insist on polynomial-time algorithms, any structural property of posets (captured by the boundedness of a numeric invariant) is negligible.

Informally, the idea of our algorithm is the following. First, given a poset $\mathbf{P}$ and a sentence ϕ, we rewrite the sentence in a simplified form (which we call a *reduced* form), equisatisfiable on $\mathbf{P}$ (Proposition 1). Next, using the properties of reduced forms, we define a syntactic notion of "depth" of a variable in ϕ and a semantic notion of "depth" of a subset of $\mathbf{P}$, and we prove that $\mathbf{P} \models \phi$ if and only if $\mathbf{P}$ verifies ϕ upon "relativizing" variables to subsets of matching depth (Lemma 1 and and Lemma 2). The key fact is that the size of the subsets of $\mathbf{P}$ used to relativize the variables of ϕ is bounded above by the width of $\mathbf{P}$ and the size of ϕ (Lemma 3), from which the main result follows (Theorem 2). We remark that the approach outlined above differs significantly from the algebraic approach used in [1]; moreover, both stages make essential use of the restriction that conjunction is the only Boolean connective allowed in the sentences.

It follows immediately that *model checking conjunctive positive logic on classes of finite posets of bounded width, parameterized by the size of the sentence, is fixed-parameter tractable* (Corollary 1). On the other hand, there exist classes of finite posets of bounded depth (the *depth* of a poset is the maximum size of a subset of pairwise comparable elements) and classes of finite posets of bounded cover-degree (the *cover-degree* of a poset is the degree of its cover relation) where model checking conjunctive positive logic is shown to be coW[2]-hard and hence not fixed parameter tractable, unless the exponential time hypothesis [8] fails, see Proposition 3. Combined with the algorithm by Seese [16], these facts complete the parameterized complexity classification of the investigated poset invariants, as depicted in Figure 1.

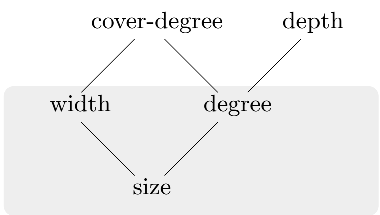

Fig. 1: On all classes of posets bounded under invariants in the gray region, model checking conjunctive positive logic is fixed-parameter tractable; on some classes of posets bounded under the remaining invariants, the problem is not fixed-parameter tractable unless FPT = coW[2].

The classification of conjunctive positive logic in this paper matches the classification of existential logic in [1], and further emphasizes the quest for a classification of full first-order logic on bounded width posets. We believe that the work presented in this paper and [1] enlightens the spectrum of phenomena that a fixed-parameter tractable algorithm for model checking the full first-order logic on bounded width posets, if it exists, has to capture.

Throughout the paper, we mark with $\star$ all statements whose proofs are omitted; we refer to [2] for a full version.

2 Preliminaries

For all integers $k \geq 1$, we let $[k]$ denote the set $\{1, \ldots, k\}$. We focus on relational first-order logic. A *vocabulary* σ is a set of *constant symbols* and *relation symbols*; each relation symbol is associated to a natural number called its *arity*; we let $\mathrm{ar}(R)$ denote the arity of $R \in \sigma$. *All vocabularies considered in this paper are finite.*

An *atom* α (over vocabulary σ) is an equality $t = t'$ or an application of a predicate $Rt_1 \ldots t_{\mathrm{ar}(R)}$, where $t, t', t_1, \ldots, t_{\mathrm{ar}(R)}$ are variable symbols (in a fixed countable set) or constant symbols, and $R \in \sigma$. We let $\mathcal{FO}$ denote the class of first-order sentences.

A *structure* $\mathbf{A}$ (over σ) is specified by a nonempty set A, called the *universe* of the structure, an element $c^{\mathbf{A}} \in A$ for each constant symbol $c \in \sigma$, and a relation $R^{\mathbf{A}} \subseteq A^{\mathrm{ar}(R)}$ for each relation symbol $R \in \sigma$. Given a structure $\mathbf{A}$ and $B \subseteq A$ such that $\{c^{\mathbf{A}} \mid c \in \sigma\} \subseteq B$, we denote by $\mathbf{A}|_B$ the substructure of $\mathbf{A}$ induced by B, defined as follows: the universe of $\mathbf{A}|_B$ is B, $c^{\mathbf{A}|_B} = c^{\mathbf{A}}$ for each $c \in \sigma$, and $R^{\mathbf{A}|_B} = R^{\mathbf{A}} \cap B^{\mathrm{ar}(R)}$ for all $R \in \sigma$. A structure is *finite* if its universe is finite and *trivial* if its universe is a singleton. *All structures considered in this paper are finite and nontrivial.*

For a structure $\mathbf{A}$ and a sentence ϕ over the same vocabulary, we write $\mathbf{A} \models \phi$ if the sentence ϕ is *true* in the structure $\mathbf{A}$. When $\mathbf{A}$ is a structure, f is a mapping from the variables to the universe of $\mathbf{A}$, and $\psi(x_1, \ldots, x_n)$ is a formula over the vocabulary of $\mathbf{A}$, we write $\mathbf{A}, f \models \psi$ or (liberally) $\mathbf{A} \models \psi(f(x_1), \ldots, f(x_n))$ to indicate that ψ is satisfied in $\mathbf{A}$ under f.

We refer the reader to [8] for the standard algorithmic setup of the model checking problem, and for standard notions in parameterized complexity theory. As for notation, the model checking problem for a class of σ-structures $\mathcal{C}$ and a class of σ-sentences $\mathcal{L} \subseteq \mathcal{FO}$ is denoted by $\mathrm{MC}(\mathcal{C}, \mathcal{L})$; it is the problem of deciding, given $(\mathbf{A}, \phi) \in \mathcal{C} \times \mathcal{L}$, whether $\mathbf{A} \models \phi$. We let $\|(\mathbf{A}, \phi)\|$, $\|\mathbf{A}\|$, and $\|\phi\|$ denote, respectively, the size of the (encoding of the) instance $(\mathbf{A}, \phi)$, the structure $\mathbf{A}$, and the sentence ϕ. The parameterization of an instance $(\mathbf{A}, \phi)$ returns $\|\phi\|$.

Conjunctive Positive Logic. In this paper, we study the *(quantified) conjunctive positive* fragment of first-order logic, in symbols $\mathcal{FO}(\forall, \exists, \wedge)$, containing first-order sentences built using only logical symbols in $\{\forall, \exists, \wedge\}$.

A conjunctive positive sentence is in *alternating prefix form* if it has the form

$$\phi = \forall x_1 \exists y_1 \ldots \forall x_l \exists y_l C(x_1, y_1, \ldots, x_l, y_l), \tag{1}$$

where $l \geq 0$ and $C(x_1, y_1, \ldots, x_l, y_l)$ is a conjunction of atoms whose variables are contained in $\{x_1, y_1, \ldots, x_l, y_l\}$; it is possible to reduce any conjunctive positive sentence to a logically equivalent conjunctive positive sentence of form (1) in polynomial time. For a simpler exposition, *every conjunctive positive sentence considered in this paper is assumed to be given in alternating prefix form (or is implicitly reduced to that form if required by the context).*

Let σ be a relational vocabulary. Let $\mathbf{A}$ be a σ-structure and let ϕ be a conjunctive positive σ-sentence as in (1). It is well known that the truth of ϕ

in $\mathbf{A}$ can be characterized in terms of the *Hintikka (or model checking) game* on $\mathbf{A}$ and ϕ. The game is played by two players, Abelard (male, the *universal* player) and Eloise (female, the *existential* player), as follows. For increasing values of i from 1 to l, Abelard assigns x_i to an element $a_i \in A$, and Eloise assigns y_i to an element $b_i \in A$; the sequence $(a_1, b_1, \ldots, a_l, b_l)$ is called a *play* on $\mathbf{A}$ and ϕ, where $(a_1, \ldots, a_l)$ and $(b_1, \ldots, b_l)$ are the plays by Abelard and Eloise respectively; Eloise wins if and only if $\mathbf{A} \models C(a_1, b_1, \ldots, a_l, b_l)$.

A *strategy for Eloise* (in the Hintikka game on $\mathbf{A}$ and ϕ) is a sequence $(g_1, \ldots, g_l)$ of functions of the form $g_i \colon A^i \to A$, for all $i \in [l]$; it *beats* a play $f \colon \{x_1, \ldots, x_l\} \to A$ by Abelard if $\mathbf{A} \models C(f(x_1), g_1(f(x_1)), \ldots, f(x_i), g_i(f(x_1), \ldots, f(x_i)), \ldots)$, where $i \in [l]$. A strategy for Eloise is *winning* (in the Hintikka game on $\mathbf{A}$ and ϕ) if it beats all Abelard plays. It is well known (and easily verified) that $\mathbf{A} \models \phi$ if and only if Eloise has a winning strategy (in the Hintikka game on $\mathbf{A}$ and ϕ).

For $X_1, Y_1, \ldots, X_l, Y_l \subseteq A$, we denote by

$$\phi' = (\forall x_1 \in X_1)(\exists y_1 \in Y_1) \ldots (\forall x_l \in X_l)(\exists y_l \in Y_l) C(x_1, y_1, \ldots, x_l, y_l), \qquad (2)$$

the relativization in ϕ of variable x_i to X_i and y_i to Y_i for all $i \in [l]$, and liberally write $\mathbf{A} \models \phi'$ meaning that ϕ' is satisfied in the intended expansion of $\mathbf{A}$. It is readily verified that, if ϕ' is as in (2), then $\mathbf{A} \models \phi'$ if and only if, in the Hintikka game on $\mathbf{A}$ and ϕ, Eloise has a strategy of the form $g_i \colon X_1 \times \cdots \times X_i \to Y_i$ for all $i \in [l]$, beating all plays f by Abelard such that $f(x_i) \in X_i$ for all $i \in [l]$.

Partially Ordered Sets. We refer the reader to [4] for the few standard notions in order theory used in the paper but not defined below.

A structure $\mathbf{G} = (G, E^{\mathbf{G}})$ with $\mathrm{ar}(E) = 2$ is called a *digraph*. Two digraphs $\mathbf{G}$ and $\mathbf{H}$ are *isomorphic* if there exists a bijection $f \colon G \to H$ such that for all $g, g' \in G$ it holds that $(g, g') \in E^{\mathbf{G}}$ if and only if $(f(g), f(g')) \in E^{\mathbf{H}}$. The *degree* of $g \in G$, in symbols $\mathrm{degree}(g)$, is equal to $|\{(g', g) \in E^{\mathbf{G}} \mid g' \in G\} \cup \{(g, g') \in E^{\mathbf{G}} \mid g' \in G\}|$, and the *degree* of $\mathbf{G}$, in symbols $\mathrm{degree}(\mathbf{G})$, is the maximum degree attained by the elements of $\mathbf{G}$.

A digraph $\mathbf{P} = (P, \leq^{\mathbf{P}})$ is a *partially ordered set* (in short, a *poset*) if $\leq^{\mathbf{P}}$ is a reflexive, antisymmetric, and transitive relation over P. For all $Q \subseteq P$, we let $\min^{\mathbf{P}}(Q)$ and $\max^{\mathbf{P}}(Q)$ denote, respectively, the set of minimal and maximal elements in the substructure of $\mathbf{P}$ induced by Q; we also write $\min(\mathbf{P})$ instead of $\min^{\mathbf{P}}(P)$, and $\max(\mathbf{P})$ instead of $\max^{\mathbf{P}}(P)$. For all $Q \subseteq P$, we let $(Q]^{\mathbf{P}}$, respectively $[Q)^{\mathbf{P}}$, denote the downset, respectively upset, of $\mathbf{P}$ induced by Q. Let $\mathbf{P}$ be a poset and let $p, q \in P$. We write $p \prec^{\mathbf{P}} q$ if q covers p in $\mathbf{P}$, and $p \parallel^{\mathbf{P}} q$ if p and q are incomparable in $\mathbf{P}$. If $\mathcal{P}$ is a class of posets, we let $\mathrm{cover}(\mathcal{P}) = \{\mathrm{cover}(\mathbf{P}) \mid \mathbf{P} \in \mathcal{P}\}$, where $\mathrm{cover}(\mathbf{P}) = \{(p, q) \mid p \prec^{\mathbf{P}} q\}$.

We introduce a family of poset invariants. Let $\mathbf{P}$ be a poset. The *size* of $\mathbf{P}$ is $|P|$. The *depth* of $\mathbf{P}$, $\mathrm{depth}(\mathbf{P})$, is the maximum size of a chain in $\mathbf{P}$. The *width* of $\mathbf{P}$, $\mathrm{width}(\mathbf{P})$, is the maximum size of an antichain in $\mathbf{P}$. The *degree* of $\mathbf{P}$, $\mathrm{degree}(\mathbf{P})$, is the degree of $\mathbf{P}$ as a digraph. The *cover-degree* of $\mathbf{P}$, $\mathrm{cover\text{-}degree}(\mathbf{P})$, is the degree of the cover relation of $\mathbf{P}$, that is, $\mathrm{degree}(\mathrm{cover}(\mathbf{P}))$. We say that a class of posets $\mathcal{P}$ is *bounded* w.r.t. the poset invariant inv if there exists $b \in \mathbb{N}$

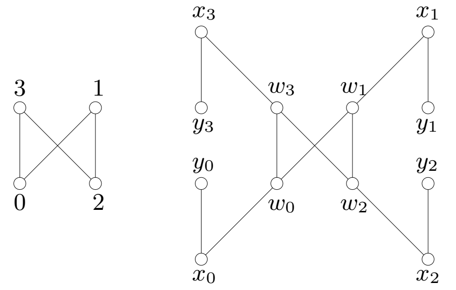

Fig. 2: The Hasse diagrams of the bowtie poset $\mathbf{B}$ (left) and of the representation $\mathbf{M}_\alpha$ of the formula α (right, see Section 4 for the interpretation of $\mathbf{M}_\alpha$) used in Theorem 1. The idea of the reduction is to simulate the constant c_i in $\psi \in \mathcal{FO}_\sigma(\exists, \wedge)$, interpreted on the element $i \in B$, by the variable w_i in $\phi \in \mathcal{FO}_\tau(\forall, \exists, \wedge)$, where $i \in \{0, 1, 2, 3\}$.

such that $\mathrm{inv}(\mathcal{P}) \leq b$ for all $\mathbf{P} \in \mathcal{P}$. The above poset invariants are ordered as in Figure 1, where $\mathrm{inv} \leq \mathrm{inv}'$ if and only if: $\mathcal{P}$ is bounded w.r.t. inv implies $\mathcal{P}$ is bounded w.r.t. inv' for every class of posets $\mathcal{P}$ [1, Proposition 3].

3 Expression Hardness

In this section we prove that conjunctive positive logic on posets is NP-hard in expression complexity. Let $\mathbf{B} = (B, \leq^{\mathbf{B}})$ be the *bowtie* poset defined by the universe $B = \{0, 1, 2, 3\}$ and the covers $0, 2 \prec^{\mathbf{B}} 1, 3$; see Figure 2.

Theorem 1. $\mathrm{MC}(\{\mathbf{B}\}, \mathcal{FO}(\forall, \exists, \wedge))$ *is* NP-*hard.*

Proof. Let $\tau = \{\leq\}$ and $\sigma = \tau \cup \{c_0, c_1, c_2, c_3\}$ be vocabularies where $\leq$ is a binary relation symbol and c_i is a constant symbol ($i \in B$). Let $\mathcal{FO}_\sigma(\exists, \wedge)$ contain first-order sentences built using only logical symbols in $\{\exists, \wedge\}$ and nonlogical symbols in σ; $\mathcal{FO}_\tau(\forall, \exists, \wedge)$ is described similarly. Let $\mathbf{B}^*$ be the σ-structure such that $B^* = B$, $(B^*, \leq^{\mathbf{B}^*})$ is isomorphic to $\mathbf{B}$ under the identity mapping, and $c_i^{\mathbf{B}^*} = i$ for all $i \in B$.

By [15, Theorem 2, Case $n = 2$], the problem $\mathrm{MC}(\{\mathbf{B}^*\}, \mathcal{FO}_\sigma(\exists, \wedge))$ is NP-hard. It is therefore sufficient to give a polynomial-time many-one reduction from $\mathrm{MC}(\{\mathbf{B}^*\}, \mathcal{FO}_\sigma(\exists, \wedge))$ to $\mathrm{MC}(\{\mathbf{B}\}, \mathcal{FO}_\tau(\forall, \exists, \wedge))$. The idea of the reduction is to simulate the constants in σ by universal quantification and additional variables; the details follow.

Let ψ be an instance of $\mathrm{MC}(\{\mathbf{B}^*\}, \mathcal{FO}_\sigma(\exists, \wedge))$, and let $\{x_i, y_i, w_i \mid i \in B\}$ be a set of 12 fresh variables (not occurring in ψ). Let ψ' be the $\mathcal{FO}_\tau(\exists, \wedge)$-sentence obtained from ψ by replacing atoms of the form $c_i \leq u$ and $u \leq c_i$, respectively, by atoms of the form $w_i \leq u$ and $u \leq w_i$ (where c_i is a constant in σ and u, w_i are variables). Let α be the conjunction of atoms defined by (see Figure 2)

$$\{w_0, w_2\} \leq \{w_1, w_3\} \wedge \bigwedge_{j \in \{0,2\}} \{x_j\} \leq \{y_j, w_j\} \wedge \bigwedge_{j \in \{1,3\}} \{y_j, w_j\} \leq \{x_j\},$$

where, for sets of variables S and S', the notation $S \leq S'$ denotes the conjunction of atoms of the form $s \leq s'$ for all $(s, s') \in S \times S'$.

We finally define the $\mathcal{FO}_\tau(\forall, \exists, \wedge)$-sentence ϕ by putting $\phi = \forall y_0 \ldots \forall y_3 \exists x_0 \ldots \exists x_3 \exists w_0 \ldots \exists w_3 (\alpha \wedge \psi')$. The reduction is clearly feasible in polynomial time; we now prove that the reduction is correct, that is, $\mathbf{B}^* \models \psi$ if and only if $\mathbf{B} \models \phi$.

An assignment $f\colon \{y_0, y_1, y_2, y_3\} \to B$ is said to be *nontrivial* if $\{f(y_0), f(y_2)\} = \{0, 2\}$ and $\{f(y_1), f(y_3)\} = \{1, 3\}$, and *trivial* otherwise; in particular, nontrivial assignments are bijective.

Claim 1 ($\star$) *$\mathbf{B}, f \models \exists x_0 \ldots x_3 w_0 \ldots w_3(\alpha \wedge \psi')$ for all trivial assignments f.*

Claim 2 ($\star$) *Let f be a nontrivial assignment. The following are equivalent.*

(i) $\mathbf{B}, f \models \exists x_0 \ldots x_3 w_0 \ldots w_3(\alpha \wedge \psi')$.
(ii) $\mathbf{B}^* \models \psi$.

We conclude the proof by showing that $\mathbf{B}^* \models \psi$ if and only if $\mathbf{B} \models \phi$. If $\mathbf{B} \not\models \phi$, then there exists an assignment f such that $\mathbf{B}, f \not\models \exists x_0 \ldots \exists x_3 \exists w_0 \ldots \exists w_3(\alpha \wedge \psi')$; by Claim 1, f is nontrivial. Then $\mathbf{B}^* \not\models \psi$ by Claim 2. Conversely, if $\mathbf{B} \models \phi$, then in particular $\mathbf{B}, f \models \exists x_0 \ldots \exists x_3 \exists w_0 \ldots \exists w_3(\alpha \wedge \psi')$ for all nontrivial assignments f, and hence $\mathbf{B}^* \models \psi$ by Claim 2. □

4 Reduced Forms

In this section, we introduce *reduced* forms for conjunctive positive sentences on posets and prove that, given a poset $\mathbf{P}$ and a sentence ϕ, a reduced form for ϕ is easy to compute and equivalent to ϕ on $\mathbf{P}$.

In the rest of this section, $\sigma = \{\leq\}$ is the vocabulary of posets, and ϕ is a conjunctive positive σ-sentence as in (1)*.* Since ϕ will be evaluated on posets, where the formulas $x \leq y \wedge y \leq x$ and $x = y$ are equivalent, we assume that no atom of the form $x = y$ occurs in ϕ; otherwise, such an atom can be replaced by the formula $x \leq y \wedge y \leq x$ maintaining logical equivalence.

We represent ϕ by the pair $(\mathbf{Q}_\phi, \mathbf{M}_\phi)$, where $\mathbf{Q}_\phi = (Q_\phi, E^{\mathbf{Q}_\phi})$ and $\mathbf{M}_\phi = (M_\phi, E^{\mathbf{M}_\phi})$ are digraphs encoding the *prefix* and the *matrix* of ϕ respectively, as follows. The universes are $Q_\phi = M_\phi = \{x_1, y_1, \ldots, x_l, y_l\}$; we let $M_\phi^\forall = \{x_1, \ldots, x_l\}$ and $M_\phi^\exists = \{y_1, \ldots, y_l\}$ denote, respectively, the set of *universal* and *existential* variables in ϕ. The structure $\mathbf{Q}_\phi$ is a chain with cover relation $x_1 \prec^{\mathbf{Q}_\phi} y_1 \prec^{\mathbf{Q}_\phi} \cdots \prec^{\mathbf{Q}_\phi} x_l \prec^{\mathbf{Q}_\phi} y_l$. The structure $\mathbf{M}_\phi$ is defined by the edge relation $E^{\mathbf{M}_\phi} = \{(x, y) \mid x \leq y \text{ is an atom of } \phi\}$. We say that ϕ is in *reduced form* if:

(i) $\mathbf{M}_\phi$ is a poset;
(ii) the substructure of $\mathbf{M}_\phi$ induced by $M_\phi^\forall$ is an antichain;
(iii) for all distinct x and x' in $M_\phi^\forall$, it holds that $[x)^{\mathbf{M}_\phi} \cap [x')^{\mathbf{M}_\phi} = (x]^{\mathbf{M}_\phi} \cap (x']^{\mathbf{M}_\phi} = \emptyset$;
(iv) for all $x \in M_\phi^\forall$ and all $y \in M_\phi^\exists \cap ((x]^{\mathbf{M}_\phi} \cup [x)^{\mathbf{M}_\phi})$, it holds that $x <^{\mathbf{Q}_\phi} y$.

Let $\phi \in \mathcal{FO}(\forall, \exists, \wedge)$. For all $Z \subseteq M_\phi$, we let $\phi|_Z$ denote the conjunctive positive sentence represented by $(\mathbf{Q}_\phi|_Z, \mathbf{M}_\phi|_Z)$. It is readily observed that, for all $Z \subseteq M_\phi$, it holds that $\phi \models \phi|_Z$.

Proposition 1 ($\star$). *Let $\mathcal{P}$ be a class of posets. There exists a polynomial-time algorithm that, given an instance $(\mathbf{P}, \phi)$ of $\mathrm{MC}(\mathcal{P}, \mathcal{FO}(\forall, \exists, \wedge))$, either correctly rejects, or returns a sentence $\phi' \in \mathcal{FO}(\forall, \exists, \wedge)$ in reduced form such that $\mathbf{P} \models \phi'$ if and only if $\mathbf{P} \models \phi$.*

5 Fixed-Parameter Tractability

In this section, we prove that model checking conjunctive positive logic is fixed-parameter tractable parameterized by the size of the sentence *and* the width of the poset; it follows, in particular, that model checking conjunctive positive logic is fixed-parameter tractable (parameterized by the size of the sentence) on classes of posets of bounded width. We refer the reader to the introduction for an informal outline of the proof idea.

In the rest of this section, $\sigma = \{\leq\}$ is the vocabulary of posets, $\mathbf{P}$ is a poset and $\phi = (\mathbf{Q}_\phi, \mathbf{M}_\phi)$ is a conjunctive positive σ-sentence as in (1) *satisfying clauses* (i) *and* (ii) *of the definition of reduced form.*

5.1 Depth in the Sentence

Using the fact that ϕ is in reduced form, we define the following. For all $y \in M_\phi^\exists$: lower-depth(y) = depth$(\mathbf{M}_\phi|_{(y]^{\mathbf{M}_\phi}})$; upper-depth$(y)$ = depth$(\mathbf{M}_\phi|_{[y)^{\mathbf{M}_\phi}})$. In words, lower-depth$(y)$ is the size of the largest chain in the substructure of $\mathbf{M}_\phi$ induced by the downset of y in $\mathbf{M}_\phi$, and upper-depth(y) is the size of the largest chain in the substructure of $\mathbf{M}_\phi$ induced by the upset of y in $\mathbf{M}_\phi$.

Next, we define a partition of $M_\phi^\exists$ into two blocks L_ϕ and U_ϕ, the *lower* and *upper* variables respectively, as follows. For all $y \in M_\phi^\exists$ let $y \in L_\phi$ if and only if there either exists $x \in M_\phi^\forall$ such that $y \leq^{\mathbf{M}_\phi} x$, or $y \parallel^{\mathbf{M}_\phi} x$ for all $x \in M_\phi^\forall$ and lower-depth$(y) \leq$ upper-depth(y). Similarly, $y \in U_\phi$ if and only if there either exists $x \in M_\phi^\forall$ such that $y \geq^{\mathbf{M}_\phi} x$, or $y \parallel^{\mathbf{M}_\phi} x$ for all $x \in M_\phi^\forall$ and lower-depth$(y) >$ upper-depth(y). In words, an existential variable y in ϕ is lower if and only if it is below a universal variable in the matrix of ϕ, or is incomparable to all universal variables in the matrix of ϕ but "closer" to the bottom of the matrix of ϕ in that lower-depth$(y) \leq$ upper-depth(y); a similar idea drives the definition of upper variables.

Finally we define, for all $y \in M_\phi^\exists$: depth(y) = lower-depth(y) if $y \in L_\phi$, and depth(y) = upper-depth(y) if $y \in U_\phi$; in words, the depth of a lower variable is its "distance" from the bottom as measured by lower-depth(y), and similarly for upper variables.

5.2 Depth in the Structure

Relative to the poset $\mathbf{P}$, we define, for all $i \geq 0$, the set P_i as follows.

- $L_0 = \min(\mathbf{P})$, $U_0 = \max(\mathbf{P}) \setminus L_0$, and $P_0 = L_0 \cup U_0$.

– Let $i \geq 1$, and let $R \subseteq P_{i-1}$ be such that $R \cap L_{i-1}$ is downward closed in $\mathbf{P}|_{L_{i-1}}$ (that is, for all $l, l' \in L_{i-1}$, if $l \in R \cap L_{i-1}$ and $l' \leq^{\mathbf{P}} l$, then $l' \in R$) and $R \cap U_{i-1}$ is upward closed in $\mathbf{P}|_{U_{i-1}}$ (that is, for all $u, u' \in U_{i-1}$, if $u \in R \cap U_{i-1}$ and $u \leq^{\mathbf{P}} u'$, then $u' \in R$). Let

$$P_{i-1,R} = \left\{ p \in P \;\middle|\; \begin{array}{l} \text{for all } l \in L_{i-1},\, l \leq^{\mathbf{P}} p \text{ if and only if } l \in R, \\ \text{for all } u \in U_{i-1},\, p \leq^{\mathbf{P}} u \text{ if and only if } u \in R \end{array} \right\};$$

in words, $p \in P_{i-1,R}$ if and only if the elements in L_{i-1} below p are exactly those in $R \cap L_{i-1}$ (and the elements in $L_{i-1} \setminus R$ are incomparable to p) and the elements in U_{i-1} above p are exactly those in $R \cap U_{i-1}$ (and the elements in $U_{i-1} \setminus R$ are incomparable to p). We now define $P_i = L_i \cup U_i$ where L_i and U_i are as follows:

$$L_i = L_{i-1} \cup \bigcup_{R \subseteq P_{i-1}} \min^{\mathbf{P}}(P_{i-1,R}),\; U_i = \big(U_{i-1} \cup \bigcup_{R \subseteq P_{i-1}} \max^{\mathbf{P}}(P_{i-1,R})\big) \setminus L_i.$$

Let $p \in P$. Let $i \geq 0$ be minimum such that $p \in P_i$ (note that for every $p \in P$ such minimum i exists, and $L_i \cap U_i = \emptyset$ by construction). If $p \in L_i$, then $p \in L_{\mathbf{P}}$ and lower-depth$(p) = i$, and if $p \in U_i$, then $p \in U_{\mathbf{P}}$ and upper-depth$(p) = i$. Note that $L_{\mathbf{P}}$ and $U_{\mathbf{P}}$ partition P into two blocks containing the *lower* and *upper* elements respectively. Finally we define, for all $p \in P$: depth(p) = lower-depth(p), if $p \in L_{\mathbf{P}}$, and depth(p) = upper-depth(p), if $p \in U_{\mathbf{P}}$.

5.3 Depth Restricted Game

We now establish and formalize the relation between the depth in ϕ and the depth in $\mathbf{P}$ (see Lemma 1); this is the key combinatorial fact underlying the model checking algorithm.

Relative to the Hintikka game on $\mathbf{P}$ and ϕ, we define the following. A pair $(y, p) \in M^{\exists}_{\phi} \times P$ is *depth respecting* if $(y, p) \in (L_\phi \times L_{\mathbf{P}}) \cup (U_\phi \times U_{\mathbf{P}})$ and depth$(p) \leq$ depth(y). A strategy $(g_1, \ldots, g_l)$ for Eloise is *depth respecting* if, for all $i \in [l]$ and all plays $f\colon \{x_1, \ldots, x_l\} \to P$ by Abelard, the pair $(y_i, g_i(f(x_1), \ldots, f(x_i)))$ is depth respecting.

Let $b \geq 0$ be the maximum depth of a variable in ϕ. A play $f\colon \{x_1, \ldots, x_l\} \to P$ by Abelard is *bounded depth* if, for all $i \in [l]$, it holds that $f(x_i) \in P_{b+1}$.

Lemma 1. *The following are equivalent (w.r.t. the Hintikka game on* $\mathbf{P}$ *and* ϕ*).*

(i) Eloise has a winning strategy.
(ii) Eloise has a depth respecting winning strategy.
(iii) Eloise has a depth respecting strategy beating all bounded depth Abelard plays.

Proof. $(ii) \Rightarrow (iii)$ is trivial. We prove $(i) \Rightarrow (ii)$ and $(iii) \Rightarrow (i)$.

$(i) \Rightarrow (ii)$: Let $\mathbf{g} = (g_1, \ldots, g_l)$ be a winning strategy for Eloise. Let the Abelard play $f\colon \{x_1, \ldots, x_l\} \to P$ and the existential variable $y_j \in M^{\exists}_{\phi}$ be a minimal witness that the above winning strategy for Eloise is not depth respecting, in the following sense: $(y_j, g_j(f(x_1), \dots, f(x_j)))$ is not depth respecting, but for all $f'\colon \{x_1, \dots, x_l\} \to P$ and all $y_{j'} \in M_\phi^\exists$ such that either $y_j, y_{j'} \in L_\phi$ and lower-depth$(y_{j'}) <$ lower-depth(y_j), or $y_j, y_{j'} \in U_\phi$ and upper-depth$(y_{j'}) <$ upper-depth(y_j), it holds that $(y_{j'}, g_j(f'(x_1), \dots, f'(x_j)))$ is depth respecting.

We define a strategy $\mathbf{g}' = (g_1, \dots, g_{j-1}, g'_j, g_{j+1}, \dots, g_l)$ for Eloise such that g'_j restricted to $P^j \setminus \{(f(x_1), \dots, f(x_j))\}$ is equal to g_j (in other words, g'_j differs from g_j only in the move after $f\colon \{x_1, \dots, x_l\} \to P$), and $(y_j, g'_j(f(x_1), \dots, f(x_j)))$ is depth respecting. There are two cases to consider, depending on whether $y_j \in L_\phi$ or $y_j \in U_\phi$. We prove the statement in the former case; the argument is symmetric in the latter case.

So, assume $y_j \in L_\phi$. Let $g_j(f(x_1), \dots, f(x_j)) = p$ and $i = \text{depth}(y_j)$. Let $R \subseteq P_{i-1}$ (with $R \cap L_{i-1}$ downward closed in $\mathbf{P}|_{L_{i-1}}$ and $R \cap U_{i-1}$ upward closed in $\mathbf{P}|_{U_{i-1}}$) be such that, for all $l \in L_{i-1}$ and $u \in U_{i-1}$, it holds that $l \leq^{\mathbf{P}} p$ if and only if $l \in R$ and $p \leq^{\mathbf{P}} u$ if and only if $u \in R$. Hence $p \in P_{i-1,R}$. Then there exists $m \in \min^{\mathbf{P}}(P_{i-1,R})$ such that $m \leq^{\mathbf{P}} p$. By construction we have depth$(m) = i$. Let $g'_j\colon P^j \to P$ be exactly as g_j with the exception that $g'_j(f(x_1), \dots, f(x_j)) = m$; note that the pair (y_j, m) is depth respecting.

Claim 3 ($\star$) *Let f' be any play by Abelard. Then $\mathbf{g}' = (g_1, \dots, g'_j, \dots, g_l)$ beats f' in the Hintikka game on $\mathbf{P}$ and ϕ.*

We obtain a depth respecting winning strategy for Eloise by iterating the above argument thanks to Claim 5.

(iii) $\Rightarrow$ *(i)*: Let $b \geq 0$ be the maximum depth of a variable in ϕ, and let $\mathbf{g} = (g_1, \dots, g_l)$ be a depth respecting strategy for Eloise beating all bounded depth plays by Abelard. We define a strategy $\mathbf{g}' = (g'_1, \dots, g'_l)$ for Eloise, as follows.

Let $f\colon \{x_1, \dots, x_l\} \to P$ be a play by Abelard, say $f(x_i) = p_i$ for all $i \in [l]$. Let $i \in [l]$ and let $R_i \subseteq P_b$ (with $R_i \cap L_{b-1}$ downward closed in $\mathbf{P}|_{L_{b-1}}$ and $R_i \cap U_{b-1}$ upward closed in $\mathbf{P}|_{U_{b-1}}$) be such that for all $l \in L_b$, it holds that $l \leq^{\mathbf{P}} p_i$ if and only if $l \in R_i$ and for all $u \in U_b$, it holds that $p_i \leq^{\mathbf{P}} u$ if and only if $u \in R_i$. By construction, there exists $r_i \in P_{b+1}$ such that for all $l \in L_b$, it holds that $l \leq^{\mathbf{P}} r_i$ if and only if $l \leq^{\mathbf{P}} p_i$ and for all $u \in U_b$, it holds that $r_i \leq^{\mathbf{P}} u$ if and only if $p_i \leq^{\mathbf{P}} u$. Let $f'\colon \{x_1, \dots, x_l\} \to P$ be the bounded depth play by Abelard defined by $f'(x_i) = r_i$ for all $i \in [l]$. Finally define, for all $i \in [l]$, $g'_i(f(x_1), \dots, f(x_i)) = g_i(f'(x_1), \dots, f'(x_i))$.

Claim 4 ($\star$) $\mathbf{g}' = (g'_1, \dots, g'_l)$ *is a winning strategy for Eloise.*

This concludes the proof of the lemma. □

5.4 Fixed-Parameter Tractability

The following two lemmas allow to establish the correctness (Lemma 2, relying on Lemma 1) and the tractability (Lemma 3) of the presented model checking algorithm, respectively.

Lemma 2 (⋆). *Let $b \geq 0$ be the maximum depth of a variable in ϕ. Let $D = P_{b+1}$ and, for all $i \in [l]$, let*

$$D_i = \begin{cases} L_{\mathrm{depth}(y_i)}, & \text{if } y_i \in L_\phi, \\ U_{\mathrm{depth}(y_i)}, & \text{if } y_i \in U_\phi. \end{cases}$$

Then, $\mathbf{P} \models \phi$ if and only if $\mathbf{P} \models (\forall x_1 \in D)(\exists y_1 \in D_1) \ldots (\forall x_l \in D)(\exists y_l \in D_l)C$.

Lemma 3 (⋆). *Let $w = \mathrm{width}(\mathbf{P})$ and let $k \geq 0$. Then, $|P_k| \leq 2w^{(3w)^k}$.*

We are now ready to describe the announced algorithm. The underlying idea is that the characterization in Lemma 2 is checkable in fixed-parameter tractable time since $|D_i| \leq |D|$ for all $i \in [l]$, and $|D|$ is bounded above by a computable function of $\mathrm{width}(\mathbf{P})$ and $\|\phi\|$.

Theorem 2 (⋆). *There exists an algorithm that, given a poset $\mathbf{P}$ and a sentence $\phi \in \mathcal{FO}(\forall, \exists, \wedge)$, decides whether $\mathbf{P} \models \phi$ in*

$$\exp^4_w(O(k)) \cdot n^{O(1)}$$

time, where $w = \mathrm{width}(\mathbf{P})$, $k = \|\phi\|$, and $n = \|(\mathbf{P}, \phi)\|$.

Corollary 1. *Let $\mathcal{P}$ be a class of posets of bounded width. Then, the problem $\mathrm{MC}(\mathcal{P}, \mathcal{FO}(\forall, \exists, \wedge))$ is fixed-parameter tractable.*

6 Fixed-Parameter Intractability

In this section, we prove that model checking conjunctive positive logic on classes of bounded depth and bounded cover-degree posets is coW[2]-hard, and hence unlikely to be fixed-parameter tractable [8].

We first observe the following. Let ϕ_k be the $\mathcal{FO}(\forall, \exists, \wedge)$-sentence ($k \geq 1$)

$$\forall x_1 \ldots \forall x_k \exists y_1 \ldots \exists y_k \exists w \left(\bigwedge_{i \in [k]} y_i \leq x_i \wedge \bigwedge_{i \in [k]} y_i \leq w \right). \tag{3}$$

Proposition 2 (⋆). *For every poset $\mathbf{P}$ and $k \geq 1$, $\mathbf{P} \models \phi_k$ iff for every k elements $p_1, \ldots, p_k \in \min(\mathbf{P})$, there exists $u \in P$ such that $p_1, \ldots, p_k \leq^{\mathbf{P}} u$.*

We now describe the reductions. Let $\mathcal{H}$ be the class of hypergraphs (a *hypergraph* is a σ-structure $\mathbf{H}$ such that $U^{\mathbf{H}} \neq \emptyset$ for all U in a unary vocabulary σ). For the depth invariant, we define a function d from $\mathcal{H}$ to a class of posets of depth at most 2 where $d(\mathbf{H}) = \mathbf{P}$ such that: $\min(\mathbf{P}) = H$; $\max(\mathbf{P}) = \sigma$; $h \prec^{\mathbf{P}} U$ for all $h \in \min(\mathbf{P})$ and $U \in \max(\mathbf{P})$ such that $h \notin U^{\mathbf{H}}$. For the cover-degree invariant, we similarly define a function c from $\mathcal{H}$ to a class of posets with cover graphs of degree at most 3 (see [2] for details). We then use Proposition 2 to obtain:

Proposition 3 (⋆). *Let $r \in \{c, d\}$. Then, $\mathrm{MC}(\{r(\mathbf{H}) \mid \mathbf{H} \in \mathcal{H}\}, \mathcal{FO}(\forall, \exists, \wedge))$ is coW[2]-hard.*

7 Conclusion

We provided a parameterized complexity classification of the problem of model checking quantified conjunctive queries on posets with respect to the invariants in Figure 1; in particular, we push the tractability frontier of the model checking problem on bounded width posets closer towards the full first-order logic. The question of whether first-order logic is fixed-parameter tractable on bounded width posets remains open.

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