Polynomial Space Hardness without Disjunction Property

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Abstract

In [HT11], Horčík and Terui show that if a substructural logic enjoys the disjunction property, then its tautology problem is PSPACE-hard. We prove that all substructural logics in the interval between intuitionistic logic and generalized Hájek basic logic have a PSPACE-hard tautology problem, which implies that uncountably many substructural logics lacking the disjunction property have a PSPACE-hard tautology problem.

Keywords: Substructural Logics; Computational Complexity.

1 Introduction

In [S79], Statman describes an interpretation of true quantified Boolean sentences into intuitionistic tautologies, thus proving that the equational theory of Heyting algebras is PSPACE-hard. The *disjunction property* (in short, DP) of the intuitionistic propositional calculus (if a disjunction is logically provable, then one disjunct is logically provable) underlies the correctness of the reduction; an observation by Hertel and Urquhart [HU11] recently lifted in the encompassing framework of residuated lattices by Horčík and Terui [HT11]: If a variety of residuated lattices has the DP, then it has a PSPACE-hard equational theory. However, there are varieties with PSPACE-hard equational theories that lack the DP, for instance Heyting algebras satisfying the weak excluded middle identity, $\neg x \vee \neg\neg x = \top$.

In this note we prove that in fact, there exist uncountably many PSPACE-hard varieties lacking the DP. The key lemma (Lemma 12) originates from a study of Statman reduction [S79] in the framework of divisible commutative residuated lattices, which encompasses the fundamental case of Heyting algebras (with weak excluded middle). We extend in this setting the idea of Statman reduction, that a decision algorithm for intuitionistic tautologies can simulate effectively a brute force decision algorithm for Boolean sentences. Our reduction combines the reduction of Boolean tautologies to lattice equations by Hunt III et al. [HRB87], and the reduction of Boolean sentences to intuitionistic tautologies by Švejdar [S03]; it uses only lattice and residual operations, thus avoiding negations and multiplications, in contrast with [S03] and [HT11] respectively.

We establish that all varieties in the interval between Heyting algebras with weak excluded middle and divisible commutative integral residuated lattices have PSPACE-hard equational theory (Theorem 13), which implies the existence of uncountably many varieties of residuated lattices that fail the DP and have a PSPACE-hard equational theory; indeed, each join reducible variety lacks the DP, and the interval contains uncountably many such varieties (Theorem 14). The weak excluded middle case further enlightens, with a concrete, purely syntactic restriction, that Statman reduction does not require the full disjunction property (see Section 3.5).

The result in [HT11] completes a previous result in [BM09]: The equational and quasiequational theories of commutative integral divisible residuated lattices are PSPACE-complete, thus establishing a large variety of PSPACE-complete residuated lattices. It would be interesting to extend this completeness result to distributive integral residuated lattices, which are PSPACE-hard by [HT11]. In recent work, Galatos [G] shows that distributive integral residuated lattices have the finite embeddability property, which implies the decidability of their quasiequational (in fact, universal) theory.

2 Background

In this section, we introduce the required formal background on residuated lattices (Section 2.1), finite models (Section 2.2), and Boolean sentences (Section 2.3). We refer the reader to [GJKO07] for a comprehensive discussion of the relation between residuated structures and substructural logics.

2.1 Residuated Lattices

Let $\sigma = (\wedge, \vee, \cdot, \to, e)$ be an algebraic language of type $(2, 2, 2, 2, 0)$. A *commutative residuated lattice* is a σ-algebra $\mathbf{A} = (A, \wedge, \vee, \cdot, \to, e)$ where $(A, \wedge, \vee)$ is a lattice, $(A, \cdot, e)$ is a commutative monoid, and *residuation* holds identically: $x \cdot z \leq y$ if and only if $z \leq x \to y$ for all $x, y, z \in A$. A commutative residuated lattice is *divisible* if $x \wedge y = x \cdot (x \to (x \wedge y))$ for all $x, y \in A$. A *commutative GBL-algebra* is a divisible commutative residuated lattice. [1] A commutative residuated lattice is: *integral*, if the monoid identity e is the maximum element in the order (in this case, we use the symbol $\top$ instead of e); *bounded*, if the order has a minimum element, identified by an additional constant symbol $\bot$. In the bounded case, we write $\neg x$ instead of $x \to \bot$. A *CIBGBL-algebra* is a commutative integral bounded GBL-algebra; we denote by $\mathcal{CIBGBL}$ the variety of CIBGBL-algebras.

As relevant examples, CIBGBL-algebras satisfying *idempotency*, $x \cdot x = x$, or *Heyting* algebras, form the equivalent algebraic semantics of intuitionistic

[1] The acronym abbreviates Generalized Basic Logic, a terminology justified in the next paragraph; compare [JT02] for the general definition of GBL-algebras relative to residuated lattices.

propositional logic (we denote by $\mathcal{H}$ the variety of Heyting algebras); CIBGBL-algebras satisfying *prelinearity*, $(x \to y) \vee (y \to x) = \top$, or *BL-algebras*, form the equivalent algebraic semantics of Hájek basic (fuzzy) propositional logic (we denote by $\mathcal{BL}$ the variety of BL-algebras); BL-algebras satisfying *involutiveness*, $\neg\neg x = x$, or *MV-algebras*, form the equivalent algebraic semantics of Łukasiewicz propositional logic; *Boolean* algebras, the semantics of classical logic, are Heyting algebras satisfying the *excluded middle*, $x \vee \neg x = \top$, or idempotent MV-algebras. Indeed, it is possible to introduce CIBGBL-algebras as the equivalent algebraic semantics of a propositional logic, the *generalized* Hájek basic logic, which is a fuzzy fragment of intuitionistic logic (worlds admit intermediate truth degrees, which are forbidden in intuitionistic logic because of idempotency, see Section 2.2), or a constructive fragment of (Hájek basic) fuzzy logic (deduction is constructive by the disjunction property, which fails in fuzzy logic because of prelinearity).

In this note, we focus on the interval (in the lattice of varieties over σ) between CIBGBL-algebras and Heyting algebras satisfying the *weak excluded middle*, $\neg x \vee \neg\neg x = \top$ (in short, HW-algebras); we denote by $\mathcal{HW}$ the variety of HW-algebras. The variety of HW-algebras forms the equivalent algebraic semantics of intuitionistic propositional logic with weak excluded middle, also known in the literature as Jankov (or De Morgan) logic.

Let $\sigma = (\wedge, \vee, \cdot, \to, \top, \bot)$ be an algebraic language of type $(2, 2, 2, 2, 0, 0)$. A *σ-term* (in short, a *term*) is either a variable in a fixed countable set X, or the constant $\top$ or $\bot$, or it has the form $(t_1 \circ t_2)$ where $\circ \in \{\wedge, \vee, \cdot, \to\}$ and t_1 and t_2 are terms. Let T be the set of σ-terms and $\mathbf{A} = (A, \wedge, \vee, \cdot, \to, \top, \bot)$ be a CIBGBL-algebra. If $t(x_1, \ldots, x_n)$ is a term with variables among $x_1, \ldots, x_n$, then t determines an n-ary operation $t^{\mathbf{A}} \colon A^n \to A$ in the usual way. Let t and s be terms with variables among $x_1, \ldots, x_n$. The equation $t = s$ holds in $\mathbf{A}$ under an assignment $\mathbf{h}$ of the variables in A such that $x_1 \mapsto a_1, \ldots, x_n \mapsto a_n$ if and only if $t^{\mathbf{A}}(a_1, \ldots, a_n) = s^{\mathbf{A}}(a_1, \ldots, a_n)$; we write $\mathbf{A}, \mathbf{h} \models t = s$. It is easy to check that $t = s$ holds in $\mathbf{A}$ under $\mathbf{h}$ if and only if $(t \to s) \wedge (s \to t) = \top$ holds in $\mathbf{A}$ under $\mathbf{h}$; thus equations in general form reduce to equations in the special form $t = \top$. The equation $t = \top$ *holds (identically)* in $\mathbf{A}$ if and only if it holds in $\mathbf{A}$ under all assignments of the variables in A.

Let $\mathcal{V}$ be a variety of CIBGBL-algebras. The equation $t = \top$ *holds* in $\mathcal{V}$ if and only if it holds in all algebras $\mathbf{A}$ in $\mathcal{V}$; we write $\mathcal{V} \models t = \top$. The *equational theory* of $\mathcal{V}$ contains exactly those terms t such that $\mathcal{V} \models t = \top$. In Section 3, we prove that any variety of integral residuated lattices between $\mathcal{HW}$ and $\mathcal{CIBGBL}$ has a PSPACE-hard equational theory.

2.2 Finite Models

A *(finite) poset* is a pair $(P, \leq_P)$ where P is a (finite) set and $\leq_P$ is a binary, reflexive, antisymmetric, and transitive relation over P. A poset $(P, \leq_P)$ is: a *chain* if each pair of distinct points in P is comparable; a *tree* if for each $p \in P$, the set $\{q \mid q \leq_P p\}$ with the inherited order is a chain; *rooted*, if there exists $p \in P$ such that $p \leq_P q$ for all $q \in P$. If $(P, \leq_P)$ is a rooted tree, then $B \subseteq P$ is a *branch* in $(P, \leq_P)$ if and only if B with the inherited order is a maximal chain (under inclusion in P).

For all $n \in \mathbb{N} = \{1, 2, \dots\}$, there exists a unique finite MV-chain,

$$\mathbf{n+1} = (\{0, 1/n, \dots, (n-1)/n, 1\}, \wedge_n, \vee_n, \cdot_n, \rightarrow_n, \top_n, \bot_n), \tag{1}$$

defined as follows. For all $m, m' \in \{0, 1/n, \dots, (n-1)/n, 1\}$:

$$\top_n = 1; \tag{2}$$
$$\bot_n = 0; \tag{3}$$
$$m \wedge_n m' = \min\{m, m'\}; \tag{4}$$
$$m \vee_n m' = \max\{m, m'\}; \tag{5}$$
$$m \cdot_n m' = \max\{0, m + m' - 1\}; \tag{6}$$
$$m \rightarrow_n m' = \min\{1, m' + 1 - m\}. \tag{7}$$

Example 1. *If $n = 1$, then $\wedge_1 = \cdot_1$, and $\mathbf{2}$ is the usual Boolean algebra on $\{0, 1\}$.*

Definition 2 (Labelled Poset)**.** *Let $S \subseteq \mathbb{N}$. A* S-labelled poset *is a triple $\mathbf{P} = (P, \leq_P, l_P)$ where $(P, \leq_P)$ is a poset and $l_P : P \to S$.*

The following notion is delicate and central.

Definition 3 (Assignment, Evaluation)**.** *Let $\mathbf{P} = (P, \leq_P, l_P)$ be a $\mathbb{N}$-labelled poset. A* (variable) assignment *in $\mathbf{P}$ is a map*

$$\mathbf{h} : P \times X \to \mathbb{Q} \cap [0, 1] \tag{8}$$

such that for all points $p \in P$ and all variables $x \in X$:

1. *$\mathbf{h}(p, x) \in \{0, 1/l_P(p), \dots, (l_P(p) - 1)/l_P(p), 1\}$;*
2. *$\mathbf{h}(p, x) = 0$ or $\mathbf{h}(q, x) = 1$ for all $q \in P$ such that $p <_P q$.*

Recalling (2)-(7), the assignment $\mathbf{h}$ extends to a unique (term) evaluation *map*

$$\mathbf{h} : P \times T \to \mathbb{Q} \cap [0, 1] \tag{9}$$

by induction on $t \in T$, as follows (where $l_P(p) = n$):

$$\mathbf{h}(p, \top) = \top_n; \tag{10}$$
$$\mathbf{h}(p, \bot) = \bot_n; \tag{11}$$
$$\mathbf{h}(p, t' \circ t'') = \mathbf{h}(p, t') \circ_n \mathbf{h}(p, t'') \text{ for all } \circ \in \{\wedge, \vee, \cdot\}; \tag{12}$$
$$\mathbf{h}(p, t' \to t'') = \begin{cases} \mathbf{h}(p, t') \rightarrow_n \mathbf{h}(p, t''), & (\forall q >_P p)(\mathbf{h}(q, t') \leq \mathbf{h}(q, t'')); \\ \bot_n, & \text{otherwise.} \end{cases} \tag{13}$$

We say that t evaluates pointwise *at p under $\mathbf{h}$ if and only if, clauses (10)-(12) apply, or the first case of clause (13) applies.*

We emphasize that if $\mathbf{h}(q, t'') < \mathbf{h}(q, t')$ for some $q \in P$ such that $p <_P q$, then $\mathbf{h}(p, t' \to t'') = 0$ independent of the values $\mathbf{h}(p, t')$ and $\mathbf{h}(p, t'')$. Note that the defining properties of an assignment inductively extend to all terms $t \in T$, in particular: $\mathbf{h}(p, t) = 0$ or $\mathbf{h}(q, t) = 1$ for all $q >_P p$. Hence, for all $t', t'' \in T$: $\mathbf{h}(p, t' \to t'') = 1$ if and only if $\mathbf{h}(q, t') \leq \mathbf{h}(q, t'')$ for all $q \geq_P p$. Throughout the paper we routinely use the previous (and other) simple properties of Definition 3, without explicit mention.

Example 4. *Let $\mathbf{P} = (P, \leq_P, l_P)$ be a $\mathbb{N}$-labelled poset such that $l_P \colon P \to \mathbb{N}$ is the constant* 1*,*

$$l_P = 1.$$

Upon displaying the assignment $\mathbf{h}$ *as a map* $\mathbf{h} \colon X \to 2^P$ *via*

$$\mathbf{h}(x) = \{p \in P \mid \mathbf{h}(p, x) = \top_1\},$$

clauses (2)-(13) define an intuitionistic Kripke model $(P, \leq_P, \mathbf{h})$ over the Kripke frame $(P, \leq_P)$, where

$$\mathbf{h}, p \models t \textit{ if and only if } \mathbf{h}(p, t) = \top_1 \tag{14}$$

for all $p \in P$ and $t \in T$. In this particular case, the general property that $\mathbf{h}(p, t) = 0$ or $\mathbf{h}(q, t) = 1$ for all $q >_P p$ is the usual monotonicity of forcing in intuitionistic logic.

Notation 5. *In light of (14) in Example 4, we freely switch between the notation in (8)-(9), and the usual intuitionistic notation. For instance, we write:*

1. $\mathbf{h}, p \models t$ *instead of* $\mathbf{h}(p, t) = \top_{l_P(p)}$*;*
2. $\mathbf{h}, p \models t' \circ t''$ *instead of* $\mathbf{h}(p, t') \circ \mathbf{h}(p, t'')$ *for* $\circ \in \{<, \leq, =\}$*.*

Let t be a term, and $\mathbf{P} = (P, \leq_P, l_P)$ a $\mathbb{N}$-labelled poset. Let $p \in P$, and $\mathbf{h}$ be an assignment in $\mathbf{P}$. We say that $t = \top$ *fails at p in* $\mathbf{P}$ *under* $\mathbf{h}$ if $\mathbf{h}, p \models t < \top$. We also say: $t = \top$ *fails at p in* $\mathbf{P}$ if there exists an assignment $\mathbf{h}$ in $\mathbf{P}$ such that $t = \top$ fails at p in $\mathbf{P}$ under $\mathbf{h}$; $t = \top$ *fails in* $\mathbf{P}$ *under* $\mathbf{h}$ if there exists $p \in P$ such that $t = \top$ fails at p in $\mathbf{P}$ under $\mathbf{h}$; $t = \top$ *fails in* $\mathbf{P}$ if there exist $p \in P$ and an assignment $\mathbf{h}$ in $\mathbf{P}$ such that $t = \top$ fails at p in $\mathbf{P}$ under $\mathbf{h}$.

It is folklore that an equation $t = \top$ fails in the variety of Heyting algebras if and only if it fails at the root of a finite Kripke model. In [JM09], Jipsen and Montagna generalize this fact to CIBGBL-algebras. The construction, a nontrivial generalization of Birkhoff representation of finite bounded distributive lattices by finite posets [B37], is based on the following definition.

Definition 6 (Poset Product)**.** *Let $\mathbf{P} = (P, \leq_P, l_P)$ be a (finite) $\mathbb{N}$-labelled poset. For all $p \in P$ such that $l_P(p) = n$, let*

$$\mathbf{C}_p = (C_p, \wedge_p, \vee_p, \cdot_p, \to_p, \top_p, \bot_p)$$

be isomorphic to $\mathbf{n+1}$*. The (finite)* poset product *over* $\mathbf{P}$*,*

$$\Xi(\mathbf{P}) = (\prod_{p \in \mathbf{P}} \mathbf{C}_p, \wedge, \vee, \cdot, \to, \top, \bot), \tag{15}$$

is defined as follows. Let

$$\prod_{p \in \mathbf{P}} \mathbf{C}_p = \{f \in \prod_{p \in P} C_p \mid (\forall p \in P)(\pi_p(f) = \bot_p \textit{ or } (\forall q >_P p)(\pi_q(f) = \top_q))\},$$

where $\prod_{p \in P} C_p$ denotes the Cartesian product of the indexed family $(C_p)_{p \in P}$, and for all indices $p \in P$, $\pi_p(f)$ denotes the projection of $f \in \prod_{p \in P} C_p$ at p.

For all $f, g \in \prod_{p \in \mathbf{P}} \mathbf{C}_p$ *and* $p \in P$*:*

$$\pi_p(\top) = \top_p; \tag{16}$$

$$\pi_p(\bot) = \bot_p; \tag{17}$$

$$\pi_p(f \circ g) = \pi_p(f) \circ_p \pi_p(g) \text{ for all } \circ \in \{\wedge, \vee, \cdot\}; \tag{18}$$

$$\pi_p(f \to g) = \begin{cases} \pi_p(f) \to_p \pi_p(g), & (\forall q >_P p)(\pi_q(f) \leq_P \pi_q(g)); \\ \bot_p, & \text{otherwise.} \end{cases} \tag{19}$$

In fact, [JM09, Section 6] shows that the map Ξ in (15) establishes a nice bijective correspondence between finite $\mathbb{N}$-labelled posets and finite CIBGBL-algebras, as follows.

Theorem 7 (Finite Representation)**.** *The map* Ξ *in (15) is (up to isomorphism) a bijection between finite CIBGBL-algebras and finite* $\mathbb{N}$*-labelled posets. If the finite CIBGBL-algebra* $\mathbf{A}$ *and the finite* $\mathbb{N}$*-labelled poset* $\mathbf{P} = (P, \leq_P, l_P)$ *correspond under* Ξ*, then:*

1. $\mathbf{A}$ *is subdirectly irreducible if and only if* $(P, \leq)$ *is rooted;*
2. $\mathbf{A}$ *is a Heyting algebra if and only if* $l_P = 1$*;*
3. $t = \top$ *fails in* $\mathbf{A}$ *if and only if* $t = \top$ *fails in* $\mathbf{P}$*, for all terms* t*.*

Example 8. *The finite CIBGBL-algebra* $\mathbf{A} = (\{0, \ldots, 7\}, \wedge, \vee, \cdot, \to, 0, 7)$*, with*

$x \cdot y$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0	0	1	0	1	1	1
2	0	0	2	0	2	2	2	2
3	0	1	0	3	1	3	3	3
4	0	0	2	1	2	4	4	4
5	0	1	2	3	4	5	5	5
6	0	1	2	3	4	5	5	6
7	0	1	2	3	4	5	6	7

and

$x \to y$	0	1	2	3	4	5	6	7
0	7	4	3	2	1	0	0	0
1	7	7	3	4	3	1	1	1
2	7	4	7	2	4	2	2	2
3	7	7	3	7	3	3	3	3
4	7	7	7	4	7	4	4	4
5	7	7	7	7	7	7	6	5
6	7	7	7	7	7	7	7	6
7	7	7	7	7	7	7	7	7

,

and the finite $\{1, 2\}$*-labelled rooted poset* $\mathbf{P} = (P, \leq_P, l_P)$*, depicted respectively on the left and on the right of Figure 1, are in correspondence under* Ξ*, that is,* $\Xi(\mathbf{P}) = \mathbf{A}$ *and* $\mathbf{P} = \Xi^{-1}(\mathbf{A})$*.*

The mapping $\mathbf{P} \mapsto \mathbf{A}$ *is a direct computation of* Ξ *in Definition 6. The inverse mapping* $\mathbf{A} \mapsto \mathbf{P}$ *is computed as follows:* $(P, \leq_P)$ *is the poset of idempotent join irreducible elements in* $\mathbf{A}$*, namely* $\{2, 3, 7\} = \{0, 2, 3, 5, 7\} \cap \{1, 2, 3, 6, 7\}$*, with inherited (reverse) order. If* $p \in P$*, then* $l_P(p) + 1$ *is the size of the interval in* $\mathbf{A}$ *between* p *and the least upper bound of all idempotent elements in* $\mathbf{A}$ *strictly below* p*. Then* $l_P(2) = |\{0, 2\}| - 1 = 1$*,* $l_P(3) = |\{0, 1, 3\}| - 1 = 2$*, and* $l_P(7) = |\{5, 6, 7\}| - 1 = 2$*.*

By universal algebraic facts [BS81, Theorem 8.6], if an equation fails in a variety, then it fails in a subdirectly irreducible algebra of the variety. Moreover, CIBGBL-algebras have the finite model property [JM09, Theorem 5.2], that is, if an equation fails in $\mathcal{CIBGBL}$, then it fails in a finite CIBGBL-algebra. These two facts, Theorem 7, and Definition 3 imply,

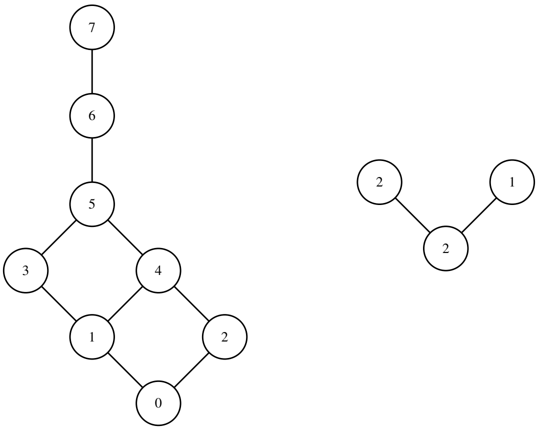

Figure 1: On the left and on the right, respectively, the algebra **A** and the poset **P** in Example 8.

Theorem 9 (Finite Model Property)**.** *Let t be a term. The following are equivalent:*

1. $t = \top$ *fails in* $\mathcal{CIBGBL}$.
2. $t = \top$ *fails at the root of some finite* $\mathbb{N}$*-labelled poset.*

2.3 Boolean Sentences

A term is in disjunctive normal form (DNF) if it is a disjunction ($\vee$) of conjunctions ($\wedge$) of variables (positive literals) and negations ($\neg$) of variables (negative literals). The *quantified Boolean formula* problem is the problem of deciding whether a sentence ϕ of the form,

$$\phi = Q_l x_l \cdots Q_2 x_2 Q_1 x_1 (\psi = \top), \tag{20}$$

where $Q_i \in \{\forall, \exists\}$ for all $i \in \{1, \dots, l\}$, and ψ is in DNF, is true in the Boolean algebra **2**. The size (of the binary encoding) of ϕ is bounded by a polynomial of the number of symbols $\neg$, $\vee$, $\wedge$ occurring in ψ, and the quantified Boolean formula problem is PSPACE-complete [Pap94].

The quantifier prefix $\pi = Q_l \cdots Q_2 Q_1$ of ϕ determines a unique finite rooted tree $(T, \leq_\pi)$ as follows. $(T, \leq_\pi)$ has $l + 1$ levels, where the root $r \in T$ has level 0, and if the point p has level k, then the covers of p have level $k + 1$ (so the leaves have level l). For $k = 0, \dots, l - 1$, let p be a point of level k. If $Q_{l-k} = \forall$, then p has exactly two covers q' and q'' in $(T, \leq_\pi)$. If $Q_{l-k} = \exists$, then p has exactly one cover q in $(T, \leq_\pi)$.

We let

$$\mathbf{T}_\pi = (T, \leq_\pi, b)$$

denote a structure where $b \colon T \setminus \{r\} \to \{0, 1\}$ is such that for all $p \in T$, if q' and q'' are two distinct covers of p, then $b(q') = 0$ and $b(q'') = 1$. Let B be a branch in $\mathbf{T}_\pi$, say, $r <_\pi q_l <_\pi q_{l-1} <_\pi \cdots <_\pi q_2 <_\pi q_1$. Define $\mathbf{h}_B \colon \{x_1, \dots, x_l\} \to \{0, 1\}$ by:

$$\mathbf{h}_B(x_i) = b(q_i) \tag{21}$$

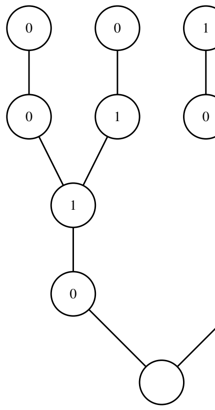

Figure 2: A countermodel $\mathbf{T}_{\forall\exists\forall\exists} = (T, \leq_{\forall\exists\forall\exists}, b)$ of ϕ in Example 10. A direct computation shows that $\mathbf{2}, \mathbf{h} \models \phi'$ for all $\mathbf{h} \in \mathbf{T}'_{\forall\exists\forall\exists}$.

for $i = 1, \ldots, l$. Define

$$\mathbf{T}'_\pi = \{\mathbf{h}_B \mid B \text{ is a branch in } \mathbf{T}_\pi\} \subseteq \{0,1\}^l. \tag{22}$$

Let ϕ be a quantified Boolean sentence as in (20). A *proof* of ϕ is a structure $\mathbf{T}_\pi$ such that for all $\mathbf{h} \in \mathbf{T}'_\pi$,

$$\mathbf{2}, \mathbf{h} \models \psi = \top.$$

A *countermodel* to ϕ is a proof of $\neg\phi$. Clearly, $\mathbf{2} \models \phi$ if and only if ϕ has a proof.

Example 10. *Let $\phi = \exists x_4 \forall x_3 \exists x_2 \forall x_1 (\psi = \top)$, where*

$$\psi = (\neg x_1 \wedge \neg x_2 \wedge x_4) \vee (x_1 \wedge x_2) \vee (x_1 \wedge \neg x_4) \vee (\neg x_3).$$

The labelled rooted tree depicted in Figure 2 is a countermodel to ϕ, or equivalently a proof for $\neg\phi = \forall x_4 \exists x_3 \forall x_2 \exists x_1 (\psi' = \top)$, where

$$\psi' = (x_1 \wedge \neg x_2 \wedge x_3 \wedge x_4) \vee (\neg x_1 \wedge x_3 \wedge \neg x_4) \vee (\neg x_1 \wedge x_2 \wedge x_3).$$

3 Hardness

In this section, we prove the main result of the paper. In Section 3.1, we give a translation of a quantified Boolean sentence ϕ to a term t_ϕ, over the restricted signature $\wedge$, $\vee$, $\to$. In Sections 3.2, 3.3, and 3.4, we prove the nontrivial parts of the statement: $\mathbf{2} \models \phi$ if and only if $\mathcal{V} \models t_\phi = \top$ for all varieties $\mathcal{V}$ between Heyting algebras with weak excluded middle and CIBGBL-algebras (Lemma 12).

3.1 Main Result

We describe a reduction that receives as input a quantified Boolean sentence ϕ specified as in (20), and returns as output a term t_ϕ. The reduction is inspired by the original interpretation of true quantified Boolean sentences into intuitionistic tautologies by Statman [S79], in the simplified version presented by Švejdar [S03]. The key idea of Statman reduction is that a decision algorithm for intuitionistic tautologies can simulate effectively a brute force decision algorithm for true quantified Boolean sentences. We note that, using the interpretation of Boolean tautologies into lattice identities by Hunt III et al. [HRB87], we obtain a version of Statman reduction that avoids negations ($\neg$); this property is necessary to prove Lemma 24 (see Example 28).

The reduction works as follows. For each variable x_i in ϕ, introduce fresh variables y_i and z_i. Let t_0 be obtained by replacing $\neg x_i$ by y_i in ψ (for $i = 1, \dots, l$), in symbols,

$$t_0 = \psi[y_1/\neg x_1, \dots, y_l/\neg x_l]. \tag{23}$$

Define, for $i = 0, \dots, l-1$,

$$t_{i+1} = \begin{cases} (t_i \to z_{i+1}) \to ((x_{i+1} \to z_{i+1}) \vee (y_{i+1} \to z_{i+1})), & \text{if } Q_{i+1} = \exists; \\ (x_{i+1} \vee y_{i+1}) \to t_i, & \text{if } Q_{i+1} = \forall. \end{cases} \tag{24}$$

Finally define

$$t_\phi = t_l. \tag{25}$$

The reduction $\phi \mapsto t_\phi$ is computable in logspace (in the size of ϕ) by a sequence of local substitutions; notice that the exponential growth of the size of the constructed terms is avoided by using the auxiliary variable z_{i+1} as a replacement for the term t_i.

Example 11. *Let ϕ be as in Example 10. Then,*

$$\begin{aligned} t_0 &= (y_1 \wedge y_2 \wedge x_4) \vee (x_1 \wedge x_2) \vee (x_1 \wedge y_4) \vee (y_3), \\ t_1 &= (x_1 \vee y_1) \to t_0, \\ t_2 &= (t_1 \to z_2) \to (x_2 \to z_2 \vee y_2 \to z_2), \\ t_3 &= (x_3 \vee y_3) \to t_2, \\ t_\phi = t_4 &= (t_3 \to z_4) \to (x_4 \to z_4 \vee y_4 \to z_4). \end{aligned}$$

Lemma 12. *Let ϕ be a quantified Boolean sentence specified as in (20), and let t_ϕ be the term specified as in (25). The following are equivalent:*

(i) $\mathcal{CIBGBL} \models t_\phi = \top$.

(ii) $\mathcal{H} \models t_\phi = \top$.

(iii) $\mathbf{2} \models \phi$.

(iv) $\mathcal{HW} \models t_\phi = \top$.

Proof. Clearly $(i) \Rightarrow (ii) \Rightarrow (iv)$. Lemma 15 in Section 3.2 proves $(iii) \Rightarrow (i)$. Lemma 22 in Section 3.3 proves $(ii) \Rightarrow (iii)$. Lemma 24 in Section 3.4 proves $(iv) \Rightarrow (ii)$. $\square$

Theorem 13. *Let $\mathcal{V}$ be a variety such that $\mathcal{HW} \subseteq \mathcal{V} \subseteq \mathcal{CIBGBL}$. Then the equational theory of $\mathcal{V}$ is PSPACE-hard.*

Proof. Let $\mathcal{V}$ be a variety such that $\mathcal{HW} \subseteq \mathcal{V} \subseteq \mathcal{CIBGBL}$. Let ϕ be a quantified Boolean sentence specified as in (20), and let t_ϕ be the term specified as in (25). If $\mathbf{2} \models \phi$, then $\mathcal{CIBGBL} \models t_\phi = \top$ by $(iii) \Rightarrow (i)$ in Lemma 12, so that $\mathcal{V} \models t_\phi = \top$ because $\mathcal{V} \subseteq \mathcal{CIBGBL}$. If $\mathcal{V} \models t_\phi = \top$, then $\mathcal{HW} \models t_\phi = \top$ because $\mathcal{HW} \subseteq \mathcal{V}$, so that $\mathbf{2} \models \phi$ by $(iv) \Rightarrow (iii)$ in Lemma 12. $\square$

Note that all varieties of residuated lattices with the DP have PSPACE-hard equational theory by [HT11], in particular those lying in the above interval. However, the interval also contains uncountably many varieties lacking the DP, but still having a PSPACE-hard equational theory.

Theorem 14. *Uncountably many varieties of residuated lattices lack the DP and have a PSPACE-hard equational theory.*

Proof. Let $\mathcal{V}$ be the join in the lattice of varieties of language σ of Heyting algebras and BL-algebras; in symbols, $\mathcal{V} = \mathcal{H} \vee \mathcal{BL}$. Observe that any subvariety of $\mathcal{V}$ satisfies

$$(x \to (x \cdot x)) \vee (y \to z) \vee (z \to y) = \top;$$

indeed, the equation holds identically in both $\mathcal{H}$ and $\mathcal{BL}$, and hence in the variety generated by $\mathcal{H} \cup \mathcal{BL}$. Let $\mathbf{A}$ be a subdirectly irreducible algebra in $\mathcal{V}$. Note that $\top^{\mathbf{A}}$ is join irreducible in $\mathbf{A}$. Therefore, either $\mathbf{A} \models x = x \cdot x$, and hence $\mathbf{A}$ is a Heyting algebra, or $\mathbf{A} \models (y \to z) \vee (z \to y) = \top$, and hence $\mathbf{A}$ is a BL-algebra: indeed, if $a \neq a \cdot a$ for some $a \in A$, then $\mathbf{A}, x \mapsto a \models x \to x \cdot x < \top$ (because $\mathbf{A}, x \mapsto a \models x > x \cdot x$), so $\mathbf{A} \models (y \to z) \vee (z \to y) = \top$ (because $\top^{\mathbf{A}}$ is join irreducible); thus $\mathbf{A}$ is a BL-algebra. Now, let $\mathcal{BL}'$ be a subvariety of $\mathcal{BL}$, and let $\mathcal{V}'$ be the subvariety of $\mathcal{V}$ generated by $\mathcal{H} \cup \mathcal{BL}'$, that is, $\mathcal{V}' = \mathcal{H} \vee \mathcal{BL}'$. Then every subdirectly irreducible algebra $\mathbf{A}$ in $\mathcal{V}' \setminus \mathcal{H}$ is a BL-algebra in $\mathcal{BL}'$: indeed, for any equation $t = \top$ such that $\mathcal{BL}' \models t = \top$ and $\mathcal{BL} \not\models t = \top$, we have that $\mathcal{V}' \models (x \to (x \cdot x)) \vee t = \top$; and since $\mathbf{A} \not\models x \to (x \cdot x) = \top$, we conclude that $\mathbf{A} \models t = \top$ as above. Therefore, any subdirectly irreducible algebra in $\mathcal{V}'$ is in $\mathcal{H}$ or in $\mathcal{BL}'$.

By [AM03, Theorem 4.12], there exist uncountably many nonidempotent subvarieties $\mathcal{BL}'$ of $\mathcal{BL}$. Since subdirectly irreducible algebras in $\mathcal{V}'$ are in $\mathcal{H}$ or in $\mathcal{BL}'$, there are uncountably many varieties $\mathcal{V}'$ of the form $\mathcal{V}' = \mathcal{H} \vee \mathcal{BL}'$, with $\mathcal{BL}'$ nonidempotent subvariety of $\mathcal{BL}$. Now $\mathcal{H} \vee \mathcal{BL}'$ is a nontrivial join decomposition of $\mathcal{V}'$ (because $\mathcal{V}'$ contains both nonprelinear and nonidempotent algebras, hence $\mathcal{V}' \neq \mathcal{BL}'$ and $\mathcal{V}' \neq \mathcal{H}$), so that $\mathcal{V}'$ is join reducible in the lattice of varieties of language σ. Clearly, $\mathcal{V}'$ lies in the interval between $\mathcal{HW}$ and $\mathcal{CIBGBL}$, so that its equational theory is PSPACE-hard by Theorem 13. However, by join reducibility $\mathcal{V}'$ fails the disjunction property. Indeed, if $s = \top$ and $t = \top$ are equations such that $\mathcal{BL}' \models s = \top$ and $\mathcal{BL}' \not\models t = \top$, and $\mathcal{H} \models t = \top$ and $\mathcal{H} \not\models s = \top$ (for instance, prelinearity and idempotency), then $\mathcal{H} \vee \mathcal{BL}' \models t \vee s = \top$ but neither $\mathcal{H} \vee \mathcal{BL}' \models s = \top$ nor $\mathcal{H} \vee \mathcal{BL}' \models t = \top$. $\square$

3.2 $(iii) \Rightarrow (i)$

In this section, we prove that

Lemma 15. $\mathbf{2} \models \phi$ *implies* $\mathcal{CIBGBL} \models t_\phi = \top$.

We establish the key fact (recall (24)).

Proposition 16. *Let $\mathbf{P} = (P, \leq_P, l_P)$ be a finite $\mathbb{N}$-labelled rooted poset, and let $\mathbf{h}$ be an assignment in $\mathbf{P}$. For all $i \in \{1, \dots, l\}$ and $p \in P$, if $\mathbf{h}, p \models t_i < \top$, then:*

1. *If $Q_i = \exists$, then there exist $q', q'' \in P$ such that, $p \leq_P q', q''$, $\mathbf{h}, q' \models t_{i-1} < x_i$, and $\mathbf{h}, q'' \models t_{i-1} < y_i$.*
2. *If $Q_i = \forall$, then there exists $q \in P$ such that, $p \leq_P q$ and, $\mathbf{h}, q \models t_{i-1} < x_i$ or $\mathbf{h}, q \models t_{i-1} < y_i$.*

Proof. For the first part, suppose $\mathbf{h}, p \models (t_{i-1} \to z_i) \to ((x_i \to z_i) \vee (y_i \to z_i)) < \top$. Then there exists $p' \geq_P p$ such that

$$\mathbf{h}, p' \models x_i \to z_i, y_i \to z_i < t_{i-1} \to z_i. \tag{26}$$

Then,

$$\mathbf{h}, q \models t_{i-1} \leq z_i \tag{27}$$

for all $q >_P p'$ (because $t_{i-1} \to z_i$ is evaluated pointwise at p' under $\mathbf{h}$), and there exist $q', q'' \geq_P p'$ such that

$$\mathbf{h}, q' \models z_i < x_i \text{ and } \mathbf{h}, q'' \models z_i < y_i; \tag{28}$$

indeed, if $\mathbf{h}, q' \models z_i \geq x_i$ for all $q' \geq_P p'$, then $x_i \to z_i$ is evaluated pointwise at p' under $\mathbf{h}$ and the pointwise evaluation gives $\mathbf{h}, p' \models x_i \to z_i = \top$, which is impossible by (26); similarly, there exists $q'' \geq_P p'$ such that $\mathbf{h}, q'' \models z_i < y_i$.

If $\mathbf{h}, p' \models t_{i-1} \leq z_i$, then we have $\mathbf{h}, q' \models t_{i-1} \leq z_i < x_i$ (by (27) if $q' >_P p'$, and by (28) if $q' = p'$) and $\mathbf{h}, q'' \models t_{i-1} \leq z_i < y_i$ (similarly). Otherwise, suppose

$$\mathbf{h}, p' \models z_i < t_{i-1}. \tag{29}$$

We claim that $x_i \to z_i$, $y_i \to z_i$, $t_{i-1} \to z_i$ are evaluated pointwise at p' under $\mathbf{h}$, so that (26) together with the definition of $\to_{l_P(p')}$ (such that $n \to_{l_P(p')} m < n' \to_{l_P(p')} m$ if and only if $n' < n$), imply that $\mathbf{h}, p' \models t_{i-1} < x_i$, and $\mathbf{h}, p' \models t_{i-1} < y_j$.

If p' is maximal in $\mathbf{P}$, then all operations evaluate pointwise at p'. Otherwise, if p' is not maximal in $\mathbf{P}$, let $q \in P$ be such that $p' <_P q$. We prove that $p' =_P q' =_P q''$. Assume for a contradiction that $p' <_P q'$. Then $\mathbf{h}, q' \models \top = t_{i-1} \leq z_i < x_i$, where the first equality follows by $\mathbf{h}, p' \models z_i < t_{i-1}$, the second inequality follows by (27), and the third inequality is (28), impossible. Then $p' =_P q'$. Similarly, $p' =_P q''$. Then by (28) and (29),

$$\mathbf{h}, p' \models z_i < t_{i-1}, x_i, y_i,$$

and by (27) and $p' <_P q$,

$$\mathbf{h}, q \models x_i = y_i = \top = t_{i-1} \leq z_i.$$

Since q is an arbitrary point above p', we have that $x_i \to z_i$, $y_i \to z_i$, $t_{i-1} \to z_i$ are all evaluated pointwise at p' under $\mathbf{h}$, which concludes the proof of the first part.

For the second part, suppose $\mathbf{h}, p \models (x_i \vee y_i) \to t_{i-1} < \top$. Then there exists $p \leq_P q$ such that $\mathbf{h}, q \models t_{i-1} < x_i \vee y_i$, so that $\mathbf{h}, q \models t_{i-1} < x_i$ or $\mathbf{h}, q \models t_{i-1} < y_i$, and we are done. $\square$

The previous proposition justifies the main construction, which is reminiscent of the idea, in Statman reduction, that the decision algorithm for intuitionistic tautologies simulates efficiently the model checking of quantified Boolean sentences.

Construction. If $t_\phi = \top$ fails in $\mathcal{CIBGBL}$, then $t_\phi = \top$ fails at the root of a finite $\mathbb{N}$-labelled rooted poset by Theorem 9. Let $\mathbf{P} = (P, \leq_P, l_P)$ be a finite $\mathbb{N}$-labelled poset and $\mathbf{h}$ an assignment in $\mathbf{P}$ such that $\mathbf{h}, r \models t_\phi < \top$, where $r \in P$ is the root of $(P, \leq)$. By induction on the level $i = 0, 1, \ldots, l$, we construct a structure of the form

$$\mathbf{T}_\pi = (T, \leq_\pi, b), \tag{30}$$

such that $T \subseteq P$, π is the quantifier prefix (of the prenex form) of $\neg\phi$, and $b\colon T \setminus \{r\} \to \{0, 1\}$, as follows.

At level $i = 0$, let the root r of $\mathbf{P}$ be the root of $(T, \leq_\pi)$. We have $\mathbf{h}, r \models t_l < \top$.

The construction of level $i + 1$ for $i \geq 0$ works as follows. Let $p \in T$ be any point at level i. Inductively, we have $\mathbf{h}, p \models t_{l-i} < \top$. If $Q_{l-i} = \exists$, by Proposition 16 there exist $q', q'' \in P$ such that, $p \leq_P q', q''$, $\mathbf{h}, q' \models t_{l-i-1} < x_{l-i}$, and $\mathbf{h}, q'' \models t_{l-i-1} < y_{l-i}$. Define q' and q'' to be the two covers of p in $(T, \leq_\pi)$, with labels $b(q') = 1$ and $b(q'') = 0$. [2] If $Q_{l-i} = \forall$, by Proposition 16 there exists $q \in P$ such that $p \leq_P q$ and $\mathbf{h}, p \models t_{l-i-1} < x_{l-i}$, or there exists $q \in P$ such that $p \leq_P q$ and $\mathbf{h}, p \models t_{l-i-1} < y_{l-i}$. In the former case, define q to be the cover of p in $(T, \leq_\pi)$, with label $b(q) = 1$; otherwise, define q to be the cover of p in $(T, \leq_\pi)$, with label $b(q) = 0$.

Example 17. *If t_ϕ is as in Example 11, the construction described above yields the structure $\mathbf{T}_{\forall\exists\forall\exists}$ in Figure 10, if the case $\mathbf{h}, p \models t_2 < x_3$ occurs twice while constructing level 2, and the case $\mathbf{h}, p \models t_0 < x_1$ occurs once while constructing level 4.*

We now prove the key property of the construction above. First note that,

Proposition 18. *For all finite $\mathbb{N}$-labelled rooted poset $\mathbf{P} = (P, \leq_P, l_P)$, all assignments $\mathbf{h}$ in $\mathbf{P}$, and all $p \in P$, $\mathbf{h}, p \models t_0 \leq t_1 \leq \cdots \leq t_l$.*

Proof. Let $i \in \{1, \ldots, l\}$ and $p \in P$. Two cases.

Case $t_i = (x_i \vee y_i) \to t_{i-1}$: By integrality, we have $\mathbf{h}, p \models (x_i \vee y_i) \cdot t_{i-1} \leq \top \cdot t_{i-1} = t_{i-1}$. Then, by residuation, $\mathbf{h}, p \models t_{i-1} \leq (x_i \vee y_i) \to t_{i-1} = t_i$.

Case $t_i = (t_{i-1} \to z_i) \to ((x_i \to z_i) \vee (y_i \to z_i))$: Applying (as above) integrality and residuation, we have $\mathbf{h}, p \models z_i \leq x_i \to z_i$ and $\mathbf{h}, p \models z_i \leq y_i \to z_i$. Then $\mathbf{h}, p \models z_i \leq (x_i \to z_i) \vee (y_i \to z_i)$. Moreover, $\mathbf{h}, p \models (t_{i-1} \to z_i) \cdot t_{i-1} \leq t_{i-1} \wedge z_i \leq z_i$, so $\mathbf{h}, p \models (t_{i-1} \to z_i) \cdot t_{i-1} \leq (x_i \to z_i) \vee (y_i \to z_i)$. Hence $\mathbf{h}, p \models t_{i-1} \leq t_i$ by residuation. □

We conclude the proof of $(iii) \Rightarrow (i)$.

[2] Taking suitable copies if $q' = p$ or $q'' = p$ or $q' = q''$. Namely, if $q' = p$ (respectively $q'' = p$), take a copy of p and rename it q' (respectively q''); and similarly, if $q' = q''$, take a copy of q' (say) and rename it q''.

Proof of Lemma 15. Assume $t_\phi = \top$ fails in $\mathcal{CIBGBL}$. Let $\mathbf{T}_\pi$ be the structure in (30), and let $\mathbf{k} \in \mathbf{T}'_\pi$. Recalling (22), $\mathbf{k}$ corresponds to a branch

$$r <_\pi q_l <_\pi q_{l-1} <_\pi \cdots <_\pi q_1$$

in $(T, \leq_\pi)$, so that $\mathbf{k}(x_i) = b(q_i)$ for $i = 1, \ldots, l$. Let, for $i = 1, \ldots, l$,

$$w_i = \begin{cases} x_i & \text{if } b(q_i) = 1, \\ y_i & \text{otherwise;} \end{cases} \tag{31}$$

$$\mathbf{f}(x_i) = \begin{cases} 1 & \text{if } w_i = x_i, \\ 0 & \text{otherwise;} \end{cases} \tag{32}$$

$$\mathbf{f}(y_i) = \begin{cases} 1 & \text{if } w_i = y_i, \\ 0 & \text{otherwise.} \end{cases} \tag{33}$$

Claim 19. $\mathbf{h}, q_1 \models t_0 < w_1, w_2, \ldots, w_l$.

Proof. Observe that, by construction, $\mathbf{h}, q_i \models t_{i-1} < w_i$ and $q_i \leq_P q_1$ for $i = 1, \ldots, l$. We claim that $\mathbf{h}, q_1 \models t_0 < w_i$ for $i = 1, \ldots, l$. We distinguish two cases. If $q_i =_P q_1$, then $\mathbf{h}, q_1 \models t_0 \leq t_{i-1} < w_i$, by Proposition 18 and the above observation. If $q_i <_P q_1$, then $\mathbf{h}, q_1 \models w_i = \top$ by Definition 3 (since $\mathbf{h}, q_i \models \bot \leq t_{i-1} < w_i$), so that $\mathbf{h}, q_1 \models t_0 < w_1 \leq \top = w_i$. Therefore $\mathbf{h}, q_1 \models t_0 < w_1, w_2, \ldots, w_l$. □

Claim 20. $\mathbf{2}, \mathbf{f} \models t_0 < \top$.

Proof. Suppose for a contradiction that $\mathbf{2}, \mathbf{f} \models t_0 = \top$. Then there is a disjunct θ_j in t_0 such that $\mathbf{2}, \mathbf{f} \models \theta_j = \top$. Then, if $\theta_j = w_{j1} \wedge \cdots \wedge w_{jm_j}$, each w_{jk} is such that $\mathbf{2}, \mathbf{f} \models w_{jk} = \top$. Thus $\mathbf{f}(w_{jk}) = 1$, which by construction and Claim 19 implies that $\mathbf{h}, q_1 \models t_0 < w_{jk}$. Therefore,

$$\mathbf{h}, q_1 \models w_{j1} \wedge \cdots \wedge w_{jm_j} = \theta_j \leq t_0 < w_{j1} \wedge \cdots \wedge w_{jm_j},$$

a contradiction. □

Claim 21. $\mathbf{2}, \mathbf{k} \models \psi < \top$.

Proof. Note that for $i = 1, \ldots, l$, $\mathbf{k}(x_i) = 1$ if and only if, $\mathbf{f}(x_i) = 1$ and $\mathbf{f}(y_i) = 0$, and $\mathbf{k}(x_i) = 0$ if and only if, $\mathbf{f}(x_i) = 0$ and $\mathbf{f}(y_i) = 1$. Then, by (23), $\mathbf{2}, \mathbf{k} \models \psi < \top$ if and only if $\mathbf{2}, \mathbf{f} \models t_0 < \top$, so that the claim follows by Claim 20. □

By Claim 21, $\mathbf{T}_\pi$ is a countermodel of ϕ, and we are done. □

3.3 *(ii)* ⇒ *(iii)*

Lemma 22. $\mathcal{H} \models t_\phi = \top$ *implies* $\mathbf{2} \models \phi$.

Proof. We prove the contrapositive. Let $(T, \leq_\pi, b)$ be a countermodel to ϕ, where π is the quantifier prefix (of the prenex form) of $\neg\phi$. We prove that $t_\phi = \top$ fails at the root $r \in T$ of the finite $\{1\}$-labelled poset $\mathbf{P} = (T, \leq_\pi, 1)$. We define an assignment $\mathbf{h}$ of the variables in $\mathbf{P}$ as follows. For all $p \in T$ and all $i = 1, \ldots, l$:

1. $\mathbf{h}, p \models x_i = \top$ if and only if, there exists $q \in T$ at level $l-i+1$ such that $q \leq_\pi p$ and $b(q) = 1$;
2. $\mathbf{h}, p \models y_i = \top$ if and only if, there exists $q \in T$ at level $l-i+1$ such that $q \leq_\pi p$ and $b(q) = 0$;
3. $\mathbf{h}, p \models z_i = t_{i-1}$.

Claim 23. $\mathbf{h}, r \models t_\phi < \top$.

Proof. By induction on $j = 0, 1, 2, \ldots, l$, we show that $\mathbf{h}, p \models t_j < \top$ for all $p \in T$ at level $l - j$. The case $j = l$ is the desired claim.

Case $j = 0$**.** We want to show that $\mathbf{h}, p \models t_0 < \top$ for all leaves in $\mathbf{P}$ (that is, for all $p \in T$ at level l). Let $p \in T$ be a leaf, and let $r <_\pi p_l <_\pi p_{l-1} <_\pi \cdots <_\pi p_1 = p$ be the branch in $\mathbf{P}$ from p to the root $r \in P$. As $(T, \leq_\pi, b)$ is a countermodel to ϕ, we have $\mathbf{2}, \{x_i \mapsto b(p_i) \mid i = 1, \ldots, l\} \models \psi < \top$. Then by (23) and the definition of $\mathbf{h}$ we have $\mathbf{h}, p \models t_0 < \top$.

Case $j > 0$**.** We want to show that $\mathbf{h}, p \models t_j < \top$ for all $p \in T$ at level $l - j$.

If $Q_j = \forall$, then $t_j = (x_j \vee y_j) \to t_{j-1}$. Let $p \in T$ at level $l - j$. Then, as $(T, \leq_\pi, b)$ is a countermodel to ϕ, there exists $q \in T$ at level $l - j + 1$ such that $p <_\pi q$ and $b(q) \in \{0, 1\}$. By the induction hypothesis, $\mathbf{h}, q \models t_{j-1} < \top$. Moreover, by the definition of $\mathbf{h}$, if $b(q) = 0$ then $\mathbf{h}, q \models y_j = \top$, and if $b(q) = 1$ then $\mathbf{h}, q \models x_j = \top$. In the former case, $\mathbf{h}, q \models t_{j-1} < y_j$, and in the latter case, $\mathbf{h}, q \models t_{j-1} < x_j$. Thus $\mathbf{h}, q \models t_{j-1} < x_j \vee y_j$. Therefore $\mathbf{h}, p \models t_j < \top$ (not pointwise).

If $Q_j = \exists$, then $t_j = (t_{j-1} \to z_j) \to (x_j \to z_j \vee y_j \to z_j)$. By the definition of $\mathbf{h}$, for all $p \in T$,

$$\mathbf{h}, p \models ((t_{j-1} \to z_j) \to ((x_j \to z_j) \vee (y_j \to z_j))) < \top$$

if and only if

$$\mathbf{h}, p \models ((x_j \to t_{j-1}) \vee (y_j \to t_{j-1})) < \top.$$

Let $p \in T$ at level $l - j$. Then, as $(T, \leq_\pi, b)$ is a countermodel to ϕ, there exist $q', q'' \in T$ at level $l - j + 1$ such that $p <_\pi q', q''$, $b(q') = 0$, and $b(q'') = 1$. By the induction hypothesis, $\mathbf{h}, q' \models t_{j-1} < \top$, and $\mathbf{h}, q'' \models t_{j-1} < \top$. Moreover, by the definition of $\mathbf{h}$, $b(q') = 0$ implies $\mathbf{h}, q' \models y_j = \top$, and $b(q'') = 1$ implies $\mathbf{h}, q'' \models x_j = \top$. Thus $\mathbf{h}, q' \models y_j \to t_{j-1} < \top$ and $\mathbf{h}, q'' \models x_j \to t_{j-1} < \top$ if the evaluation is pointwise (otherwise, the inequalities hold trivially). Therefore $\mathbf{h}, p \models y_j \to t_{j-1} < \top$ and $\mathbf{h}, p \models x_j \to t_{j-1} < \top$. Then $\mathbf{h}, p \models t_j < \top$.

The claim is settled. □

By Theorem 7 and Theorem 9, the statement is proved. □

3.4 $(iv) \Rightarrow (ii)$

Lemma 24. $\mathcal{HW} \models t_\phi = \top$ *implies* $\mathcal{H} \models t_\phi = \top$.

Since t_ϕ does not contain occurrences of $\bot$, the above lemma follows directly from the following fact.

Proposition 25. *Let t be a term not containing occurrences of $\bot$. If $t = \top$ holds in $\mathcal{HW}$, then $t = \top$ holds in $\mathcal{H}$.*

Proof. We prove the contrapositive. Assume $t = \top$ fails in $\mathcal{H}$. By Theorem 9 and Theorem 7, there exist a finite labelled poset $\mathbf{P} = (P, \leq_P, 1)$ with root $r \in P$, and an assignment $\mathbf{h}$ in $\mathbf{P}$ such that $\mathbf{h}, r \models t < \top$. Let $\mathbf{Q} = (Q, \leq_Q, 1)$ be the finite labelled poset obtained by adjoining a fresh top m to $\mathbf{P}$. Recalling the map Ξ in Theorem 7, we have the following claim.

Claim 26. $\Xi(\mathbf{Q}) \in \mathcal{HW}$.

Proof. It is sufficient to show that $\mathbf{k}, r \models \neg x \vee \neg\neg x = \top$ for all assignments $\mathbf{k}$ over $\mathbf{Q}$ and all variables $x \in X$. If $\mathbf{k}, r \models \neg x = \top$, we are done. We claim that if $\mathbf{k}, r \models \neg x < \top$, then $\mathbf{k}, r \models \neg\neg x = \top$. Recall that $\neg x = x \to \bot$. Note that $\mathbf{k}, m \models x = \top$, because otherwise $\mathbf{k}, q \models x = \bot$ for all $q \in Q$, and then $\mathbf{k}, r \models \neg x = \top$ with $\neg x$ evaluated pointwise at r under $\mathbf{k}$. Thus, for all $q \in Q \setminus \{m\}$, $\neg x$ does not evaluate pointwise at q under $\mathbf{k}$, that is, $\mathbf{k}, q \models \neg x = \bot$. Also, clearly, $\mathbf{k}, m \models \neg x = \bot$. Thus $\neg\neg x$ evaluates pointwise at r under $\mathbf{k}$. Since $\mathbf{k}, r \models \neg x = \bot$, we have $\mathbf{k}, r \models \neg\neg x = \top$. $\square$

Let $\mathbf{k}$ be the assignment in $\mathbf{Q}$ such that for all $x \in X$, $\mathbf{k}(m, x) = 1$, and $\mathbf{k}(q, x) = \mathbf{h}(q, x)$ for all $q \in Q \setminus \{m\}$.

Claim 27. $\mathbf{k}, r \models t < \top$.

Proof. By induction on t, we show that $\mathbf{k}(q, t) = \mathbf{h}(q, t)$ for all $q \in Q \setminus \{m\}$, which implies the statement. The only nontrivial case is the inductive case $t = t' \to t''$. Since t' and t'' do not contain $\bot$, we have $\mathbf{k}, m \models t' = t'' = \top$. By the induction hypothesis, $\mathbf{k}(q, t') = \mathbf{h}(q, t')$ and $\mathbf{k}(q, t'') = \mathbf{h}(q, t'')$ for all $q \in Q \setminus \{m\}$. Then, for all $q \in Q \setminus \{m\}$, t evaluates pointwise at q in $\mathbf{Q}$ under $\mathbf{k}$ if and only if t evaluates pointwise at q in $\mathbf{P}$ under $\mathbf{h}$. Therefore $\mathbf{k}(q, t) = \mathbf{h}(q, t)$ for all $q \in Q \setminus \{m\}$. $\square$

By Claim 27, $t = \top$ fails in $\mathbf{Q}$, so that by Theorem 9, $t = \top$ fails in $\Xi(\mathbf{Q})$, which is in $\mathcal{HW}$ by Claim 26. $\square$

The following example shows that negations break Statman reduction in the weak excluded middle case.

Example 28. *Let $\phi = \exists x(x \wedge \neg x = \top)$ be a quantified Boolean sentence. In this case, (Švejdar version of) the original Statman reduction [S03] yields*

$$t_\phi = ((x \wedge \neg x) \to z) \to ((x \to z) \vee (\neg x \to z)).$$

Clearly ϕ is false in **2** *(and $t_\phi = \top$ fails in $\mathcal{H}$), but a straightforward computation shows that $t_\phi = \top$ holds in $\mathcal{HW}$.*

3.5 Disjunction Property

For a variety $\mathcal{V}$ of integral commutative residuated lattices, the DP is the property that for all terms t and t', if $t \vee t' = \top$ holds in $\mathcal{V}$, then $t = \top$ holds in $\mathcal{V}$ or $t' = \top$ holds in $\mathcal{V}$.

It is straightforward to check that CIBGBL-algebras have the DP. Assume that $t = \top$ and $t' = \top$ fail in $\mathcal{CIBGBL}$. By Theorem 9, there exist finite $\mathbb{N}$-labelled posets $\mathbf{P}$ and $\mathbf{P}'$, and assignments $\mathbf{h}$ in $\mathbf{P}$ and $\mathbf{h}'$ in $\mathbf{P}'$, such that $\mathbf{h}, r \models t < \top$ and $\mathbf{h}', r' \models t' < \top$, where $r \in P$ and $r' \in P'$ are the roots of $\mathbf{P}$ and $\mathbf{P}'$ respectively. Let the finite $\mathbb{N}$-labelled poset $\mathbf{Q}$ be obtained by adjoining a fresh root $s \in Q$ to the disjoint union of $\mathbf{P}$ and $\mathbf{P}'$, and let $\mathbf{k}$ be any assignment in $\mathbf{Q}$ such that for all $x \in X$:

1. $\mathbf{k}(x, p) = \mathbf{h}(x, p)$ for all $p \in P$;
2. $\mathbf{k}(x, p') = \mathbf{h}'(x, p')$ for all $p' \in P'$.

Then by construction, $\mathbf{k}(r, t) = \mathbf{h}(r, t)$ and $\mathbf{k}(r', t') = \mathbf{h}'(r', t')$. Then $\mathbf{k}, r \models t < \top$ and $\mathbf{k}, r' \models t' < \top$. Then by Definition 3, $\mathbf{k}, s \models t \vee t' < \top$. Thus $t \vee t' = \top$ fails in $\mathcal{CIBGBL}$ by Theorem 9, and we are done. If the labelling of $\mathbf{P}$ and $\mathbf{P}'$ above is the constant 1, then it is possible to construct $\mathbf{Q}$ such that its labelling is the constant 1. Then by Theorem 7, Heyting algebras have the DP, or in other words, idempotency maintains the DP.

In fact the main construction in Section 3.2, which essentially shows the correctness of Statman reduction, is based on the above step (or property). The idea is that, if $Q_{l-i} = \exists$, then any point $p \in T$ at level i has exactly two covers q' and q'' in $(T, \leq_\pi)$ at level $i+1$, such that $p \leq_P q', q''$, $\mathbf{h}, q' \models x_{l-i} \to t_{l-i-1} < \top$ and $\mathbf{h}, q'' \models y_{l-i} \to t_{l-i-1} < \top$, and then $\mathbf{h}, p \models (x_{l-i} \to t_{l-i-1}) \vee (y_{l-i} \to t_{l-i-1}) < \top$ by the DP. Therefore, Statman reduction does not require the full disjunction property, but only the aforementioned special case.

A concrete example of this fact is provided by the weak excluded middle equation. The DP fails upon adding the weak excluded middle equation to the $\mathcal{CIBGBL}$ axiomatization (for instance, both $\neg x = \top$ and $\neg\neg x = \top$ fail in $\mathbf{2}$). However, the DP is maintained on a syntactic fragment that is large enough to implement Statman reduction. Along the lines of Proposition 25, it is possible to show that adding the weak excluded middle equation generates subvarieties of CIBGBL-algebras and Heyting algebras which are conservative with respect to terms not containing $\perp$. Thus, CIBGBL-algebras with weak excluded middle and HW-algebras have the DP with respect to such terms, which suffices to show PSPACE-hardness via Statman reduction.

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