Finite RDP-Algebras: Duality, Coproducts, and Logic

Simone Bova
Department of Mathematics
Vanderbilt University (Nashville, TN, USA)
simone.bova@vanderbilt.edu

Diego Valota
Dipartimento di Scienze dell'Informazione
Università degli Studi di Milano (Milan, Italy)
valota@dsi.unimi.it

Abstract

The variety of RDP-algebras forms the algebraic semantics of RDP-logic, the many-valued propositional logic of the revised drastic product left-continuous triangular norm and its residual. We prove a Priestley duality for finite RDP-algebras, and obtain an explicit description of coproducts of finite RDP-algebras. In this light, we give a combinatorial representation of free finitely generated RDP-algebras, which we exploit to construct normal forms, strongest deductive interpolants, and most general unifiers. We prove that RDP-unification is unitary, and that the tautology problem for RDP-logic is coNP-complete.

1 Introduction

The variety of RDP-algebras forms the algebraic semantics of the *RDP-logic*, a propositional many-valued logic that naturally arises as a boundary case in the setting of *triangular norms logics*.

A *triangular norm* T is a binary, associative and commutative $[0,1]$-valued operation on the unit square $[0,1]^2$ that is monotone, has 1 as identity, and has 0 as annihilator ($y \leq z$ implies $T(x,y) \leq T(x,z)$, $T(x,1) = x$, and $T(x,0) = 0$). Under these conditions, the *drastic product* triangular norm, $D(x,y)$, [1] and the *minimum* triangular norm, $\min\{x,y\}$, are the *strongest* and *weakest* triangular norms in that every triangular norm T satisfies the inequality

$$D(x,y) \leq T(x,y) \leq \min\{x,y\},$$

for every $x, y \in [0,1]$. In the theory of fuzzy sets, triangular norms and their duals, triangular conorms, model respectively intersections and unions of fuzzy

[1] The drastic product triangular norm, $D(x,y)$, is introduced in [20], and defined by $D(x,y) = 0$ for every $x, y \in [0,1)$ and $D(x,y) = \min\{x,y\}$ otherwise.

sets and hence provide natural interpretations for conjunctions and disjunctions of propositions whose truth values range over the unit interval. If a triangular norm T is left-continuous, then the operation $R = \max\{z \mid T(x,z) \leq y\}$, called the *residual* of T, is the unique binary $[0,1]$-valued operation on the unit square that satisfies the residuation equivalence,

$$T(x,y) \leq z \text{ if and only if } x \leq R(y,z),$$

and hence, arguably acts as the logical implication induced by the interpretation of T as a logical conjunction (for instance, it implies right-distributivity of R over T). The variety of MTL-algebras forms the algebraic counterpart of the *MTL-logic*, the logic of all left-continuous triangular norms and their residuals [12, 16], and the RDP-logic lies in the hierarchy of its schematic extensions. For an axiomatization of MTL-logic and RDP-logic, we refer the reader to [12] and [23] respectively. [2]

Historically, however, the RDP-logic has been introduced semantically, by Jenei. In [15], the author applies a generalization of the ordinal sum theorem of semigroups to the construction of new families of left-continuous triangular norms as ordinal sums of triangular subnorms. As a remarkable example of this machinery, the *revised drastic product* left-continuous triangular norm arises by displaying the left-discontinuous drastic product triangular norm, identified above as the strongest triangular norm, as an ordinal sum of the trivial triangular subnorm and the minimum triangular norm. In these terms, RDP-logic is a natural boundary case among the family of triangular norm based logics.

In the present paper, we extensively study RDP-logic, the logic of the revised drastic product and its residual (sketched in Section 1.1, Figure 1) from the point of view of algebraic and categorical logic. As the lattice reduct of a (finite) MTL-algebra is a (finite) bounded distributive lattice, it is natural to study the dual space of such algebras building upon the Priestley (or Birkhoff, emphasizing finiteness) duality between finite bounded distributive lattices and bounded lattice homomorphisms, and finite posets and monotone maps [9, and references therein]. In [11], Esakia establishes a duality for Heyting algebras and their homomorphisms. In the finite case, the dual category consists of finite posets and monotone maps sending downsets to downsets (which we call *open* maps here, despite the original terminology); such maps dualize exactly those lattice homomorphisms that preserve the residual of the lattice meet, namely, intuitionistic implication. Diverting the intuitionistic paradigm, the role of many-valued implication over MTL-algebras is played by the residual of the monoidal operation T discussed above, which is added to the lattice (in the general setting, this monoidal operation is usually called *fusion*). Therefore, to dualize subvarieties of MTL-algebras, plain posets and open maps are not sufficient, even when one restricts attention to finite objects only. Suitable additional structure does become necessary. This line of research has been pursued in [1], where an enriched Priestley duality for the finite objects in a pertinent locally finite subvariety of MTL-algebras has been presented. [3] In the same vein, we develop in this paper a Priestley duality for finite RDP-algebras,

[2]Insisting on the continuity of T, the hierarchy of many-valued logics extending Hájek's Basic logic arises [14].

[3]It is worth mentioning that in recent work, Cabrer and Celani, building on [5, 21], give spectral dualities for several algebraic varieties of bounded distributive lattices with additional (logical) operators, including non locally finite varieties and in particular, MTL-algebras [4].

and prove a categorical equivalence between finite RDP-algebras and a suitably defined combinatorial category. Finite RDP-algebras display a rich spectral theory, based on Gödel algebras [8].

The results presented, together with previous related results in the hierarchy of locally finite subvariety of MTL-algebras, notably NM-algebras and NMG-algebras [1, 2], encourage an investigation of the variety of WNM-algebras in the same spirit. Indeed, WNM-algebras form the algebraic semantics of a many-valued propositional logic, the logic of the *weak nilpotent minimum* triangular norm and its residual [12]. A reason of interest towards this logic is that in recent work [6], Ciabattoni et al. present a uniform method for generating analytic logical calculi from given axiom schemata, and the WNM-logic represents a hard case (in a sense that can be made precise) where the method succeeds.

The paper is organized as follows. In Section 1.1, we collect from the literature some background theory on RDP-algebras, and start investigating the structure of finite RDP-algebras. In Section 2.1, we give a Priestley duality for finite RDP-algebras: we define a combinatorial category, the category HF of finite *hall forests* and their morphisms, and we prove that it is dually equivalent to the category FRDP of finite RDP-algebras. As a benchmark of the manageability and usefulness of the presented duality, in Section 2.2 we give algorithmic constructions for finite products in HF and we obtain explicit descriptions of coproducts of finite RDP-algebras. We thus attain an amenable combinatorial representation of free finitely generated RDP-algebras (Section 2.3). In Section 3, we exploit such representation to provide explicit constructions of a number of objects relevant from the point of view of the logical interpretation RDP-algebras: normal forms (Section 3.1), strongest deductive interpolants (Section 3.2), and most general unifiers (Section 3.3). We prove that RDP-unification is unitary, establishing the first result in unification theory above WNM-logic, and broadening the scope of previous work of Dzik on Hájek's Basic logic [10]. We prove that the tautology problem for RDP-logic is coNP-complete.

1.1 Background

In this section, we introduce some background theory on RDP-algebras. If A is an algebra, [4] and t is an algebraic term on the signature of A over the variables $x_1, \ldots, x_n$, we let t^A denote the n-ary term operation in A defined by t.

A *commutative integral bounded residuated lattice* is an algebra

$$A = (A, \wedge, \vee, \odot, \rightarrow, \bot, \top)$$

of type $(2, 2, 2, 2, 0, 0)$ such that $(A, \wedge, \vee, \bot, \top)$ is a bounded lattice, with top $\top$ and bottom $\bot$, $(A, \odot, \top)$ is a commutative monoid, and the *residuation* equivalence, $x \odot y \leq z$ if and only if $x \leq y \rightarrow z$, holds. Commutative integral bounded

Their very general technique, motivated by the topological characterization of congruences in these varieties, relies upon the systematic translation of the equations defining the target algebraic class into (possibly first-order) relational conditions over the dual Priestley space. We believe that similar dualities can be attained for diverse locally finite subvarieties of MTL-algebras, including several subvarieties of WNM-algebras. In the spirit of the present work, it would be interesting to understand whether such general methods support explicit descriptions of algebraic coproducts and free algebras on the primal side; this would potentially enlighten widely open problems such as, for instance, a satisfactory representation of free finitely generated MTL-algebras.

[4] We disregard trivial algebras.

residuated lattice form an algebraic variety [13]. If the lattice order is total, A is called a *chain*. An *MTL-algebra* is a commutative integral bounded residuated lattice satisfying the *prelinearity* equation, $(x \to y) \vee (y \to x) = \top$. A *Gödel* algebra is an *idempotent* MTL-algebra, that is, an MTL-algebra satisfying $x \odot x = x$. The unary term operation $\neg x$ is defined by $x \to \bot$. A *WNM-algebra* is an MTL-algebra satisfying the *weak nilpotent minimum* equation,

$$\neg(x \odot y) \vee ((x \wedge y) \to (x \odot y)) = \top, \tag{1}$$

and an RDP-algebra is a WNM-algebra satisfying the *revised drastic product* equation,

$$\neg\neg x \vee (x \to \neg x) = \top. \tag{2}$$

Notice that Gödel algebras are idempotent RDP-algebras.

In every RDP-algebra, the operations $\wedge$ and $\vee$, and the constant $\top$ are definable as term operations over $\odot$, $\to$, $\bot$ [23, Proposition 3.2]. In the sequel, for notation compactness, we freely write $x \leftrightarrow y$ instead of $(x \to y) \odot (y \to x)$, x^n instead of $x \odot \cdots \odot x$ (n times), and $\bar{x}$ instead of $\neg x$.

By [23, Theorem 3.7 and Theorem 3.8], the variety of RDP-algebras is singly generated by the algebra

$$[0,1] = ([0,1], \wedge^{[0,1]}, \vee^{[0,1]}, \odot^{[0,1]}, \to^{[0,1]}, \bot^{[0,1]}, \top^{[0,1]}), \tag{3}$$

where, for every $x, y \in [0,1]$, we let $x \wedge^{[0,1]} y = \min\{x, y\}$, $x \vee^{[0,1]} y = \max\{x, y\}$, $\bot^{[0,1]} = 0$, $\top^{[0,1]} = 1$, and for some arbitrary but fixed $0 < a < 1$,

$$x \odot^{[0,1]} y = \begin{cases} 0 & x, y \leq a, \\ \min\{x, y\} & \text{otherwise,} \end{cases} \tag{4}$$

$$x \to^{[0,1]} y = \begin{cases} 1 & x \leq y, \\ a & y < x \leq a, \\ y & \text{otherwise.} \end{cases} \tag{5}$$

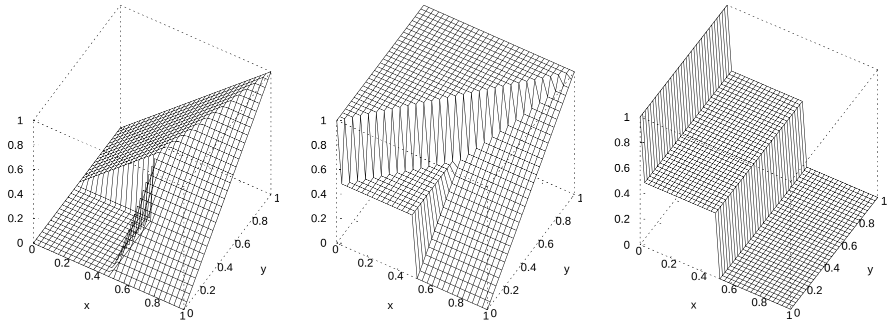
By direct computation, for every $x \in [0,1]$,

$$\neg^{[0,1]} x = \begin{cases} 1 & x = 0, \\ a & 0 < x \leq a, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Note that for all $x, y \in [0,1]$, if $x \leq y$, then $\neg^{[0,1]} y \leq \neg^{[0,1]} x$, that is, the operation $\neg^{[0,1]}$ is antitone. Also note that the operation $\to^{[0,1]}$ is the unique binary operation over the real interval $[0,1]$ satisfying the residuation equivalence with respect to $\odot^{[0,1]}$.

By universal algebraic facts [3], the free n-generated RDP-algebra, F_n, is the clone of n-ary term operations of the algebra $[0,1]$ in (3), equipped with operations defined pointwise by the basic operations of $[0,1]$. [5] The algebra F_n is the Lindenbaum-Tarski algebra of RDP-logic, the many-valued propositional

[5] The *clone* of n-ary term operations over $[0,1]$ is the smallest set of n-ary operations over $[0,1]$ containing the n-ary projections $x_1, \ldots, x_n$, and closed under arbitrary compositions with the basic operations of the generic algebra.



$x \odot^{[0,1]} y$. $\quad$ $x \to^{[0,1]} y$. $\quad$ $\neg^{[0,1]} x = x \to^{[0,1]} \bot^{[0,1]}$.

Figure 1: The revised drastic product left-continuous triangular norm and its residual, with $a = 1/2$ in (4)-(6).

logic discussed in the introduction. So, an RDP-term t is a *tautology* of RDP-logic, that is, $t^{[0,1]}(a_1, \ldots, a_n) = 1$ for every $(a_1, \ldots, a_n) \in [0,1]^n$, if and only if $t^{[0,1]} = \top^{[0,1]}$.

Notice that F_n is finite, because the variety of RDP-algebras is locally finite. Indeed, the subdirectly irreducible members of subvarieties of MTL-algebras are chains [12], and WNM-chains are locally finite, thus the variety of WNM-algebras is locally finite [19]; it follows that the variety of RDP-algebras is locally finite. Therefore, finitely generated RDP-algebras and finite RDP-algebras coincide. To see this directly, observe that RDP-chains are locally finite: Indeed, let $C = (C, \wedge, \vee, \odot, \to, \bot, \top)$ be a RDP-chain generated by $x_1, \ldots, x_n$. Then, since C is (isomorphic to) a subalgebra of $[0,1]$, for all $x, y \in C$, by equations (4), (5) and (6),

$$x \odot y = \begin{cases} \bot & x, y \leq \neg x, \neg y, \\ \min\{x, y\} & \text{otherwise}, \end{cases} \tag{7}$$

$$x \to y = \begin{cases} \top & x \leq y, \\ \neg x & y < x \leq \neg x, \\ y & \text{otherwise}. \end{cases} \tag{8}$$

Let t be a RDP-term over variables $x_1, \ldots, x_n$. By induction on t, and direct inspection of equations (7) and (8),

$$t^C \in \{\bot^C, \top^C, x_i^C, \neg x_i^C \mid i \in [n]\};^6$$

hence, $|C| \leq 2(n+1)$.

We now establish some useful facts on finite RDP-algebras. Let A be a finite RDP-algebra. By the subdirect representation theorem [3, Theorem 8.6], and the fact that subdirectly irreducible RDP-algebras are chains [12], A is a subdirect product of an indexed family $(C_i)_{i \in I}$ of RDP-chains. For every $y \in A$, we let y_i denote the projection of y over index $i \in I$.

We say that A has *fixpoint* if there exists $y \in A$ such that $y = \neg y$.

[6] As a notation, for $n \geq 1$, we let $[n] = \{1, \ldots, n\}$.

Proposition 1. *If A is an RDP-algebra, then A has at most one fixpoint.*

Proof. Each RDP-chain C has at most one fixpoint, since if x and y are fixpoints of C, say without loss of generality $x \leq y$, then $y = \neg y \leq \neg x = x$ by antitonicity, and $x = y$. Let A be an RDP-algebra, displayed as the subdirect product of the indexed family $(C_i)_{i \in I}$ of RDP-chains. Now, if x is a fixpoint of A, the ith projection x_i of x is the unique fixpoint of C_i (for all $i \in I$), and then, x is unique. $\square$

We now record key properties of finite directly indecomposable RDP-algebras (that is, RDP-algebras not representable as the direct product of two nontrivial RDP-algebras), with and without a fixpoint: we show that a finite directly indecomposable RDP-algebra is either a Gödel algebra, or its nonidempotent elements form a chain below the fixpoint.

Proposition 2. *Let A be a finite directly indecomposable RDP-algebra. If x is the fixpoint of A, then $\{y \in A \mid \bot < y \leq x\} = \{y \in A \mid y^2 < y\}$ is a chain. If A has no fixpoint, then $\{y \in A \mid y^2 < y\}$ is empty.*

Proof. Let A be the subdirect product of the indexed family $(C_i)_{i \in I}$ of RDP-chains.

For the first part, suppose for a contradiction that the downset of x is not a chain. Let $y, z \leq x$ be incomparable in the downset of x. Let J and K be subsets of I such that $y_j \leq z_j$ for all $j \in J$, and $z_k < y_k$ for all $k \in K$. Let A' and A'' be the nontrivial RDP-algebras generated by $\{(a_j)_{j \in J} \mid a \in A\}$ and $\{(a_k)_{k \in K} \mid a \in A\}$ respectively, with coordinatewise defined operations (for nontriviality, notice that there exist $j \in J$ such that $y_j < z_j$ and $k \in K$ such that $z_k < y_k$). We show that A is the direct product of A' and A''. A straightforward computation on the subdirect representation of A, using (4) and (5), shows that the element

$$a = (y \to z) \to \neg(y \to z)$$

of A is such that $a_j = \bot_j$ for all $j \in J$ and $a_k = \top_k$ for all $k \in K$; thus, $\neg a$ is such that $\neg a_j = \top_j$ for all $j \in J$ and $\neg a_k = \bot_k$ for all $k \in K$. Let $a' \in A'$ and $a'' \in A''$ be any two elements, and let $b' \in A$ and $b'' \in A$ be such that $b'_j = a'_j$ for all $j \in J$ and $b''_k = a''_k$ for all $k \in K$. Notice that b' and b'' exist in A by construction. By direct computation,

$$b = (\neg a \wedge b') \vee (a \wedge b'')$$

is an element of A such that $b_j = b'_j = a'_j$ for all $j \in J$ and $b_k = b''_k = a''_k$ for all $k \in K$. The equality $\{y \in A \mid \bot < y \leq x\} = \{y \in A \mid y^2 < y\}$ is now easy to check on the subdirect representation of A: Every $\bot \neq y \in A$ below x is nonidempotent, and every $y \in A$ strictly above x is idempotent.

For the second part, we show a preliminary fact. Let C be an RDP-chain. We claim that if C has no fixpoint, then C is idempotent. Let $w \in C$, so that $w \neq \neg w$. As C is (isomorphic to) a subalgebra of $[0, 1]$, by (4), if $\neg w < w$, then $w^2 = w$; and if $w < \neg w$, then $w = \bot$ (in fact, $\bot < w < \neg w$ implies $\neg\neg w = \neg w$ by (5), contradiction as C has no fixpoint), so $w^2 = w$.

We now show that if A is not idempotent, then A has a fixpoint. Let $J = \{i \in I \mid C_i \text{ has a fixpoint}\}$ and $K = \{i \in I \mid C_i \text{ has no fixpoint}\}$. Let $y \in A$

be such that $y^2 < y$, and let $i \in I$ such that $y_i^2 < y_i$. Then C_i is nonidempotent, and by the preliminary fact, C_i has a fixpoint; hence $J \neq \emptyset$.

Suppose $J = I$ (or, $K = \emptyset$). We claim that A has a fixpoint. Indeed, for all $j \in J \neq \emptyset$, let $z_j \in A$ be such that the jth projection $(z_j)_j$ of z_j is the fixpoint of C_j (such z_j's exist by subdirect representation). Then,

$$f = \bigvee_{j \in J} \neg z_j$$

is the fixpoint of A: For, notice that for all $j \in J$, $(\neg z_j)_j$ is equal to the fixpoint of C_j, and for all $j' \neq j \in J$, $(\neg z_j)_{j'}$ is less than or equal to the fixpoint of $C_{j'}$, so that, for all $j \in J$, f_j is equal to the fixpoint of C_j.

Otherwise, suppose that $J \subset I$ (or, $K \neq \emptyset$). Let A' and A'' be the RDP-algebras generated by $\{(a_j)_{j \in J} \mid a \in A\}$ and $\{(a_k)_{k \in K} \mid a \in A\}$ respectively, with coordinatewise defined operations. Note that $J \neq \emptyset$ implies that A' is nontrivial. Also, $|A''| \geq 1$. If $|A''| > 1$, we claim that A is the direct product of nontrivial RDP-algebras A' and A''. As above, for all $j \in J \neq \emptyset$, let $z_j \in A$ be such that the jth projection $(z_j)_j$ of z_j is the fixpoint of C_j (such z_j's exist by subdirect representation). Using (5) and (6), a direct computation on the subdirect representation of A shows that the element

$$a = \bigvee_{j \in J} (z_j \leftrightarrow \neg z_j)$$

of A is such that $a_j = \top_j$ for all $j \in J$ and $a_k = \bot_k$ for all $k \in K$; thus, $\neg a$ is such that $\neg a_j = \bot_j$ for all $j \in J$ and $\neg a_k = \top_k$ for all $k \in K$. Let $a' \in A'$ and $a'' \in A''$ be any two elements, and let $b' \in A$ and $b'' \in A$ be such that $b'_j = a'_j$ for all $j \in J$ and $b''_k = a''_k$ for all $k \in K$. Notice that b' and b'' exist in A by construction. By direct computation,

$$b = (a \wedge b') \vee (\neg a \wedge b'')$$

is an element of A such that $b_j = b'_j = a'_j$ for all $j \in J$ and $b_k = b''_k = a''_k$ for all $k \in K$. But this is a contradiction with the fact that A is directly indecomposable. Then, $|A''| = 1$, and the element f computed above, is again the fixpoint of A: with respect to $k \in K$, simply notice that $f_k = (\neg f)_k$, because $|A''| = 1$ implies $|C_k| = 1$.

This settles the proposition. $\square$

Let A be a finite directly indecomposable RDP-algebra. By Proposition 2, we introduce the following terminology. The *type* of A, in symbols $\mathrm{type}(A)$, is the nonnegative integer uniquely determined by letting,

$$\mathrm{type}(A) = |\{y \in A \mid y^2 < y\}| = |\{y \in A \mid \bot < y \leq x,\ x \text{ fixpoint of } A\}|; \quad (9)$$

in words, the type of A is the number of nonidempotent elements in the universe of A, or equivalently, the cardinality of the chain below the fixpoint of A (excluding the bottom). In particular, the type of A is equal to 0 if all elements of A are idempotent, or equivalently, if A has no fixpoint.

Proposition 3. *Let A and B be finite directly indecomposable RDP-algebras, and let $h\colon A \to B$ be a homomorphism. Then,* $\mathrm{type}(A) \leq \mathrm{type}(B)$.

Proof. If $\text{type}(A) = 0$, then the statement holds trivially. Otherwise, suppose $\text{type}(A) > 0$. Let y be the fixpoint of A, that is $y = \neg y$. As h is a homomorphism, h is to respect the fixpoint of A, namely, $z = h(y) = h(\neg y) = \neg h(y) = \neg z$. Let z be the fixpoint of B. Also, h is clearly to send each nonidempotent point below the fixpoint of A to a nonidempotent point below the fixpoint of B. Moreover, h is to respect the chain of nonidempotent elements below the fixpoint of A: For otherwise, suppose for a contradiction that $\perp < x < x' < y$ in A but $h(x') = w' \leq w = h(x)$ in B. Then, $\top > z = h(y) = h(x' \to x) = h(x') \to h(x) = w' \to w = \top$, contradiction. Then, the cardinality of the chain below the fixpoint of A is at most equal to the cardinality of the chain below the fixpoint of B, that is, $\text{type}(A) \leq \text{type}(B)$. This concludes the proof. □

2 Spectral Duality

In this section, we prove a Priestley duality between the category of finite RDP-algebras and their homomorphisms, FRDP, and the category HF of finite *hall forests*, whose objects are (pairs of) certain finite posets, and whose morphisms are (pairs of) open maps between them. Recall that, if P and Q are posets, an *open* map is a monotone map from P to Q that sends downsets of P to downsets of Q. [7] The key lemma (Lemma 1) establishes a duality between finite directly indecomposable RDP-algebras and *hall trees*, yielding the following representation: if A is a finite directly indecomposable RDP-algebra, then the hall tree (T, J), dual to A, is such that the ordinal sum $J \oplus T$ of posets J and T is order isomorphic to the prime filters of the lattice reduct of A ordered by reverse inclusion; and conversely, if (T, J) is a hall tree, then the algebra A, dual to (T, J), is order isomorphic to the downsets of the poset $J \oplus T$ ordered by inclusion. [8]

2.1 Categorical Equivalence

Let A be a commutative integral bounded residuated lattice. A *filter* of A is a nonempty *upset* F of A (that is, for all $x, y \in A$, if $x \leq y$ and $x \in F$, then $y \in F$), closed under $\odot$ (that is, for all $x, y \in F$, $x \odot y \in A$). We call $\bigwedge_{x \in F} x$ the *generator* of the filter F. A filter F of A is *prime* if $F \neq A$ and for all $x, y \in A$, either $x \to y$ or $y \to x$ is in F. We call the poset of prime filters of A ordered by reverse inclusion, the *prime spectrum* of A.

The main result of this section exploits the structural resemblance between RDP-algebras and Gödel algebras. Let A be a directly indecomposable RDP-algebra. It is possible to describe the prime spectrum of a A in terms of the prime spectrum of a certain Gödel algebra A_G, specified as follows. First notice that the idempotent elements of A,

$$I(A) = \{x \in A \mid x^2 = x\},$$

form a subuniverse of A (since the idempotent elements in any RDP-chain, $\perp$ or elements x such that $\neg x < x$, are closed under the RDP-operations in (7)

[7] If P is a poset, and $S \subseteq P$, then S is a *downset* of P if for all $x, y \in P$, if $x \leq y$ and $y \in S$ then $x \in S$.

[8] If P and Q are disjoint posets, then their *ordinal sum* $P \oplus Q$ is the poset over $P \cup Q$ such that $x \leq y$ in $P \oplus Q$ if and only if, either $x \in P$ and $y \in Q$, or $x \leq y$ in P, or $x \leq y$ in Q.

and (8), and each RDP-algebra is representable as the subdirect product of a family of RDP-chains), hence the algebra

$$A_G = (I(A), \wedge, \vee, \odot, \to, \bot, \top),$$

is a subalgebra of A and in fact a Gödel algebra. Also, we claim that A_G is directly indecomposable. Indeed, if A has no fixpoint, this is trivial because $I(A) = A$ by Proposition 2. If x is the fixpoint of A, since $I(A) = \{\bot\} \cup \{y \in A \mid x < y\}$ is a subalgebra of A, it follows straightforwardly that $\{y \in A \mid x < y\}$ is the unique maximal nontrivial filter of $I(A)$, then A_G is directly indecomposable.

Let A and B be directly indecomposable RDP-algebras, and let $h\colon A \to B$ be a homomorphism. Then, it is straightforward to verify that the restriction of h to $I(A)$, for short h_G, is a homomorphism from A_G to B_G.

We record the categorical equivalence between the category of finite Gödel algebras and their homomorphisms, FG, and the category of finite forests and open maps, F, presented in [8]. The equivalence is based on the fact that a finite Gödel algebra is directly indecomposable if and only if its prime spectrum is a tree.

Theorem 1. *FG and F are dually equivalent via the contravariant functor Θ, defined as follows: for every object A in FG,*

$$\Theta(A) = (\{F \subseteq A \mid F \textit{ prime filter}\}, \supseteq);$$

for every morphism $h\colon A \to B$ in FG, $\Theta(h)$ is the open map sending each prime filter F in $\Theta(B)$ to the prime filter in $\Theta(A)$ defined as follows:

$$(\Theta(h))(F) = \{a \in A \mid h(a) \in F\}. \tag{10}$$

Proposition 4. *Let A be a finite directly indecomposable RDP-algebra. Then, the prime spectrum of A is order isomorphic to $\Theta(A_G)$.*

Proof. The claim is trivial if A has no fixpoint, because in this case $A = A_G$. Let x be the fixpoint of A. It is sufficient to prove that F is a prime filter of A if and only if F is a prime filter of A_G.

Let F be a prime filter of A, and let $y \in F$. We claim that $y \in I(A)$. Indeed, suppose that y is not in $I(A)$, that is, $\bot < y \leq x$. By Proposition 2 the downset of x in A is a chain; hence, $y \odot y = \bot$ by (7). Thus, $\bot \in F$. But then, $F = A$, and F is not a prime filter, contradiction. Therefore, F is a prime filter of A_G, because the operations of A_G are the operations of A restricted to $I(A)$.

Let F be a prime filter of A_G, and let $z \in I(A)$ be the generator of F. Notice that $\bot < z$, as F is prime. Therefore, F is a prime filter of A, because all elements greater than or equal to z in A are in $I(A)$, and the operations of A, restricted to $I(A)$, behave exactly as the operations of A_G. □

Proposition 5. *Let $h\colon A \to B$ be a homomorphism of finite directly indecomposable RDP-algebras A and B, and let $E(h)$ be the set of homomorphisms h' from A to B such that $h_G = h'_G$. If $1 < \mathrm{type}(A) = n \leq m = \mathrm{type}(B)$, then $|E(h)| = \binom{m}{n}$, otherwise $|E(h)| = 1$.*

Proof. By Proposition 2, $\mathrm{type}(A) \leq \mathrm{type}(B)$. If $\mathrm{type}(A) = 0$, then $h = h_G$ and then, $|E(h)| = 1$. If $\mathrm{type}(A) = 1 < \mathrm{type}(B)$, then the only extension of h_G to a homomorphism from A to B is the unique map that sends the fixpoint of A to the fixpoint of B. Hence, $|E(h)| = 1$.

If $1 \leq \text{type}(A) = n \leq m = \text{type}(B)$, then the extension of h_G to a homomorphism from A to B is not unique (unless $n = m$). Each extension sends the fixpoint of A to the fixpoint of B, each nonidempotent point below the fixpoint of A to a nonidempotent point below the fixpoint of B, and respects the chain of nonidempotent elements below the fixpoint of A. Since the chain of nonidempotent elements below the fixpoint of A has n points, and the chain of nonidempotent elements below the fixpoint of B has $m \geq n$ points, there are exactly $\binom{m}{n}$ mappings that respect the chain of nonidempotent elements below the fixpoint of A. □

In order to achieve a correct definition of the category dual to the category of directly indecomposable finite RDP-algebras, it is necessary to consider two facts. First, there exist nonisomorphic directly indecomposable finite RDP-algebras A and B having order isomorphic prime spectra. For instance, an RDP-chain of three elements with fixpoint and an RDP-chain of two elements (hence, with no fixpoint) have the same prime spectrum but are not RDP-isomorphic. Second, by Proposition 5, there exist distinct homomorphisms h' and h'' of directly indecomposable finite RDP-algebras that have the same behavior upon restriction to idempotent elements, and hence induce the same open map between the corresponding prime spectra. For these reasons, objects in the dual category will be suitable pairs of posets, and morphisms will be suitable pairs of morphisms, acting componentwise, as follows.

Definition 1 (Hall Forest)**.** *A* (finite) hall tree *is a pair* (T, J) *where* T *is a tree and* J *is a chain. A* (finite) hall forest *is a (finite) multiset* $\{(T_1, J_1), \dots, (T_n, J_n)\}$ *of (finite) hall trees.* [9]

For every pair (T, J) *and* (T', J') *of hall trees a* morphism (of hall trees) *is a pair* (f, g) *where* $f: T \to T'$ *and* $g: J \to J'$ *are (partial) open maps, such that* $g(\max(J)) = \max(J')$. [10] *For every pair* F *and* F' *of hall forests, a* morphism (of hall forests) *is a map from the hall trees of* F *to the hall trees of* F'*, acting treewise as a morphism of hall trees.*

For every pair of morphism of hall trees $(f_1, g_1): (T_1, J_1) \to (T_2, J_2)$*, and* $(f_2, g_2): (T_2, J_2) \to (T_3, J_3)$*, the* composition *of* (f_1, g_1) *and* (f_2, g_2) *is the morphism of hall trees*

$$(f, g) = (f_2, g_2) \circ (f_1, g_1): (T_1, J_1) \to (T_3, J_3)$$

such that $f = f_2 \circ f_1$ *and* $g = g_2 \circ g_1$*. The composition of morphisms of hall forests is determined by the treewise composition of the underlying morphism of hall trees.*

Upon noticing that finite posets and open maps form a category, it is easy to check that by Definition 1 compositions of morphism (of hall forests) are associative and preserve identities. Hence, (finite, hall) forests and their morphisms form a category, HF. We now prove the announced categorical equivalence between FRDP and HF.

[9] A *multiset* is a family whose members have multiple instances (a set is a multiset whose members have exactly one instance).

[10] Note that, if $g: J \to J'$ is an open map such that $g(\max(J)) = \max(J')$, then $|J'| \leq |J|$.

First, let HT denote the full subcategory of (finite, hall) trees and their morphisms, and FDRDP denote the category of finite directly indecomposable RDP-algebras and their homomorphisms. In light of Proposition 4, Proposition 5, and Theorem 1, we introduce a contravariant functor, Ξ, from FDRDP to HT, as follows. Let A be a finite directly indecomposable RDP-algebra. Then,

$$\Xi(A) = (\Theta(A_G), A_P),$$

where

$$A_P = (\{\{x \in A \mid y \leq x\} \mid \bot < y \leq z, z \text{ fixpoint of } A\}, \supseteq).$$

In words, A_P is the structure formed by the filters (with respect to the lattice order of A) generated by the nonidempotent elements of A, ordered by reverse inclusion. By Proposition 2, A_P is a chain, and by (9), $|A_P| = \text{type}(A)$. Let $f \colon A \to B$ be a morphism in FDRDP. We let

$$\Xi(f) = (\Theta(f_G), f_P)$$

be the morphism (of hall trees) from $\Xi(B) = (\Theta(B_G), B_P)$ to $\Xi(A) = (\Theta(A_G), A_P)$ such that for every $F \in \Theta(B_G)$,

$$\Theta(f_G)(F) \in \Theta(A_G),$$

and, for every $F \in B_P$,

$$f_P(F) = \{x \in A \mid f(x) \in F\} \in A_P. \tag{11}$$

By Proposition 2, the dual of f satisfies the definition of morphism of (finite, hall) trees.

It is routine to verify that Ξ is a contravariant functor from FDRDP to HT.

Lemma 1. *The category* FDRDP *is dually equivalent to the category* HT *via the contravariant functor* Ξ.

Proof. It is sufficient to show that $\Xi \colon \mathsf{FDRDP} \to \mathsf{LT}$ is full, faithful, and essentially surjective [18, Theorem 4.4.1].

First we prove that Ξ is essentially surjective, that is, for every object (T, J) in HT, there exists an object A in FDRDP such that $\Xi(A)$ is isomorphic to (T, J) in HT. Let (T, J) be in HT. By Theorem 1, let B be a finite directly indecomposable Gödel algebra such that $\Theta(B)$ is isomorphic to T in the category of finite forests F. If $|J| = |\emptyset| = 0$, let A be a finite directly indecomposable RDP-algebra such that $A = A_G = B$. Then, (T, J) is isomorphic in HT to $\Xi(A)$. If $|J| > 0$, let A be the finite directly indecomposable RDP-algebra obtained as follows: Replace the minimum element $\bot$ of B with a chain $\bot < \cdots < x$ of $|J| + 1$ elements (whose maximum and minimum are designed respectively as the bottom and the fixpoint of A); define the operations $\odot$ and $\to$ over A by extending $\odot$ and $\to$ over B to the new $|J| + 1$ elements of A as follows: if $y, y' \leq x$ in A, then $y \odot y' = \bot$, otherwise $y \odot y' = y \wedge y'$; if $y \leq y'$ in A then $y \to y' = \top$, otherwise if $y' < y \leq x$ in A then $y \to y' = x$, otherwise $y \to y' = y'$. By construction, $\Theta(A_G)$ is order isomorphic to T, and A_P is order isomorphic to J, so that (T, J) is isomorphic in HT to $\Xi(A)$.

Now we prove that Ξ is full, that is, for every morphism (f, g) in HT, there exists a morphism h in FDRDP such that $\Xi(h) = (f, g)$. Let $(f, g) \colon (T, J) \to$

(T', J') be a morphism in HT so that $|J'| \leq |J|$. We construct h, as follows. Since Ξ is essentially surjective, there exists objects A and B in FDRDP such that $(T, J) = \Xi(B)$ and $(T', J') = \Xi(A)$, that is, $T = \Theta(B_G)$ and $J = B_P$, and $T' = \Theta(A_G)$ and $J' = A_P$. Note that $\text{type}(A) \leq \text{type}(B)$. By Theorem 1, there exists an homomorphism h_G from A_G to B_G such that $\Theta(h_G)$ is equal to open map f from T to T'. Now, $h \colon A \to B$ is the extension of h_G to nonidempotent elements in A defined in terms of g, as follows. Let x be a nonidempotent element in A, and let $F \in A_P$ be the filter generated by x with respect to the lattice order of A. As $g^{-1}(F) \subseteq B_P$ is a chain, with respect to the order of B_P, let F' be the maximum in $g^{-1}(F)$, and let y be the generator of F' in B. Then, $h(x) = y$. It is routine to check that, by the definitions, h is a homomorphism from A to B.

Finally we prove that Ξ is faithful, that is, for every pair $f \colon A \to B$ and $g \colon A \to B$ of morphisms in FDRDP, if $\Xi(f) = \Xi(g)$, then $f = g$. Suppose that f and g are distinct, say $f(y) \neq g(y)$ for some $y \in A$. We distinguish two cases. If $y \in I(A)$, then the open maps that f_G and g_G induce by (10) are distinct. But then $\Xi(f) = (\Theta(f_G), \cdot) \neq (\Theta(g_G), \cdot) = \Xi(g)$, because by Theorem 1, $\Theta(f_G) \neq \Theta(g_G)$. Otherwise, if $y \notin I(A)$, then y lies in the chain below the fixpoint of A above the bottom (because the homomorphisms f and g are to send the bottom of A to the bottom of B, and the fixpoint of A to the fixpoint of B). Also, the length of the chain below the fixpoint of B is strictly greater than the length of the chain below the fixpoint of A (because the homomorphisms f and g are to respect the chain below the fixpoint of A, but send the point y to distinct points in the chain below the fixpoint of B). But then, the open maps that f and g induce by (11) are distinct. Then, $\Xi(f) = (\cdot, f') \neq (\cdot, g') = \Xi(g)$, because $f' \neq g'$. □

We extend the contravariant functor $\Xi \colon \mathsf{FDRDP} \to \mathsf{HT}$ to the entire category FRDP. For objects, let A be a finite RDP-algebra, and let $(A_i)_{i \in I}$ be its direct decomposition. Then, $\Xi(A)$ is the hall forest given by the disjoint union (accounting for multiplicity) of the hall trees $\Xi(A_i)$, for all $i \in I$. For morphisms, let $f \colon A \to B$ be a homomorphism of finite RDP-algebras. Let A and B be directly decomposed by $(A_i)_{i \in I}$ and $(B_j)_{j \in J}$ respectively, let $\Xi(B)$ and $\Xi(A)$ be the disjoint union (accounting for multiplicity) of $\Xi(B_j)$ for $j \in J$ and $\Xi(A_i)$ for $i \in I$ respectively. Let $j \in J$. If F is a prime lattice filter of B_j, then $G = \{a \in A \mid f(a)_j \in F\}$ is a prime lattice filter of A. By primality, if x is the generator of G, then there exists a unique $i \in I$ such that $\bot_i < x_i$. Moreover, i is independent of the choice of F, that is, if F' is a prime lattice filter of B_j and x' is the generator of $G' = \{a \in A \mid f(a)_j \in F'\}$, then $\bot_i < x'_i$. Let $f_j \colon A_i \to B_j$ be the map defined by $f_j(x) = (f(\bot_1, \ldots, \bot_{i-1}, x, \bot_{i+1}, \ldots, \bot_{|I|}))_j$, for all $x \in A_i$; it is easy to check that f_j is an RDP-homomorphism, and that $f_j(a_i) = f(a)_j$. The morphism of hall forests $\Xi(f) \colon \Xi(B) \to \Xi(A)$ is defined treewise by the action of the morphisms of hall trees $\Xi(f_j)$, for all $j \in J$. Compare Example 3.

Theorem 2. *The category* FRDP *is dually equivalent to the category* HF *via the contravariant functor* Ξ.

Proof. By universal algebraic facts [3, Theorem 7.10], every finite RDP-algebra is isomorphic to the direct product of a finite family of directly indecomposable finite RDP-algebras, and this direct decomposition is unique (modulo isomorphism). The fact that Ξ is full, faithful, and essentially surjective follows by appealing to Lemma 1. $\square$

Aiming at a combinatorial representation of the free n-generated RDP-algebra, we now define explicitly a contravariant functor $\Psi\colon \mathsf{HF} \to \mathsf{FRDP}$, adjoint to $\Xi\colon \mathsf{FRDP} \to \mathsf{HF}$, such that: for every finite hall forest F, $\Psi(F)$ is a finite RDP-algebra; and, for every morphism (f, g) from the hall forest F' to the hall forest F'', $\Psi((f, g))$ is a homomorphism from the finite RDP-algebra $\Psi(F'')$ to the finite RDP-algebra $\Psi(F')$.

We provide a construction in two stages of the finite RDP-algebra $\Psi(F)$: first, on the basis of the finite hall forest F, we compute a finite *augmented* forest F'; then, we obtain the finite RDP-algebra by equipping the maximal antichains over F' with suitably defined operations. [11]

Step 1: For each hall tree (T, J) in F, the *augmented* forest F' contains an *augmented* tree T'. T' is a copy of T, with the following modifications. If the maximal points of T are $x_1, \dots, x_n$, then T' contains new points $y_1, \dots, y_n$ such that $x_i < y_i$ in T', for all $i \in [n]$. Also, if $|J| \geq 1$ and the minimum element of T is y, then the chain J is adjoined below y in T' (that is, y covers the maximal element of J in T'), and in this case, the point y is called the *fixpoint* of T', in symbols, $y = \operatorname{fixpoint} T'$.

Step 2: Let $\mathbf{A}_F$ be the set of maximal antichains in F', and let $\mathbf{C}_F$ be the set of maximal chains in F'. Since each maximal chain $C \in \mathbf{C}_F$ is contained in some augmented tree T' of F', if T' has a fixpoint, then C contains such fixpoint, which we denote by $\operatorname{fixpoint} C$. We interpret the binary operations $\wedge$, $\vee$, $\odot$, and $\to$, and the constants $\bot$ and $\top$ over $\mathbf{A}_F$ as follows ($A, A' \in \mathbf{A}_F$ and $C \in \mathbf{C}_F$):

$$A \wedge_F A' \cap C = \min\{A \cap C, A' \cap C\}, \tag{12}$$

$$A \vee_F A' \cap C = \max\{A \cap C, A' \cap C\}, \tag{13}$$

$$A \odot_F A' \cap C = \begin{cases} \min C & A \cap C, A' \cap C \leq \operatorname{fixpoint} C, \\ \min\{A \cap C, A' \cap C\} & \text{otherwise}, \end{cases} \tag{14}$$

$$A \to_F A' \cap C = \begin{cases} \max C & A \cap C \leq A' \cap C, \\ \operatorname{fixpoint} C & A' \cap C < A \cap C \leq \operatorname{fixpoint} C, \\ A' \cap C & \text{otherwise}, \end{cases} \tag{15}$$

$\bot_F \cap C = \min C$, and $\top_F \cap C = \max C$. As maximal antichains in $\mathbf{A}_F$ are uniquely determined by their intersections with maximal chains in $\mathbf{C}_F$, the previous definition is sound. Also, notice the resemblance between (14) and (15) above and (4) and (5) respectively.

Example 1. *If $F = \{(T_1, \emptyset), (T_2, J_2)\}$ is the finite hall forest on the left, then $\mathbf{A}_F$ is the algebra of maximal antichains over the augmented forest $F' = \{T_1', T_2'\}$ on the right, where $\min T_1' = \bot\bar{x}\bar{y}$ and $\min T_2' = \bot\bar{x}$; notation is displayed for further reference.*

Let F be a finite hall forest. The key of the construction is to establish a bijection

$$m\colon \mathbf{A}_F \to \hom(F, \Xi(F_1)), \tag{16}$$

[11]A *maximal antichain* (*chain*, respectively) in a poset is a maximal set of pairwise incomparable (comparable, respectively) points.

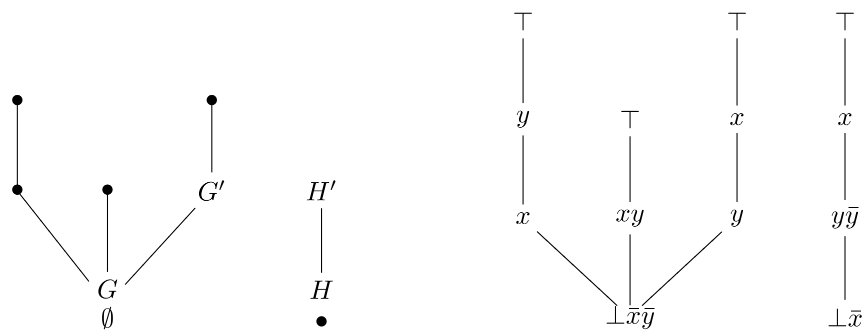


Figure 2: Example 1 and Example 2.

from the maximal antichains in $\mathbf{A}_F$, to the morphisms from the hall forest F to the hall forest $\Xi(F_1)$ corresponding to the prime spectrum of the free 1-generated RDP-algebra. For presentation sake, we defer to Proposition 6 the description of F_1 and the construction of $\Xi(F_1)$. Here, we assume that $\Xi(F_1)$ is as in Figure 3. The bijection m is defined as follows. Let h be a morphism from F to $\Xi(F_1)$.

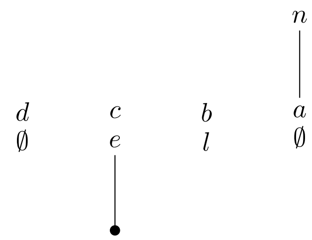


Figure 3: $\Xi(F_1)$ with notation for the discussion of bijection m displayed. For each hall tree (T, J) in $\Xi(F_1)$, the component J is displayed below T.

Let (T, J) be a hall tree in F, and let (f, g) be the morphism implementing the behavior of h on (T, J). Let T' be the augmented tree corresponding to T. Then, the maximal antichain $m^{-1}(h)$, corresponding to the labelled morphism h, restricted to T', satisfies the following conditions. If $f^{-1}(a)$ is empty, then the antichain $m^{-1}(h) \cap T' = \min T'$. Otherwise, if $f^{-1}(b)$ is equal to T, then $m^{-1}(h) \cap T' = \text{fixpoint}\, T'$. Otherwise, if $f^{-1}(c)$ is equal to T, then $m^{-1}(h) \cap T'$ is determined by $g^{-1}(e)$, as follows: if the maximum element in $g^{-1}(e)$ is the kth smallest element of J, then $m^{-1}(h) \cap T'$ is the $(k+1)$th smallest element of T'. Otherwise, if $f^{-1}(a)$ is nonempty, $m^{-1}(h) \cap T'$ contains the covers in F' of the maximal points in $f^{-1}(a)$ (these points are in F' by construction). As there are no other cases, the definition of m is complete.

Example 2. *First compare the hall tree $(T_1, \emptyset)$ in Example 1. By Definition 1, there are 19 morphisms $h = (f, g)$ from $(T_1, \emptyset)$ to $\Xi(F_1)$, indexed by the 19 maximal antichains in T_1'. Comparing Figure 3, for instance, if $f(T_1) = d$ in $\Xi(F_1)$, then $m^{-1}(h)$ is the maximal antichain $\{\bot\bar{x}\bar{y}\}$ in T_1'; if $f(T_1) = a$, then $m^{-1}(h) = \{\top, \top, \top\}$; if $f(\{G, G'\}) = a$ and $f(T_1 \setminus \{G, G'\}) = n$, then $m^{-1}(h) = \{x, xy, x\}$.*

Next compare the hall tree (T_2, J_2) in Example 1. By Definition 1, there are 4 morphisms $h = (f, g)$, from (T_2, J_2) to $\Xi(F_1)$, indexed by the 4 maximal antichains in T_2', as follows. If $f(T_2) = d$ in $\Xi(F_1)$, then $m^{-1}(h) = \{\bot\bar{x}\}$ in T_2'; if $f(T_2) = b$ and $g(J_2) = l$, then $m^{-1}(h) = \{y\bar{y}\}$; if $f(H) = a$ and $f(H') = n$, then $m^{-1}(h) = \{x\}$; and, if $f(T_2) = a$, then $m^{-1}(h) = \{\top\}$.

Given m, a contravariant functor $\Psi\colon \mathsf{HF} \to \mathsf{FRDP}$ is easily obtained, along the lines of [1], as follows: If F is a finite hall forest, then

$$\Psi(F) = (\mathbf{A}_F, \wedge_F, \vee_F, \odot_F, \to_F, \bot_F, \top_F) \tag{17}$$

is a finite RDP-algebra. If g is a morphism from the finite hall forest F' to the finite hall forest F'', then $\Psi(g)$ is the homomorphism from $\Psi(F'') = \mathbf{A}_{F''}$ to $\Psi(F') = \mathbf{A}_{F'}$, such that for every $a \in \mathbf{A}_{F''}$,

$$(\Psi(g))(a) = m^{-1}(m(a) \circ g) \in \mathbf{A}_{F'}. \tag{18}$$

The verification that $\Psi(g)\colon \mathbf{A}_{F''} \to \mathbf{A}_{F'}$ is an RDP-homomorphism is a burdening computation.

Example 3. *Let $F' = \{(T_1, J_1), (T_2, J_2)\}$ and $F'' = \{(T_3, \emptyset)\}$ be the hall forests depicted on the left, where $|T_1| = 1$, $|T_2| = 2$, $|T_3| = 6$. Let $\Psi(F') = \mathbf{A}_{F'}$ and $\Psi(F'') = \mathbf{A}_{F''}$ be the algebras of maximal antichains over the augmented forests $\{T_1', T_2'\}$ and $\{T_3'\}$ depicted on the right, where $|T_1'| = 3$, $|T_2'| = 4$, $|T_3'| = 9$.*

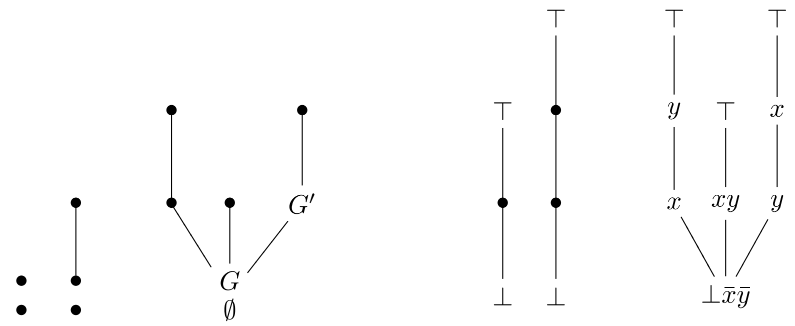


Figure 4: Example 3.

Let g be the morphism that sends T_1 and T_2 to $\min T_3$; then, $\Psi(g)\colon \mathbf{A}_{F''} \to \mathbf{A}_{F'}$ is defined by (18). We compute $\Psi(g)$ on two samples.

Let $a = \{\bot\bar{x}\bar{y}\} \in \Psi(F'')$. Along the lines of Example 2, $m(a)$ is a morphism (f_a, g_a) from F'' to $\Xi(F_1)$ such that $f_a(T_3) = d$ (recall Figure 3). Then, the composition $m(a) \circ g$ is a morphism from F' to $\Xi(F_1)$ that sends T_1 and T_2 to d. Then, by the definition of m,

$$(\Psi(g))(a) = m^{-1}(m(a) \circ g) = \{\bot, \bot\}.$$

Let $a = \{x, xy, x\} \in \Psi(F'')$. Along the lines of Example 2, $m(a)$ is a morphism (f_a, g_a) from F'' to $\Xi(F_1)$ such that $f_a(\{G, G'\}) = a$ and $f_a(T_3 \setminus \{G, G'\}) = n$. Then, the composition $m(a) \circ g$ is a morphism from F' to $\Xi(F_1)$ that sends T_1 and T_2 to a. By the definition of m,

$$(\Psi(g))(a) = m^{-1}(m(a) \circ g) = \{\top, \top\}.$$

Let $a = \{x, xy, x\} \in \Psi(F'')$. In light of the previous computations, we show that $\Psi(g)$ preserves the negation of a,

$$\begin{aligned}\Psi(g)(\neg_{F''}a) &= \Psi(g)(\neg_{F''}\{x, xy, x\})\\ &= \Psi(g)(\{\bot\bar{x}\bar{y}\})\\ &= \{\bot, \bot\}\\ &= \neg_{F'}\{\top, \top\}\\ &= \neg_{F'}(\Psi(g)(\{x, xy, x\}))\\ &= \neg_{F'}(\Psi(g)(a));\end{aligned}$$

analogous computations show that in fact, $\Psi(g)$ is an RDP-homomorphism.

2.2 Coproducts of RDP-Algebras

In this section, we describe explicitly the (binary) product operation, $\times$, in the category of finite hall forests. Then, the coproduct of finite RDP-algebras A and B will be given by

$$\Psi(\Xi(A) \times \Xi(B)),$$

where Ξ and Ψ are the adjoint contravariant functors between finite RDP-algebras and finite hall forests given in Section 2.1.

Let F and F' be finite hall forests. We will describe the product $F \times F'$, and the projections π and π' of $F \times F'$ onto F and F' respectively. Each of F and F' is a multiset of finite hall trees, say $F = \{(T_i, J_i) \mid i \in [k]\}$ and $F' = \{(T'_i, J'_i) \mid i \in [k']\}$. In general, the result of the product $F \times F'$, and its projections, are uniquely determined by the result of the individual products $(T_m, J_m) \times (T'_n, J'_n)$ for every pair $(m, n) \in [k] \times [k']$. Hence, it is sufficient to describe the product $(T_m, J_m) \times (T'_n, J'_n)$, and its projections. In the present setting, the result of the product $(T_m, J_m) \times (T'_n, J'_n)$ is uniquely determined by the result of the individual products $T_m \times T'_n$ and $J_m \times J'_n$, and their projections, as follows. The product $T_m \times T'_n$ and its projections is computed in [8], and yields a finite tree S and its projections $\varsigma_{m,n}$ and $\varsigma'_{m,n}$ onto T_m and T'_n respectively. The product $J_m \times J'_n$ and its projections, explained below, yields a finite collection of $N(|J_m|, |J'_n|) \geq 1$ many chains K_o, together with their projections $\rho_{m,n,o}$ and $\rho'_{m,n,o}$ onto J_m and J'_n respectively ($1 \leq o \leq N(|J_m|, |J'_n|)$). Finally, the product $(T_m, J_m) \times (T'_n, J'_n)$ is the finite collection of $N(|J_m|, |J'_n|)$ many hall trees (S, K_o) with projections $(\varsigma_{m,n}, \rho_{m,n,o})$ and $(\varsigma'_{m,n}, \rho'_{m,n,o})$ onto (T_m, J_m) and (T'_n, J'_n) respectively ($1 \leq o \leq N(|J_m|, |J'_n|)$).

Aiming at the proof of the universal property, we give a careful description of the aforementioned chains $K_1, \ldots, K_{N(|J|,|J'|)}$, for a given pair of chains J and J'. If $j \leq 1$ or $j' \leq 1$, then $N(|J|, |J'|) = 1$ and $|K_1| = \max\{j, j'\}$. Otherwise, suppose that $j > 1$ and $j' > 1$. Roughly, given two chains J and J' of cardinality j and j' respectively, the problem is to describe the chains over the points in the union of $J \setminus \max(J)$ and $J' \setminus \max(J')$ that respect the order of J and J'; without loss of generality, $J \cap J' = \emptyset$. Below, we let C_i denote a chain of length i. Clearly, it is possible to obtain chains of minimum length $m = \max\{j, j'\} - 1$ and maximum length $M = j + j' - 2$. Hence, the problem is equivalent to describing the surjective maps f from

$$D = (J \setminus \max(J)) \cup (J' \setminus \max(J'))$$

to chains C_i of length i ranging from m to M that *respect* the order of J and J', that is, if $x < y$ in J or J', then $f(x) < f(y)$ in C_i. We first enumerate these maps, and then, for each such map, we compute the corresponding chain K together with its projections onto J and J'.

The number of maps from $J \setminus \max(J)$ to C_i that respect the order of J is $\binom{i}{j-1}$, and the number of maps from $J' \setminus \max(J')$ to C_i that respect the order of J' is $\binom{i}{j'-1}$, hence the number of maps from D to C_i that respect simultaneously the order of J and J' is

$$OrdPres(i,j,j') = \binom{i}{j-1}\binom{i}{j'-1}.$$

We now establish the number of non-surjective maps from D to C_i that preserve the order of J and J', for short $NotSurj(i,j,j')$, to conclude that

$$N(i,j,j') = OrdPres(i,j,j') - NotSurj(i,j,j').$$

Any non-surjective map from D to C_i neglects k points in C_i, for some k between 1 to $i - m$. Clearly, there are $\binom{i}{k}$ possible choices for these k neglected points, and for each choice, the number of order-preserving non-surjective maps from D to C_i coincide with the number of order-preserving surjective maps from D to C_{i-k}, that is, $N(i-k,j,j')$. Hence, we obtain the recurrence,

$$NotSurj(i,j,j') = \sum_{k=1}^{i-m} \binom{i}{k} N(i-k,j,j'),$$

whose base case is $NotSurj(m,j,j') = 0$, because in this case, the sum is the empty sum. Summarizing, given two chains J and J' of cardinality j and j' respectively, letting $m = \max\{j,j'\} - 1$ and maximum length $M = j + j' - 2$,

$$N(j,j') = \sum_{i=m}^{M} N(i,j,j').$$

Now, for finite hall forests $F = \{(T_i, J_i) \mid i \in [k]\}$ and $F' = \{(T'_i, J'_i) \mid i \in [k']\}$, let $(m,n) \in [k] \times [k']$, and let J_m and J'_n be the chain components of two hall trees (T_m, J_m) and (T'_n, J'_n). Let f be the oth map in some fixed order over the $N(|J_m|, |J'_n|)$ many surjective order-preserving maps from the union of $J_m \setminus \max(J_m)$ and $J'_n \setminus \max(J'_n)$ to chains of length $\max\{|J_m|, |J'_n|\} - 1 \leq i \leq |J_m| + |J'_n| - 2$. Then, we let the oth chain K_o in the collection of chains returned by $J_m \times J'_n$ be the chain of $i+1$ points, whose projections onto J_m and J'_n are respectively $\rho_{m,n,o}$ and $\rho'_{m,n,o}$, defined as follows. The projection onto the left factor J_m is defined by: $\rho_{m,n,o}(\max(K_o)) = \max(J_m)$; for $x \in K_o$, if $x \in J_m$, then $\rho_{m,n,o}(x)$ is equal to x; otherwise, $\rho_{m,n,o}(x)$ is equal to $\rho_{m,n,o}(y)$ where y is the smallest element of K_o above x such that $y \in J_m$. The projection onto the right factor J'_n is similarly defined by: $\rho_{m,n,o}(\max(K_o)) = \max(J'_n)$; for $x \in K_o$, if $x \in J'_n$, then $\rho_{m,n,o}(x)$ is equal to x; otherwise, $\rho_{m,n,o}(x)$ is equal to $\rho_{m,n,o}(y)$ where y is the smallest element of K_o above x such that $y \in J'_n$.

We now show that the product operation described above has the universal property.

Theorem 3. *Let $F = \{(T_i, J_i) \mid i \in [k]\}$ and $F' = \{(T'_i, J'_i) \mid i \in [k']\}$ be finite hall forests. Then,*

$$F \times F' = \{(T_m, J_m) \times (T'_n, J'_n) \mid (m, n) \in [k] \times [k']\},$$

with projections π and π' onto F and F' given by,

$$\pi = \{(\varsigma_{m,n}, \rho_{m,n,1}), \dots, (\varsigma_{m,n}, \rho_{m,n,N(|J_m|,|J'_n|)}) \mid (m, n) \in [k] \times [k']\},$$
$$\pi' = \{(\varsigma'_{m,n}, \rho'_{m,n,1}), \dots, (\varsigma'_{m,n}, \rho'_{m,n,N(|J_m|,|J'_n|)}) \mid (m, n) \in [k] \times [k']\},$$

is the product of F and F' in the category HF.

Proof. The morphisms under consideration split into two components, the first acting on trees as by [8], and the second acting on chains. For the first component we rely upon the universal property of products of finite trees [8]. Hence, we reduce to prove the universal property of products of finite chains. The details follow.

It suffices to prove that if J, J' and J'' are chains, g' and g'' are morphisms from J to J' and J'' respectively, and π' and π'' are the projections of $J' \times J''$ onto J' and J'' respectively, then there exists a unique morphism h from J to $J \times J'$ such that $\pi' \circ h = g'$ and $\pi'' \circ h = g''$.

We establish a bijection between pairs of morphism g' and g'' from J to J' and J'' respectively, and morphisms h from J to $J' \times J''$. The bijection has the property that if h corresponds to g' and g'', then $\pi' \circ h = g'$ and $\pi'' \circ h = g''$. It follows that there exists a unique morphism h that factorizes g' and g'' through π' and π''.

The bijection is given by the following explicit construction of the morphism h, given morphisms g' and g''. The range of h is the chain K_o in $J' \times J''$ defined as follows (h sends J to a single chain in $J' \times J''$, as it is an open map). The chain K_o is the restriction of chain J to the points $x \in J$ such that one of the following four (disjoint and exhaustive) cases occur. Case 1: x is the maximum in $g'^{-1}(y)$ for some $y \in J'$ and x is the maximum in $g''^{-1}(z)$ for some $z \in J''$; in this case, we label x by $\{y, z\}$, and we let $h(x) = \{y, z\}$. Case 2: x is the maximum in $g'^{-1}(y)$ for some $y \in J'$; in this case, we label x by $\{y\}$, and we let $h(x) = \{y\}$. Case 3: x is the maximum in $g''^{-1}(z)$ for some $z \in J''$; in this case, we label x by $\{z\}$, and we let $h(x) = \{z\}$. Case 4: For the remaining $x \in J$, we let $h(x) = h(x')$ where x' is the smallest element above x in J such that $h(x')$ is defined by the above clauses (note that at least, $h(x')$ is defined if $x' = \max(J)$). Clearly, given g' and g'', the map h is uniquely determined. Moreover, by construction, $\pi' \circ h = g'$ and $\pi'' \circ h = g''$.

For injectivity, we prove that if $(f', f'') \neq (g', g'')$ are distinct pairs of morphisms from J to J' and J'' respectively, then the maps obtained from the above construction, say h' and h'', are distinct. If h' and h'' have distinct range, then they are distinct. Otherwise, if they have the same range, we claim that there exists $x \in J$ such that $h'(x) \neq h''(x)$. Suppose for a contradiction that $h' = h''$. Then, $f' = \pi' \circ h' = \pi' \circ h'' = g'$ and $f'' = \pi'' \circ h' = \pi'' \circ h'' = g''$, contradiction. For surjectivity, trivially, if h is a map from J to $J' \times J''$, then there exists a pair of morphisms g' and g'' from J to J' and J'' respectively: simply let, $g' = \pi' \circ h$ and $g'' = \pi'' \circ h$.

The proof is complete. ☐

It follows that HF has all finite products. In fact, by [18, Proposition 3.5.1], a category has all finite products if it has binary products and a terminal object; but, HF has binary products, and it is easy to check that the finite hall forest $\{(\bullet, \emptyset)\}$ is a terminal object (dually, the RDP-algebra $\perp < \top$ homomorphically maps to any RDP-algebra). Therefore, for S a finite hall forest in HF, we denote by S^n the product in HF of n copies of S, and by π_i the projection of S^n onto the ith factor S ($n \geq 1$).

In the next section, we will exploit the ability to compute finite coproducts of finitely generated RDP-algebras to provide a combinatorial representation of free finitely generated RDP-algebras.

2.3 Free Finitely Generated RDP-Algebras

In this section, exploiting the categorical machinery developed, we give a combinatorial representation of the free n-generated RDP-algebra F_n, for $n \geq 1$.

As a preliminary step, we describe the free 1-generated RDP-algebra, F_1 (compare Figure 5). Recall from Section 1.1 that F_1 is finite. Hence, by universal algebraic facts [3, Theorem 9.6], the RDP-algebra F_1 is isomorphic to a subdirect product of a finite number of subdirectly irreducible finite RDP-algebras. As subdirectly irreducible finite RDP-algebras are finite RDP-chains [12], F_1 is isomorphic to a subdirect product of a finite family of singly generated finite RDP-chains. By direct computation over (3), there are exactly five pairwise nonisomorphic singly generated factors (that is, homomorphic images of subalgebras) of the generic algebra, namely, there are exactly five pairwise nonisomorphic singly generated RDP-chains : C_1 is $\perp = x < \neg x = \top$, C_2 is $\perp < x < \neg x = \neg\neg x < \top$, C_3 is $\perp < x = \neg x < \top$, C_4 is $\perp = \neg x < x < \top$, C_5 is $\perp = \neg x < x = \top$ (where x is the generator). Then, there is a subdirect embedding of F_1 into the direct product of a finite family $A_1, \ldots, A_m$ of RDP-chains, where each A_i is either C_1, C_2, C_3, C_4, or C_5. Up to isomorphism, we can remove from the finite family $A_1, \ldots, A_m$ all copies of C_5 (C_5 is a proper quotient of C_4, via the map that sends x to $\top$), and multiple copies of C_i for $i = 1, 2, 3, 4$.

Summarizing, there is a subdirect embedding of F_1 into the direct product $A = C_1 \times C_2 \times C_3 \times C_4$, so that $|F_1| \leq |A| = 72$. It is possible to check that $|F_1| = 72$. The idea is the following: Given a tuple $(a_1, a_2, a_3, a_4) \in A$, construct an RDP-term t over the variable x such that the ith projection of t^A is equal to a_i for $i = 1, 2, 3, 4$. For instance, by direct computation, the RDP-terms $x \to \neg x$, $t = \neg((x \leftrightarrow \neg x)^2)$, $t \to \neg\neg x$, and $\neg((\neg x)^2)$ realize respectively $(\top, \top, \top, \perp)$, $(\top, \top, \perp, \top)$, $(\top, \perp, \top, \top)$, and $(\perp, \top, \top, \top)$. The details of the construction are given in Section 3.1. As F_1 is the largest singly generated RDP-algebra (every singly generated RDP-algebra is a quotient of F_1 [3, Corollary 10.11]), we conclude that $F_1 = A$.

Proposition 6. $\Xi(F_1) = S_1$ *is the finite hall forest displayed in Figure 6.*

Proof. We adopt the terminology and notation introduced in the above discussion. Notice that C_1, C_2, C_3, C_4 are finite, directly indecomposable RDP-algebras. By definition: $\Xi(C_1) = (G_1, J_1)$, where G_1 is the prime filter of F_1 generated by $(\neg x, \perp, \perp, \perp)$, and $|J_1| = \mathrm{type}(C_1) = 0$; $\Xi(C_2) = (G_2, J_2)$, where G_2 is the prime filter of F_1 generated by $(\perp, \top, \perp, \perp)$, and $|J_2| = \mathrm{type}(C_2) = 2$; $\Xi(C_3) = (G_3, J_3)$, where G_3 is the prime filter of F_1 generated by $(\perp, \perp, \top, \perp)$,

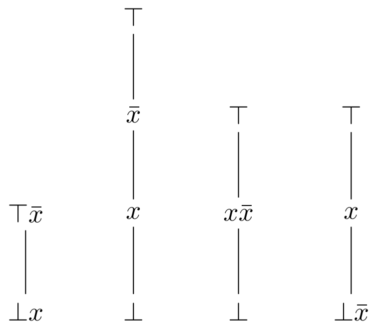


Figure 5: The free 1-generated RDP-algebra F_1 is the algebra of maximal antichains in the depicted forest, equipped with the operations defined in (14)-(15).

and $|J_3| = \text{type}(C_3) = 1$; $\Xi(C_4) = (G_4 \supseteq G_5, J_4)$, where G_4 and G_5 are the prime filters of F_1 generated respectively by $(\bot, \bot, \bot, x)$ and $(\bot, \bot, \bot, \top)$, and $|J_4| = \text{type}(C_4) = 0$. As $\Xi(F_1)$ is the disjoint union of $\Xi(C_i)$ for $i = 1, 2, 3, 4$, the statement is proved. $\square$

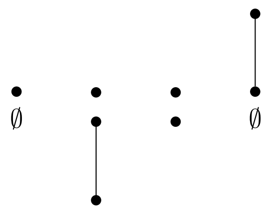


Figure 6: The hall forest $S_1 = \Xi(F_1)$. For each hall tree (T, J) in S_1, the component J is displayed below T.

Lemma 2. *The prime spectrum $\Xi(F_n)$ of the free n-generated RDP-algebra F_n, over the free generators $x_1, \ldots, x_n$, is the finite hall forest S_1^n.*

Proof. As in any variety, the free n-generated RDP-algebra, F_n, is the coproduct of n copies of the free 1-generated RDP-algebra, F_1. By Proposition 6, $\Xi(F_1)$ is the finite hall forest S_1. The statement now follows from the categorical equivalence of HF and FRDP via the contravariant functor Ξ (Theorem 2). $\square$

Theorem 4. *The free n-generated RDP-algebra F_n, over the free generators $x_1, \ldots, x_n$, is isomorphic to $\Psi(S_1^n)$.*

Proof. Note that the functor Ψ is the contravariant adjoint to the functor Ξ, and that, by Lemma 2, the finite hall forest S_1^n is exactly $\Xi(F_n)$, that is, the prime spectrum of the free n-generated RDP-algebra F_n over the free generators $x_1, \ldots, x_n$. Recall that $\Psi(S_1^n)$ is the algebra of maximal antichains in $\mathbf{A}_{S_1^n}$ specified by (17). To identify the maximal antichains in $\mathbf{A}_{S_1^n}$ corresponding to the free generators $x_1, \ldots, x_n$, let π_i be the projection of S_1^n onto the ith factor S_1, and let m be the bijection in (16); the maximal antichain corresponding to the free generator x_i of F_n is $m^{-1}(\pi_i)$, for $i \in [n]$. $\square$

To sample the general case, we now describe in a sequence of examples the product of two copies of the finite hall forest S_1 depicted in Figure 7, namely, the product $F \times F'$ where

$$\begin{aligned} F &= \{(T_1, J_1), (T_2, J_2), (T_3, J_3), (T_4, J_4)\} \\ &= \{(\{\perp\}, \emptyset), (\{\perp\}, \{x < \bar{x}\}), (\{\perp\}, \{x = \bar{x}\}), (\{\perp < x\}, \emptyset)\}; \\ F' &= \{(T_1', J_1'), (T_2', J_2'), (T_3', J_3'), (T_4', J_4')\} \\ &= \{(\{\perp\}, \emptyset), (\{\perp\}, \{y < \bar{y}\}), (\{\perp\}, \{y = \bar{y}\}), (\{\perp < y\}, \emptyset)\}. \end{aligned}$$

The adopted labelling of factors is useful to describe the product operation and the projection maps.

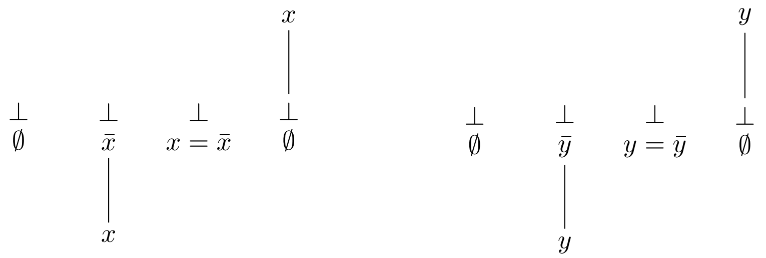


Figure 7: Two copies of S_1 suitably labelled in view of the description of $S_1 \times S_1$. For each hall tree (T, J) in S_1, the component J is displayed below T.

The general behavior of products of trees is described in [8]. In the sample case under consideration, we have the following.

Example 4. *We study the action of product $F \times F'$ over the tree components of pairs of hall trees in F and F'. Precisely, for each $(m, n) \in [4] \times [4]$, we compute the product $T_m \times T_n'$, together with its projections onto the left and right factor. The result is the following.*

For $j = 1, 2, 3$ and $i = 1, 2, 3$, $T_j \times T_i'$ yields the tree $S_{j,i} = \{\perp\}$, whose projection $\varsigma_{j,i}$ onto T_j is $\perp \mapsto \perp$, and whose projection $\varsigma_{j,i}'$ onto T_i' is $\perp \mapsto \perp$.

For $j = 1, 2, 3$, $T_j \times T_4'$ yields the tree $S_{j,4} = \{\perp < y\}$, whose projections $\varsigma_{j,4}$ and $\varsigma_{j,4}'$ are respectively, $\perp \mapsto \perp, y \mapsto \perp$, and $\perp \mapsto \perp, y \mapsto y$.

For $i = 1, 2, 3$, $T_4 \times T_i'$ yields the tree $S_{4,i} = \{\perp < x\}$, whose projections $\varsigma_{4,i}$ and $\varsigma_{4,i}'$ are respectively, $\perp \mapsto \perp, x \mapsto \perp$, and $\perp \mapsto \perp, x \mapsto x$.

$T_4 \times T_4'$ yields the tree $S_{4,4}$ given by the chains $\perp < \{x = y\}$, $\perp < x < \{x < y\}$, $\perp < y < \{y < x\}$, whose projections $\varsigma_{4,4}$ and $\varsigma_{4,4}'$ are respectively, $\perp \mapsto \perp, \{x = y\} \mapsto x, x \mapsto x, \{x < y\} \mapsto \perp, y \mapsto \perp, \{y < x\} \mapsto x$, and $\perp \mapsto \perp, \{x = y\} \mapsto y, x \mapsto \perp, \{x < y\} \mapsto y, y \mapsto y, \{y < x\} \mapsto \perp$.

The action of the product $F \times F'$ over the chain components of pairs of hall trees in F and F' is the following.

Example 5. *We study the action of product $F \times F'$ over the chain components of pairs of hall trees in F and F'. Precisely, for each $(m, n) \in [4] \times [4]$, we compute the product $J_m \times J_n'$, together with its projections onto the left and right factor. The result is the following.*

$J_1 \times J'_1$ yields the chain $K_{1,1} = \emptyset$, whose projection $\rho_{1,1}$ onto J_1 is the empty function, and whose projection $\rho'_{1,1}$ onto J'_1 is the empty function.

$J_1 \times J'_2$ yields $K_{1,2} = \{y < \bar{y}\}$, whose projections $\rho_{1,2}$ and $\rho'_{1,2}$ are respectively, the empty function, and $y \mapsto y, \bar{y} \mapsto \bar{y}$.

$J_1 \times J'_3$ yields $K_{1,3} = \{\{y = \bar{y}\}\}$, whose projections $\rho_{1,3}$ and $\rho'_{1,3}$ are respectively, the empty function, and $\{y = \bar{y}\} \mapsto \{y = \bar{y}\}$.

$J_1 \times J'_4$ yields $K_{1,4} = \emptyset$, whose projections $\rho_{1,4}$ and $\rho'_{1,4}$ are respectively, the empty function, and the empty function.

$J_2 \times J'_1$ yields $K_{2,1} = \{x < \bar{x}\}$, whose projections $\rho_{2,1}$ and $\rho'_{2,1}$ are respectively, $x \mapsto x, \bar{x} \mapsto \bar{x}$, and the empty function.

$J_2 \times J'_2$ yields the following three chains: $K_{2,2,1} = \{x = y < \bar{x} = \bar{y}\}$, whose projections $\rho_{2,2,1}$ and $\rho'_{2,2,1}$ are respectively, $x = y \mapsto x, \bar{x} = \bar{y} \mapsto \bar{x}$, and $x = y \mapsto y, \bar{x} = \bar{y} \mapsto \bar{y}$; $K_{2,2,2} = \{x < y < \bar{x} = \bar{y}\}$, whose projections $\rho_{2,2,2}$ and $\rho'_{2,2,2}$ are respectively, $x \mapsto x, y \mapsto \bar{x}, \bar{x} = \bar{y} \mapsto \bar{x}$, and $x \mapsto y, y \mapsto y, \bar{x} = \bar{y} \mapsto \bar{y}$; and $K_{2,2,3} = \{y < x < \bar{x} = \bar{y}\}$, whose projections $\rho_{2,2,3}$ and $\rho'_{2,2,3}$ are respectively, $y \mapsto x, x \mapsto x, \bar{x} = \bar{y} \mapsto \bar{x}$, and $y \mapsto y, x \mapsto \bar{y}, \bar{x} = \bar{y} \mapsto \bar{y}$.

$J_2 \times J'_3$ yields $K_{2,3} = \{x < \bar{x} = y = \bar{y}\}$, whose projections $\rho_{2,3}$ and $\rho'_{2,3}$ are respectively, $x \mapsto x, \bar{x} = y = \bar{y} \mapsto \bar{x}$, and $x \mapsto y = \bar{y}, \bar{x} = y = \bar{y} \mapsto y = \bar{y}$.

$J_2 \times J'_4$ yields $K_{2,4} = \{x < \bar{x}\}$, whose projections $\rho_{2,4}$ and $\rho'_{2,4}$ are respectively, $x \mapsto x, \bar{x} \mapsto \bar{x}$, and the empty function.

$J_3 \times J'_1$ yields $K_{3,1} = \{x = \bar{x}\}$, whose projections $\rho_{3,1}$ and $\rho'_{3,1}$ are respectively, $x = \bar{x} \mapsto x = \bar{x}$, and the empty function.

$J_3 \times J'_2$ yields $K_{3,2} = \{y < x = \bar{x} = \bar{y}\}$, whose projections $\rho_{3,2}$ and $\rho'_{3,2}$ are respectively, $y \mapsto x = \bar{x}, x = \bar{x} = \bar{y} \mapsto x = \bar{x}$, and $y \mapsto y, x = \bar{x} = \bar{y} \mapsto \bar{y}$.

$J_3 \times J'_3$ yields $K_{3,3} = \{x = \bar{x} = y = \bar{y}\}$, whose projections $\rho_{3,3}$ and $\rho'_{3,3}$ are respectively, $x = \bar{x} = y = \bar{y} \mapsto x = \bar{x}$, and $x = \bar{x} = y = \bar{y} \mapsto y = \bar{y}$.

$J_3 \times J'_4$ yields $K_{3,4} = \{x = \bar{x}\}$, whose projections $\rho_{3,4}$ and $\rho'_{3,4}$ are respectively, $x = \bar{x} \mapsto x = \bar{x}$, and the empty function.

$J_4 \times J'_1$ yields $K_{4,1} = \emptyset$, whose projections $\rho_{4,1}$ and $\rho'_{4,1}$ are respectively, the empty function, and the empty function.

$J_4 \times J'_2$ yields $K_{4,2} = \{y < \bar{y}\}$, whose projections $\rho_{4,2}$ and $\rho'_{4,2}$ are respectively, the empty function, and $y \mapsto y, \bar{y} \mapsto \bar{y}$.

$J_4 \times J'_3$ yields $K_{4,3} = \{y = \bar{y}\}$, whose projections $\rho_{4,3}$ and $\rho'_{4,3}$ are respectively, the empty function, and $y = \bar{y} \mapsto y = \bar{y}$.

$J_4 \times J'_4$ yields $K_{4,4} = \emptyset$, whose projections $\rho_{4,4}$ and $\rho'_{4,4}$ are respectively, the empty function, and the empty function.

Figure 8 displays $F \times F'$. The projections π and π' of $F \times F'$, onto F and F' respectively, are uniquely determined by their restrictions to each pair of hall trees, as specified in the following example.

Example 6. *For each $(m, n) \in [4] \times [4]$, we compute the product $(T_m, J_m) \times (T'_n, J'_n)$, together with its projections onto the left and right factor. The result is the following.*

If $m = n = 2$, $(T_2, J_2) \times (T'_2, J'_2)$ yields three hall trees, namely, for $j = 1, 2, 3$, $(S_{2,2}, K_{2,2,j})$, whose projections are $\pi_{2,2,j} = (\varsigma_{2,2}, \rho_{2,2,j})$ and $\pi'_{2,2,j} = (\varsigma'_{2,2}, \rho'_{2,2,j})$. Otherwise, $(T_m, J_m) \times (T'_n, J'_n)$ yields the hall tree $(S_{m,n}, K_{m,n})$ whose projections are $\pi_{m,n} = (\varsigma_{m,n}, \rho_{m,n})$ and $\pi'_{m,n} = (\varsigma'_{m,n}, \rho'_{m,n})$.

We conclude this section by displaying in Figure 9 a (suitably) labelled version of $\Psi(S^2_1)$, paralleling Figure 5 in the 2-generated case. This labelling

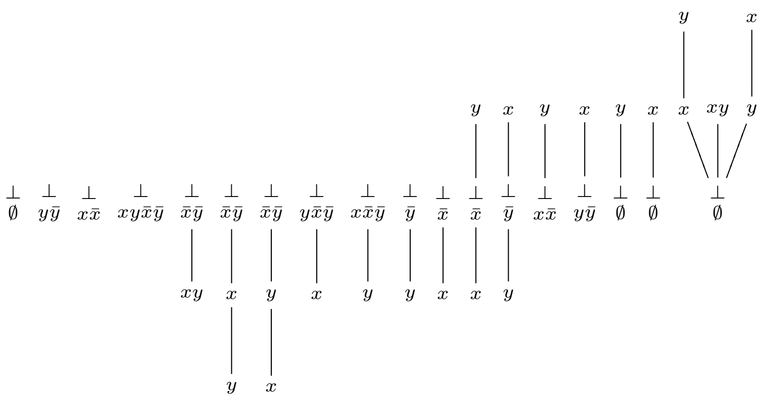


Figure 8: The finite hall forest $S_1^2 = S_1 \times S_1$. The labelling allows for recovering the projection maps of the first and second factor, displayed in Figure 8. For each hall tree (T, J) in S_1, the component J is displayed below T.

method (formalized in the next section) will allow for a streamlined investigation of several logical problems related to the free finitely generated RDP-algebra.

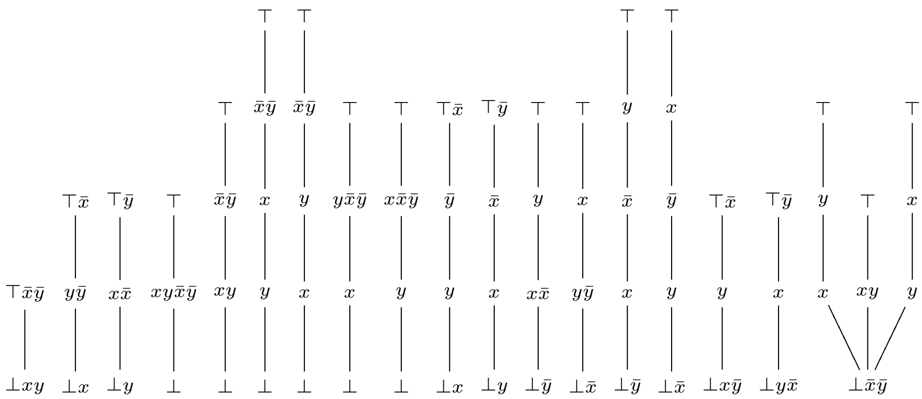


Figure 9: Display of $\Psi(S_1^2)$, by Theorem 4 isomorphic to F_2, where the maximal antichains corresponding to the free generators x and y of F_2 are those containing points whose label include x and y respectively.

The combinatorial representation of F_n achieved is amenable for investigation under several respects, substantially sampled by the logical applications in the next section. In addition, we mention that the given representation yields a recurrence relation for the computation the cardinality of F_n. We omit the details [22], and limit to report that, for instance, $|F_1| = 72$, $|F_2| = 94556160000$, $|F_3| \sim 4.06 \cdot 10^{71}$, and $|F_4| \sim 1.478733152865106 \cdot 10^{543}$. The first two statements are easy to check by directly count the maximal antichains in the forests displayed in Figure 5 and Figure 9.

3 Logical Properties

In this section, we apply the theory of finitely generated RDP-algebras developed in the previous two sections to obtain a number of results on the logical counterpart of RDP-algebras discussed in the introduction.

In Section 2.3, we characterize the free n-generated RDP-algebra F_n as the algebra $\Psi(S_1^n)$, that is, the algebra of maximal antichains in $\mathbf{A}_{S_1^n}$ over the augmented forest of S_1^n specified by (17). In the rest of this section, it is convenient to adopt a labelled display of the augmented forest of S_1^n, where each point is labelled with subsets of $\{\bot, \top, x_1, \neg x_1, \ldots, x_n, \neg x_n\}$, satisfying the following conditions:

(i) x_i belongs to the label of each point in the maximal antichain corresponding to the free generator x_i of F_n (compare Theorem 4).

(ii) The label of each root contains $\bot$, and the label of each leaf contains $\top$.

(iii) $\neg x_i$ belongs to the label of each point in the negation in $\mathbf{A}_{S_1^n}$ of the antichain corresponding to the free generator x_i.

Example 7. *Figure 5 displays the labelled augmented forest corresponding to S_1, and Figure 9 displays the labelled augmented forest corresponding to S_1^2. The maximal antichain corresponding to the free generator x_1 (respectively, x_2) is the set of points whose labels contain x_1 (respectively, x_2).*

Let $C \in \mathbf{C}_{S_1^n}$ be a maximal chain in the labelled augmented forest of S_1^n. Note that C is a homomorphic image of F_n; indeed, the map $h\colon \mathbf{A}_{S_1^n} \to C$ such that for every $A \in \mathbf{A}_{S_1^n}$ and $c \in C$, $h(A) = c$ if and only if $A \cap C = c$ is a surjective RDP-homomorphism. Hence, C is an RDP-chain. In the adopted display, C is an ordered partition $B_1 < \cdots < B_k$ of $\{\bot, \top, x_1, \neg x_1, \ldots, x_n, \neg x_n\}$, such that: $\bot \in B_1$ (the bottom of C), $\top \in B_k$ (the top of C), there exists at most one index $1 < f < k$ such that some $\neg x_i$'s belong to B_f (the fixpoint of C), and each B_i that is neither the bottom, nor the fixpoint, nor the top of C contains at least one of $x_1, \ldots, x_n$. Note that any point $c \in C$ can be regarded as a block amongst $B_1, \ldots, B_k$.

Now, let $t(x_1, \ldots, x_n)$ be a RDP-term over variables $x_1, \ldots, x_n$. Then, the maximal antichain t^{F_n} that corresponds to t in the labelled display of F_n is inductively defined as follows. For every $C = B_1 < \cdots < B_k \in \mathbf{C}_{S_1^n}$: If $t = x_j$, then $x_j \in t^{F_n} \cap C$; if $t = \bot$, then $\bot \in t^{F_n} \cap C$; for $\circ \in \{\odot, \to\}$, if $t = t' \circ t''$, $t'^{F_n} \cap C = B'$, and $t''^{F_n} \cap C = B''$, then $t'^{F_n} \cap C = B' \circ B''$, where the operation $\circ$ on $\{B_1, \ldots, B_k\}$ is defined by making the block that contains x (respectively, $\neg x$, y, $\neg y$, $\bot$, $\top$) acting as x (respectively, $\neg x$, y, $\neg y$, $\bot$, $\top$) in (7) and (8). Compare Figure 10.

For the sake of notation, in the sequel we let

$$t(C) = t^{F_n} \cap C.$$

A routine induction on t shows that t is a tautology of RDP-logic if and only if $t(C) = \max C$ for every maximal chain $C \in \mathbf{C}_{S_1^n}$, and by the standard completeness theorem [23], it follows that t is a theorem of RDP-logic, in symbols $\vdash_{RDP} t$.

The computational complexity of deciding the tautology problem of RDP-logic is as expected.

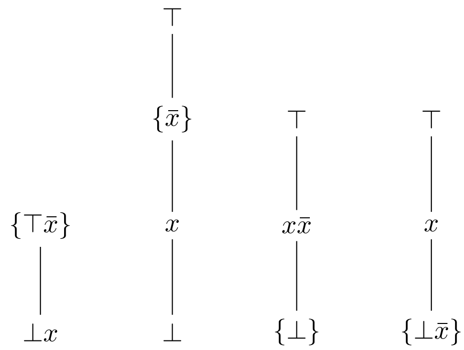


Figure 10: Displaying terms in F_1 as maximal antichains in the labelled augmented forest of S_1: $(\neg(\neg x \to x))^{F_1}$ is the bracketed maximal antichain in the diagram.

Proposition 7. *The RDP-tautology problem is coNP-complete (under logspace many-one reductions).*

Proof. Let t be an RDP-term on the variables $x_1, \dots, x_n$. For the upper bound, the algorithm receives in input a maximal chain in $\mathbf{C}_{S_1^n}$ and returns in output "Yes" if $t(C) = \max C$, and "No" otherwise. For the lower bound, we interpret the Boolean tautology problem. The reduction, given a Boolean term $t(x_1, \dots, x_n)$, say on conjunction $\odot$, implication $\to$, and zero $\bot$, outputs the RDP term $s = t(r_1, \dots, r_n)$, obtained by replacing uniformly variable x_i with term $r_i = (\neg\neg x_i) \odot (\neg\neg x_i)$ in t, for all $i \in [n]$. The substitution is feasible in logspace, and it is easy to check that t is a Boolean tautology (that is, $t = \top$ in $\mathbf{2}$) if and only if s is an RDP-tautology (that is, $s = \top$ in the generic RDP-algebra $[0,1]$ given by (3)).

Indeed, assume that t is a Boolean tautology. Let $\mathbf{a} \in [0,1]^n$. Noticing that $(r_1^{[0,1]}(\mathbf{a}), \dots, r_n^{[0,1]}(\mathbf{a})) = \mathbf{b} \in \{0,1\}^n$, and that for any term q, the operations $q^{\mathbf{2}}$ and $q^{[0,1]}$ coincide upon restriction to $\{0,1\}$, we have,

$$s^{[0,1]}(\mathbf{a}) = t^{[0,1]}(r_1^{[0,1]}(\mathbf{a}), \dots, r_n^{[0,1]}(\mathbf{a})) = t^{[0,1]}(\mathbf{b}) = t^{\mathbf{2}}(\mathbf{b}) = \top^{\mathbf{2}} = \top^{[0,1]},$$

so s is an RDP-tautology. Conversely, if t is not a Boolean tautology, say $t^{\mathbf{2}}(\mathbf{b}) = \bot^{\mathbf{2}}$ for $\mathbf{b} = (b_1, \dots, b_n) \in \{0,1\}^n$, since $r_i^{[0,1]}(\mathbf{b}) = b_i$ for all $i \in [n]$, we similarly have,

$$s^{[0,1]}(\mathbf{b}) = t^{[0,1]}(r_1^{[0,1]}(\mathbf{b}), \dots, r_n^{[0,1]}(\mathbf{b})) = t^{[0,1]}(\mathbf{b}) = t^{\mathbf{2}}(\mathbf{b}) = \bot^{\mathbf{2}} = \bot^{[0,1]},$$

so s is not an RDP-tautology. $\square$

Let r and s be MTL-terms over the variables $x_1, \dots, x_n$. The *local* deduction theorem of MTL-logic [7] states that for some $n \geq 1$,

$$r \vdash_{MTL} s \text{ if and only if } \vdash_{MTL} r^n \to s;$$

since the equation $x^3 = x^2$ holds in every WNM-algebra, the local deduction theorem holds in RDP-logic with $n = 2$, namely,

$$r \vdash_{RDP} s \text{ if and only if } \vdash_{RDP} r^2 \to s. \tag{19}$$

In this light, we say that RDP-logic *proves* s from r, in symbols $r \vdash_{RDP} s$, if $r^2 \to s$ is a theorem of RDP-logic.

3.1 Normal Forms

In this section, we compute *normal forms* for the elements of the free n-generated RDP-algebra F_n. The construction naturally generalizes disjunctive normal forms for the elements of the free n-generated Boolean algebra, exploiting the representation of F_n as the algebra of maximal antichains in the augmented forest of S_1^n specified by (17).

In the Boolean case, a minterm t over variables $x_1, \dots, x_n$ is a conjunction of the form $l_1 \wedge \cdots \wedge l_n$ where l_i is either the variable x_i or its negation $\neg x_i$, for $i \in [n]$; it is clear that t evaluates to 1 under exactly one assignment of the variables in $\{0, 1\}$. Therefore, it is possible to express every Boolean function of n variables as the disjunction of the minterms corresponding to the assignments of the variables that evaluate the function to 1.

This intuition smoothly migrates in the setting of the free n-generated RDP-algebra F_n, as follows. Let C be a maximal chain in the augmented forest of S_1^n, let c be a point in C, and let A' be the smallest maximal antichain in $\mathbf{A}_{S_1^n}$ satisfying $A' \cap C = c$. An n-ary *RDP-minterm* is an RDP-term t_c over the variables $x_1, \dots, x_n$ such that $t_c^{F_n} = A'$. Now, let A be any maximal antichain in $\mathbf{A}_{S_1^n}$, let $C_1, \dots, C_k$ be the maximal chains in $\mathbf{C}_{S_1^n}$, and let $A \cap C_i = c_i$ for $i \in [k]$. Then, the RDP-term

$$t_A = t_{c_1} \vee \cdots \vee t_{c_k} \tag{20}$$

provides the desired disjunctive normal form for A, indeed, $t_A^{F_n} = A$.

In light of the previous remark, it is sufficient to provide an explicit construction of the RDP-minterm t_c for every maximal chain $C \in \mathbf{C}_{S_1^n}$ and every $c \in C$.

Fix an RDP-chain $C = B_1 < \cdots < B_f < \cdots < B_k$ in $\mathbf{C}_{S_1^n}$, and let B_f be the fixpoint of C, where $f > 1$; if C has no fixpoint, we stipulate that $f = 0$. For $i = 1, \dots, f$, fix a point $z_i \in B_i$, and define the following RDP-terms:

(N1) $\xi_{B_i} = \bigwedge_{x \in B_i} \neg((z_i \leftrightarrow x) \to \neg(z_i \leftrightarrow x))$;

(N2) $\xi'_{B_i} = (z_{i+1} \to z_i) \to \neg(z_{i+1} \to z_i)$;

(N3) $\xi''_{B_i} = z_i \to \neg z_i$.

For $i = f+1, \dots, k$, fix a point $z_i \in B_i$, and define the following RDP-terms:

(I1) $\zeta_{B_i} = \bigwedge_{x \in B_i} (z_i \leftrightarrow x)$;

(I2) $\zeta'_{B_i} = (z_{i+1} \to z_i) \to z_{i+1}$ for $i < k$;

(I3) $\zeta''_{B_i} = \neg(z_i \to \neg z_i)$ for $i > 1$.

Example 8 ($n = 3$)**.** *We construct the terms in (N1)-(N3) and (I1)-(I3) picking two samples C in $\mathbf{C}_{S_1^3}$. The first sample is an RDP-chain C with fixpoint, $C = \bot\bar{x_2}\bar{x_3} < x_1 < \bar{x_1} < x_2 < x_3 < \top$. Fix $z_1 = \bot$, $z_2 = x_1$, $z_3 = \bar{x_1}$, $z_4 = x_2$, $z_5 = x_3$ and $z_6 = \top$. Then:*

(N1) $\xi_{\bot\bar{x_2}\bar{x_3}} = \neg((\bot \leftrightarrow \neg x_2) \to \neg(\bot \leftrightarrow \neg x_2)) \wedge \neg((\bot \leftrightarrow \neg x_3) \to \neg(\bot \leftrightarrow \neg x_3))$;

$\xi_{x_1} = \neg((x_1 \leftrightarrow x_1) \to \neg(x_1 \leftrightarrow x_1));$

$\xi_{\bar{x_1}} = \neg((\neg x_1 \leftrightarrow \neg x_1) \to \neg(\neg x_1 \leftrightarrow \neg x_1));$

(N2) $\xi'_{\bot\bar{x_2}\bar{x_3}} = (x_1 \to \bot) \to \neg(x_1 \to \bot);$

$\xi'_{x_1} = (\neg x_1 \to x_1) \to \neg(\neg x_1 \to x_1);$

$\xi'_{\bar{x_1}} = (x_2 \to \neg x_1) \to \neg(x_2 \to \neg x_1);$

(N3) $\xi''_{\bot\bar{x_2}\bar{x_3}} = \bot \to \top;$

$\xi''_{x_1} = x_1 \to \neg x_1;$

$\xi''_{\bar{x_1}} = \neg x_1 \to \neg\neg x_1.$

(I1) $\zeta_{x_2} = (x_2 \leftrightarrow x_2);$

$\zeta_{x_3} = (x_3 \leftrightarrow x_3);$

$\zeta_{\top} = (\top \leftrightarrow \top);$

(I2) $\zeta'_{x_2} = (x_3 \to x_2) \to x_3;$

$\zeta'_{x_3} = (\top \to x_3) \to \top;$

(I3) $\zeta''_{x_2} = \neg(x_2 \to \neg x_2);$

$\zeta''_{x_3} = \neg(x_3 \to \neg x_3).$

$\zeta''_{\top} = \neg(\top \to \bot).$

The second sample is an RDP-chain D with no fixpoint, $D = \bot\bar{x_1}\bar{x_2}\bar{x_3} < x_1 < x_2 < x_3 < \top$. Note that in this case, the terms (N1)-(N3) do not exist. Fix $z_1 = \bot$, $z_2 = x_1$, $z_3 = x_2$, $z_4 = x_3$ and $z_5 = \top$. Then:

(I1) $\zeta_{\bot\bar{x_1}\bar{x_2}\bar{x_3}} = (\bot \leftrightarrow \neg x_1) \wedge (\bot \leftrightarrow \neg x_2) \wedge (\bot \leftrightarrow \neg x_3);$

$\zeta_{x_1} = (x_1 \leftrightarrow x_1);$

$\zeta_{x_2} = (x_2 \leftrightarrow x_2);$

$\zeta_{x_3} = (x_3 \leftrightarrow x_3);$

$\zeta_{\top} = (\top \leftrightarrow \top);$

(I2) $\zeta'_{\bot\bar{x_1}\bar{x_2}\bar{x_3}} = (x_1 \to \bot) \to x_1;$

$\zeta'_{x_1} = (x_2 \to x_1) \to x_2;$

$\zeta'_{x_2} = (x_3 \to x_2) \to x_3;$

$\zeta'_{x_3} = (\top \to x_3) \to \top;$

(I3) $\zeta''_{x_1} = \neg(x_1 \to \neg x_1);$

$\zeta''_{x_2} = \neg(x_2 \to \neg x_2);$

$\zeta''_{x_3} = \neg(x_3 \to \neg x_3);$

$\zeta''_{\top} = \neg(\top \to \bot)$.

The following facts hold by direct computation of the value of the involved RDP-terms over the involved RDP-chains. First, we study how the terms in (N1)-(N3) and (I1)-(I3) behave on C.

Fact 1. *The terms in (N1)-(N3) and (I1)-(I3) evaluate to* $\max C$ *over* C.

Example 9 ($n = 3$)**.** *Let C be the RDP-chain in Example 8. For instance, we evaluate the term $\xi_{\bot\bar{x_2}\bar{x_3}}$ over C:*

$$\begin{aligned}\xi_{\bot\bar{x_2}\bar{x_3}}(C) = {} & \neg((\bot(C) \leftrightarrow \neg x_2(C)) \to \neg(\bot(C) \leftrightarrow \neg x_2(C))) \wedge \\ & \neg((\bot(C) \leftrightarrow \neg x_3(C)) \to \neg(\bot(C) \leftrightarrow \neg x_3(C))) \\ = {} & \neg((\top(C) \to \neg\top(C))) \wedge \neg((\top(C) \to \neg\top(C))) \\ = {} & \neg\bot(C) \wedge \neg\bot(C) \\ = {} & \neg\bot(C) \\ = {} & \top(C) = \max C.\end{aligned}$$

Also,

$$\begin{aligned}\zeta_{x_2}(C) &= (x_2 \leftrightarrow x_2) \\ &= (x_2 \to x_2) \wedge (x_2 \to x_2) \\ &= \top(C) = \max C.\end{aligned}$$

Next, we study how RDP-terms in (N1)-(N3) and (I1)-(I3) behave on an RDP-chain $C' \in \mathbf{C}_{S_1^n}$ different from C, entering an exhaustive case distinction.

The first case we consider is the following: Either C has a fixpoint B_f, C' has a fixpoint $B_{f'}$, and the first f' blocks of C' are equal to the first f blocks of C; or, C and C' have no fixpoint. In this case, by [2, Theorem 5.5], we have

Fact 2. *The terms in (N1)-(N3) and (I3) evaluate to* $\max C'$ *over* C'*; the terms in (I1) and (I2) evaluate to the smallest* $c' \in C'$ *such that* $c' \parallel \max C$ *in the augmented forest of* S_1^n.

Example 10 ($n = 3$)**.** *Let C be the RDP-chain in Example 8, and let $C' \in \mathbf{C}_{S_1^3}$ be the RDP-chain $\bot\bar{x_2}\bar{x_3} < x_1 < \bar{x_1} < x_3 < x_2 < \top$, so that C and C' share the downset of the fixpoint. Then, $\xi_{\bot\bar{x_2}\bar{x_3}}$ evaluates to $\max C'$ over C',*

$$\begin{aligned}\xi_{\bot\bar{x_2}\bar{x_3}}(C') = {} & \neg((\bot(C') \leftrightarrow \neg x_2(C')) \to \neg(\bot(C') \leftrightarrow \neg x_2(C'))) \wedge \\ & \neg((\bot(C') \leftrightarrow \neg x_3(C')) \to \neg(\bot(C') \leftrightarrow \neg x_3(C'))) \\ = {} & \neg((\top(C') \to \neg\top(C'))) \wedge \neg((\top(C') \to \neg\top(C'))) \\ = {} & \neg\bot(C') \wedge \neg\bot(C') \\ = {} & \neg\bot(C') \\ = {} & \top(C') = \max C';\end{aligned}$$

and, ζ'_{x_2} evaluates to the smallest $c' \in C'$ such that $c' \parallel \max C$, namely,

$$\begin{aligned}\zeta'_{x_2}(C') &= (x_3(C') \to x_2(C')) \to x_3(C') \\ &= \top(C') \to x_3(C') \\ &= x_3(C').\end{aligned}$$

The second case we consider is the following: Either C has a fixpoint B_f, C' has a fixpoint $B_{f'}$, and the first f' blocks of C' are not equal to the first f blocks of C; or, C has a fixpoint B_f, and C' has no fixpoint.

Fact 3. *At least one term in (N1)-(N3) or in (I3) evaluates to* $\min C'$ *over* C'.

Example 11 ($n = 3$)**.** *Let C be the RDP-chain in Example 8, and let $C' \in \mathbf{C}_{S_1^3}$ be the RDP-chain $\bot\bar{x_3} < x_1 < x_2\bar{x_2}\bar{x_1} < x_3 < \top$. Then, C and C' have fixpoint, but the downsets of the fixpoints is not equal. Indeed, $\xi_{\bot\bar{x_2}\bar{x_3}}$ evaluates to* $\min C'$ *over C',*

$$\begin{aligned}
\xi_{\bot\bar{x_2}\bar{x_3}}(C') = \neg((\bot(C') \leftrightarrow \neg x_2(C')) \rightarrow \neg(\bot(C') \leftrightarrow \neg x_2(C')))\wedge \\
\neg((\bot(C') \leftrightarrow \neg x_3(C')) \rightarrow \neg(\bot(C') \leftrightarrow \neg x_3(C'))) \\
= \neg((\bot(C') \rightarrow \neg\bot(C'))) \wedge \neg((\top(C') \rightarrow \neg\top(C'))) \\
= \neg\top(C') \wedge \neg\bot(C') \\
= \bot(C') \wedge \top(C') \\
= \bot(C') = \min C'.
\end{aligned}$$

The last case is where C has no fixpoint and C' has a fixpoint.

Fact 4. *At least one term in (I1)-(I3) evaluates to* $\min C'$ *over* C'.

Example 12 ($n = 3$)**.** *Let C and D be the RDP-chains in Example 8, so that C has a fixpoint and D has no fixpoint. Indeed, ζ''_{x_1}, defined in the second part of Example 8, evaluates to* $\min C$ *over C,*

$$\begin{aligned}
\zeta''_{x_1}(C) &= \neg(x_1(C) \rightarrow \neg x_1(C)) \\
&= \neg\top(C) \\
&= \bot(C) = \min C.
\end{aligned}$$

In light of the previous facts, we complete the construction of the RDP-minterm t_c, and prove its correctness.

If $c = B_1$, then $t_c = \bot$; otherwise, if $c = B$ and x_j belongs to B, we let

$$t_C = \bigwedge_{i=1}^{f} \xi_{B_i} \wedge \bigwedge_{i=1}^{f-1} \xi'_{B_i} \wedge \bigwedge_{i=1}^{f} \xi''_{B_i} \wedge \bigwedge_{i=f+1}^{k} \zeta_{B_i} \wedge \bigwedge_{i=f+1}^{k-1} \zeta'_{B_i} \wedge \bigwedge_{i=f+1}^{k} \zeta''_{B_i}, \tag{21}$$

and

$$t_c = x_j \wedge t_C. \tag{22}$$

Proposition 8. *Let $C \in \mathbf{C}_{S_1^n}$, let $c \in C$, and let $A \in \mathbf{A}_{S_1^n}$ be the smallest maximal antichain such that $A \cap C = c$. Then,*

$$t_c^{F_n} = A.$$

Proof. By Fact 1, $t_C(C) = \max C$ hence,

$$t_c^{F_n} \cap C = t_c(C) = (x_j \wedge t_C)(C) = x_j(C) \wedge t_C(C) = B \wedge B_k = c \wedge \max C = c.$$

Also, let $C' \in \mathbf{C}_{S_1^n}$ be different from C. Then, by either Fact 3, or Fact 4, or Fact 2, $t_C(C')$ evaluates to either $\min C'$ or to the smallest $c' \in C'$ such that $c' \parallel \max C$, and hence $c' \parallel c$, in the augmented forest of S_1^n. In both cases, $t_C(C') \leq x_j(C')$, so that $t_c(C') = t_C(C')$. Summarizing, for each $C' \in \mathbf{C}_{S_1^n}$ different from C, $t_c^{F_n} \cap C'$ is equal to the smallest $c' \in C'$ such that $c' \parallel c$ in the augmented forest of S_1^n.

□

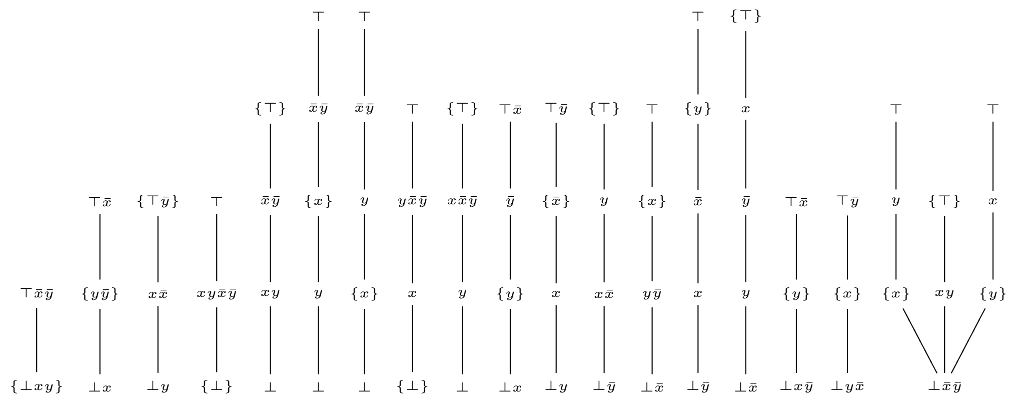


Figure 11: Sampling Proposition 8. The RDP-term $t(x, y) = t_{\perp xy} \vee t_{y\bar{y}} \vee t_{\top\bar{y}} \vee t_{\perp} \vee t_{\top} \vee t_x \vee t_x \vee t_{\perp} \vee t_{\top} \vee t_y \vee t_{\bar{x}} \vee t_{\top} \vee t_x \vee t_y \vee t_{\top} \vee t_y \vee t_x \vee t_x \vee t_{\top} \vee t_y$, is such that t^{F_2} is the maximal antichain highlighted (bracketed) in the labelled augmented forest S_1^2 in the figure.

3.2 Interpolation Properties

In this section, we prove that RDP-logic has the deductive interpolation property, and provide an explicit construction of strongest deductive interpolants.

Let X, Y, and Z be pairwise disjoint sets of variables. Let r and s be RDP-terms over $X \cup Z$ and $Y \cup Z$ respectively. The pair $r = x \wedge \neg x$ and $s = y \vee \neg y$ witnesses the failure of *Craig interpolation* in RDP-logic, as direct inspection of F_2 in Figure 9 shows: indeed, $\vdash_{RDP} r \to s$, but there not exists a ground term t such that $\vdash_{RDP} r \to t$ and $\vdash_{RDP} t \to s$. However, building upon the representation of free finitely generated RDP-algebras given in Section 2.3, and the construction of normal forms given in Section 3.1, we now provide a constructive proof that RDP-logic enjoys a weaker interpolation property, the *deductive interpolation* property: If $r \vdash_{RDP} s$, then there exists an RDP-term t over the variables Z such that $r \vdash_{RDP} t$ and $t \vdash_{RDP} s$. We describe an explicit construction of the *strongest* deductive interpolant t to r and s in RDP-logic, namely, a deductive interpolant t to r and s such that for every deductive interpolant t' to r and s, $t \vdash_{RDP} t'$.

For W a set of variables, we display the free $|W|$-generated RDP-algebra F_W as the RDP-algebra of labelled maximal antichains over the augmented forest of S_1^W discussed in the introduction of Section 3. If t is an RDP-term on W, we let $A_t \in \mathbf{A}_{S_1^W}$ denote the maximal labelled antichain F_W corresponding to t, that is, $t^{F_W} = A_t$. Let $V \subseteq W$. If $B \subseteq \{\perp, \top, x, \neg x \mid x \in W\}$, we let $B|_V = B \setminus \{x, \neg x \mid x \in W \setminus V\}$ denote the V-*structure* of B. Let $D = D_1 < \cdots < D_m \in \mathbf{C}_{S_1^V}$. Then, $C = C_1 < \cdots < C_n \in \mathbf{C}_{S_1^W}$ is said to be V-*equivalent* to D if $C_1|_V < \cdots < C_n|_V$, after eliminating empty blocks, is equal to $D_1 < \cdots < D_m$. Let $A' \in \mathbf{A}_{S_1^V}$. Then, $A \in \mathbf{A}_{S_1^W}$ is said the *cylindrification* of A' over $W \setminus V$ if for all $D \in \mathbf{C}_{S_1^V}$, for all $C \in \mathbf{C}_{S_1^W}$ V-equivalent to D, it holds that $(A \cap C)|_V = A' \cap D$; note that $A' \in \mathbf{A}_{S_1^V}$ guarantees that the right hand side of the equality is nonempty.

Assume $r \vdash_{RDP} s$, or equivalently, $\vdash_{RDP} r^2 \to s$, where r and s are specified as above. Let $W = X \cup Y \cup Z$. Then,

$$A_{r^2} \leq A_s$$

holds in F_W. Let A_t be the *smallest* maximal antichain in $\mathbf{A}_{S_1^Z}$ such that

$$A_{r^2} \leq A_t$$

holds in F_W; here, with slight abuse of notation, $A_t \in \mathbf{A}_{S_1^W}$ denotes the cylindrification of $A_t \in \mathbf{A}_{S_1^Z}$ over $X \cup Y$. We now show that A_t corresponds to the desired interpolant.

Claim 1. *$A_{t^2} \leq A_s$ in F_W.*

Proof. Suppose for a contradiction that $A_{t^2} \leq A_s$ does not hold in F_W. Then, there exists $C \in \mathbf{C}_{S_1^W}$ such that $A_{t^2} \cap C > A_s \cap C$ over C. By the choice of A_t, $A_t \cap C$ is the smallest point $d \in C$ such that $A_{r^2} \cap C \leq d$ and $d|_Z \neq \emptyset$; in words, d is the smallest point in C lying above $A_{r^2} \cap C$ and having nonempty Z-structure (otherwise, if $d' \in C$ is a point such that $A_{r^2} \cap C \leq d' < d$ and $d'|_Z \neq \emptyset$, the maximal antichain $A_{t'}$ such that $A_{t'} \cap D = d'$ for all maximal chains $D \in \mathbf{C}_{S_1^W}$ that are $X \cup Z$-equivalent to C, and equal to A_t otherwise, would satisfy $A_{r^2} \leq A_{t'} < A_t$, contradicting the minimality of A_t).

Observe that $\min C < A_{r^2} \cap C = A_r \cap C$: Indeed, if $\min C = A_{r^2} \cap C$, then $A_t \cap C = \min C$ (as $\min C$ has nonempty Z-structure, since $\perp \in \min C$); but $A_t \cap C = \min C$ implies $A_{t^2} \cap C = \min C$, contradiction with $A_{t^2} \cap C > A_s \cap C$. Moreover, $A_{r^2} \cap C < A_r \cap C$ implies $\min C = A_{r^2} \cap C$, again impossible along the above lines.

By the previous observation $A_{r^2} \cap C$ is idempotent, and since $A_{r^2} \cap C \leq A_t \cap C$ by the choice of A_t, we have $A_{t^2} \cap C = A_t \cap C$. The choice of $A_t \cap C$ is such that the right-open interval $\mathcal{I} = [A_{r^2} \cap C, A_{t^2} \cap C)$ in C has no Z-structure, that is, each point in the interval has empty Z-structure. Note that $A_{r^2} \cap C \leq A_s \cap C < A_{t^2} \cap C$ implies that $A_s \cap C$ lies in $\mathcal{I}$; also, by the observation in the previous paragraph, the interval $\mathcal{I}$ lies above the fixpoint of C if such fixpoint exists, or above $\min C$ if such fixpoint does not exists. Say that $\mathcal{I}$ has the form

$$A_{r^2} \cap C = B_1 < \cdots < B_n < A_{t^2} \cap C,$$

with $B_i = X_i \cup Y_i$, where X_i and Y_i denote the X-structure and the Y-structure of B_i respectively, for $i \in [n]$; note that $\perp \notin B_1$ and $\top \notin B_n$, as $\mathcal{I}$ lies above the bottom of C and below $A_{t^2} \cap C \leq \max C$, thus the X-structure and Y-structure of each B_i are disjoint. We know that $A_{r^2} \cap C = B_1$; suppose that $A_s \cap C = B_i$ for some $1 \leq i \leq n$. Let C' be the maximal chain in $\mathbf{C}_{S_1^W}$, obtained by replacing in C the interval $B_1 < \cdots < B_n$ with the interval (for instance)

$$Y_1 < \cdots < Y_i < \cdots < Y_n < X_1 < \cdots < X_n,$$

disregarding empty X_k's and Y_k's; by the above, Y_i and X_1 are nonempty. By construction, C' is $X \cup Z$-equivalent and $Y \cup Z$-equivalent to C. But then, $A_s \cap C' = Y_i < X_1 = A_{r^2} \cap C'$, contradiction with the fact that $A_{r^2} \leq A_s$ holds in F_W, and hence in particular over C'. $\square$

Therefore, $A_{r^2} \leq A_t$ by the choice of A_t, and $A_{t^2} \leq A_s$ by the claim. We use the normal forms construction in Section 3.1 to compute an RDP-term over variables in Z that corresponds to A_t; with slight abuse of notation, let t denote such term, that is, $t^{F_Z} = A_t$. We immediately have $\vdash_{RDP} r^2 \to t$ and $\vdash_{RDP} t^2 \to s$, and by (19), $r \vdash_{RDP} t$ and $t \vdash_{RDP} s$. So, t is a deductive interpolant to r and s in RDP-logic, in fact the strongest such, by the choice of A_t. Summarizing,

Theorem 5. *RDP-logic has the deductive interpolation property.* [12]

3.3 Unification Type

In this section, we prove that the variety of RDP-algebras has unitary unification type. If a given RDP-unification instance is solvable, we provide an explicit exponential-time construction of the most general RDP-unifier (which is likely to be optimal, since the problem in NP-hard).

Let T_n denote the RDP-algebra of terms over the variables $x_1, \ldots, x_n$. An instance to the RDP-unification problem is a term $t \in T_n$, and the question is whether there exists a *unifier* for t, that is, an endomorphism h of T_n such that

$$\vdash_{RDP} h(t).$$

A unifier h for $t \in T_n$ such that $h(x_i) \in \{\bot, \top\}$ for $i \in [n]$ is said *ground.*

Proposition 9. *Let $t \in T_n$. Then, t is unifiable if and only if t has a ground unifier.*

Proof. Let h be a unifier for t, and let C in $\mathbf{C}_{S_1^n}$ be the labelled maximal chain of the form $\{\bot, x_1, \ldots, x_n\} < \{\top, \neg x_1, \ldots, \neg x_n\}$. Let h' be the endomorphism of T_n such that, for $i \in [n]$,

$$h'(x_i) = \begin{cases} \bot & \text{if } \bot \in (h(x_i))(C), \\ \top & \text{if } \top \in (h(x_i))(C). \end{cases} \tag{23}$$

It is easy to check that h' is a ground unifier for t. The converse is trivial. $\square$

Let h and h' be unifiers for t. Then, h' is *less general* than h, in symbols $h' \leq h$, if there exists an endomorphism h'' of T_n such that

$$\vdash_{RDP} h'(x_i) \leftrightarrow h''(h(x_i))$$

for $i \in [n]$. A unifier h for t such that every unifier for t is less general than h is said a *most general* unifier for t.

In the rest of this section, we prove that the type of RDP-unification is *unitary*, that is, every unifiable RDP-term has a most general unifier. The proof provides an explicit construction of most general unifiers.

An RDP-term $t \in T_n$ is said to be *projective* if there exists a unifier h for t such that, for $i \in [n]$,

$$t \vdash_{RDP} x_i \leftrightarrow h(x_i). \tag{24}$$

[12] Equivalently, RDP-algebras enjoy the *injective generalized amalgamation* property [17].

Proposition 10. *Let $t \in T_n$. If t is projective, then t has a most general unifier.*

Proof. Suppose that t is projective with h witnessing (24), and let h' be a unifier for t. It is easy to check that $h' \leq h$. Indeed, by instantiating (24) through h', $h'(t) \vdash_{RDP} h'(x_i \leftrightarrow h(x_i))$; as h' commutes over the RDP-signature, $h'(t) \vdash_{RDP} h'(x_i) \leftrightarrow h'(h(x_i))$; as $\vdash_{RDP} h'(t)$, we conclude that $\vdash_{RDP} h'(x_i) \leftrightarrow h'(h(x_i))$. Therefore, h is a most general unifier for t. □

The following characterization of projectivity, which parallels the Boolean case, is key to prove that RDP-unification is unitary.

Lemma 3. *Let $t \in T_n$. Then, t is unifiable if and only if t is projective.*

Proof. Suppose that t is unifiable (the other direction is trivial). By Proposition 9, t has a ground unifier g. We prove that the endomorphism h_t of T_n such that, for $i \in [n]$,

$$h_t(x_i) = (t^2 \to x_i) \odot (\neg t^2 \to g(x_i)) \tag{25}$$

is a witnesses of the projectivity of t, and in fact, by Proposition 10, a most general unifier for t. [13]

Claim 2. *$\vdash_{RDP} h_t(t)$, that is, $(h_t(t))(C) = \max C$ for every $C \in \mathbf{C}_{S_1^n}$; and $\vdash_{RDP} t^2 \to (x_i \leftrightarrow h_t(x_i))$, that is $t^2(C) \leq (x_i \leftrightarrow h_t(x_i))(C)$ for every $C \in \mathbf{C}_{S_1^n}$.*

Proof. Let $C \in \mathbf{C}_{S_1^n}$. We enter a case distinction.

Case 1. Assume $\bot(C) = t(C)$ or $\bot(C) = t^2(C)$. In this case, for $i \in [n]$,

$$\begin{aligned}
(h_t(x_i))(C) &= ((t^2 \to x_i) \odot (\neg t^2 \to g(x_i)))(C) \\
&= (\bot(C) \to x_i(C)) \odot (\top(C) \to g(x_i)(C)) \\
&= \top(C) \odot g(x_i)(C) \\
&= g(x_i)(C).
\end{aligned}$$

Then, $(h_t(t))(C) = t(h_t(x_1), \ldots, h_t(x_n))(C) = t(g(x_1), \ldots, g(x_n))(C) = (g(t))(C) = \max C$, as g is a unifier for t. Clearly, $\bot(C) = t^2(C) \leq (x_i \leftrightarrow h_t(x_i))(C)$ for $i \in [n]$.

Case 2. Assume $t(C) = \top(C)$. In this case, for $i \in [n]$,

$$\begin{aligned}
(h_t(x_i))(C) &= ((t^2 \to x_i) \odot (\neg t^2 \to g(x_i)))(C) \\
&= (\top(C) \to x_i(C)) \odot (\bot(C) \to g(x_i)(C)) \\
&= x_i(C) \odot \top(C) \\
&= x_i(C).
\end{aligned}$$

Then, $(h_t(t))(C) = t(h_t(x_1), \ldots, h_t(x_n))(C) = t(x_1, \ldots, x_n)(C) = t(C) = \top(C) = \max C$. Also, $t^2(C) = \top(C) = (x_i \leftrightarrow h_t(x_i))(C)$ for $i \in [n]$.

[13] This application of (25) generalizes previous work of Dzik [10].

Case 3. Assume $\bot(C) < t^2(C) = t(C) < \top(C)$. We prove that, for $i \in [n]$,

$$(h_t(x_i))(C) = \begin{cases} x_i(C) & \text{if } x_i(C) < t(C), \\ \top(C) & \text{if } t(C) \leq x_i(C). \end{cases} \tag{26}$$

Suppose that $\bot(C) \leq x_i(C) < t(C)$. Then,

$$\begin{aligned} (h_t(x_i))(C) &= ((t^2 \to x_i) \odot (\neg t^2 \to g(x_i)))(C) \\ &= (t(C) \to x_i(C)) \odot (\neg t(C) \to g(x_i)(C)) \\ &= (t(C) \to x_i(C)) \odot (\bot(C) \to g(x_i)(C)) \\ &= x_i(C) \odot \top(C) \\ &= x_i(C). \end{aligned}$$

Now suppose that $\bot(C) < t(C) \leq x_i(C)$. Then,

$$\begin{aligned} (h_t(x_i))(C) &= ((t^2 \to x_i) \odot (\neg t^2 \to g(x_i)))(C) \\ &= (t(C) \to x_i(C)) \odot (\neg t(C) \to g(x_i)(C)) \\ &= \top(C) \odot (\bot(C) \to g(x_i)(C)) \\ &= \top(C) \odot \top(C) \\ &= \top(C). \end{aligned}$$

For the first part, we prove that $(h_t(t))(C) = \max C$. Suppose for a contradiction that $(h_t(t))(C) < \top(C)$. Now, $\bot(C) < t(C) < \top(C)$ implies $t(C) = x_i(C)$ or $t(C) = (\neg x_i)(C)$ for some $i \in [n]$. However, the first case does not occur (if $t(C) = x_i(C)$ for some $i \in [n]$, then $(h_t(t))(C) = (h_t(x_i))(C) = \top(C)$ by the above), therefore $t(C) = (\neg x_i)(C)$ for some $i \in [n]$. But $(\neg x_i)(C) < \top(C)$ implies $\bot(C) = ((\neg x_i)^2)(C)$, contradiction with $\bot(C) < t^2(C)$.

For the second part, we prove that $t^2(C) \leq (x_i \leftrightarrow h_t(x_i))(C)$. By (26), we distinguish two cases. Let $i \in [n]$. If $x_i(C) < t(C)$, then $(h_t(x_i))(C) = x_i(C)$ so that $t^2(C) \leq \top(C) = (x_i \leftrightarrow h_t(x_i))(C)$. If $t(C) \leq x_i(C)$, then $(h_t(x_i))(C) = \top(C)$ so that $x_i(C) \leq (x_i \leftrightarrow h_t(x_i))(C)$, and we are done noticing that $t^2(C) = t(C) \leq x_i(C)$.

The claim is settled. □

The lemma is settled. □

Theorem 6. *RDP-unification is unitary.*

Proof. Every RDP-term $t \in T_n$ has at most one most general unifier, indeed if t is unifiable, then t has a ground unifier by Proposition 9, then t is projective by Lemma 3, and hence, t has a most general unifier by Proposition 10. □

Note that the complexity of computing the most general unifier h for t via (25) is dominated by the complexity of computing the ground unifier g for t. It is easy to check that t has a ground unifier (as an RDP-term) if and only if t is satisfiable (as a Boolean term), hence, by Proposition 9, deciding the RDP-unification problem is NP-hard, and in fact, NP-complete: given a ground unifier h for t, it is sufficient to check if the equation $h(t) = \top$ holds.

Acknowledgments. The authors thank the anonymous reviewer for careful comments, and Stefano Aguzzoli and Vincenzo Marra for helpful discussions.

References

[1] S. Aguzzoli, M. Busaniche, and V. Marra. Spectral Duality for Finitely Generated Nilpotent Minimum Algebras, with Applications. *Journal of Logic and Computation*, 17(4):749–765, 2007.

[2] S. Aguzzoli and B. Gerla. Normal Forms and Free Algebras for Some Extensions of MTL. *Fuzzy Sets and Systems*, 159(10):1131–1152, 2008.

[3] S. Burris and H.P. Sankappanvar. *A Course in Universal Algebra*. Springer-Verlag, 1981.

[4] L.M. Cabrer and S.A. Celani. Priestley Dualities for Some Lattice-Ordered Algebraic Structures, Including MTL, IMTL and MV-algebras. *Central European Journal of Mathematics*, 4:600–623, 2006.

[5] S.A. Celani. Bounded Distributive Lattices with Fusion and Implication. *Southeast Asian Bulletin of Mathematics*, 28:999–1010, 2004.

[6] A. Ciabattoni, N. Galatos, and K. Terui. From Axioms to Analytic Rules in Nonclassical Logics. pages 229–240, 2008. In Proceedings of 23rd IEEE Symposium on Logic in Computer Science (LICS'08).

[7] J. Czelakowski. Logical Matrices and the Amalgamation Property. *Studia Logica*, 41(4):329–341, 1982.

[8] O. M. D'Antona and V. Marra. Computing Coproducts of Finitely Presented Gödel Algebras. *Annals of Pure and Applied Logic*, 142(1-3):202–211, 2006.

[9] B.A. Davey and H.A. Priestley. *Introduction to Lattices and Order*. Cambridge University Press, second edition, 2002.

[10] W. Dzik. Unification in Some Substructural Logics of BL-Algebras and Hoops. *Reports on Mathematical Logic*, 43:73–83, 2008.

[11] L. Esakia. Topological Kripke Models. *Soviet Mathematics Doklady*, 15:147–151, 1974.

[12] F. Esteva and L. Godo. Monoidal t-Norm Based Logic: Towards a Logic for Left-Continuous t-Norms. *Fuzzy Sets and Systems*, 124(3):271–288, 2001.

[13] N. Galatos, P. Jipsen, T. Kowalski, and H. Ono. *Residuated Lattices: An Algebraic Glimpse at Substructural Logics*. Elsevier, 2007.

[14] P. Hájek. *Metamathematics of Fuzzy Logic*. Kluwer, 1998.

[15] S. Jenei. A Note on the Ordinal Sum Theorem and its Consequence for the Construction of Triangular Norms. *Fuzzy Sets and Systems*, 126(2):199–205, 2002.

[16] S. Jenei and F. Montagna. A Proof of Standard Completeness for Esteva and Godo's Logic MTL. *Studia Logica*, 70(2):183–192, 2002.

[17] H. Kihara and H. Ono. Interpolation Properties, Beth Definability Properties and Amalgamation Properties for Substructural Logics. *Journal of Logic and Computation*, 2009.

[18] S. MacLane. *Categories for the Working Mathematician*. Springer-Verlag, second edition, 1998.

[19] C. Noguera, F. Esteva, and J. Gispert. On Triangular Norm Based Axiomatic Extensions of the Weak Nilpotent Minimum logic. *Mathematical Logic Quarterly*, 54(4):387–409, 2008.

[20] B. Schweizer and A. Sklar. Associative Functions and Abstract Semigroups. *Publicationes Mathematicae Debrecen*, 10:69–81, 1963.

[21] A. Urquhart. Duality for Algebras of Relevant Logics. *Studia Logica*, 56(1-2):263–276, 1996.

[22] D. Valota. Poset Representation for Free RDP-Algebras. In H. Hosni and F. Montagna, editors, *Probability, Uncertainty and Rationality*, volume 10 of *CRM Series*. Edizioni della Scuola Normale Superiore, Pisa 2010.

[23] S. Wang. A fuzzy logic for the revised drastic product t-norm. *Soft computing*, 11(6):585–590, 2007.