Combinatorics of Interpolation in Gödel Logic

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March 20, 2009

Abstract

We investigate the combinatorics of interpolation in Gödel logic, the propositional logic whose algebraic semantics is the variety of Heyting algebras generated by chains.

1 Motivation

In recent work, Busaniche and Mundici prove that Łukasiewicz logic has the Robinson property RP [4], exploiting the geometry of prime filters in the free finitely generated MV-algebra. In this note we recast their techniques in the combinatorial setting of Gödel logic, proving the RP for that logic (Theorem 1). As the algebraic semantics of Łukasiewicz logic and Gödel logic, respectively MV-algebras and Gödel algebras, are varieties of commutative residuated lattices, the RP is equivalent to the deductive interpolation property DIP [8].

The results presented can be obtained by the literature. Indeed, the RP of Gödel logic follows from the stronger, classical result, that Gödel logic has the Craig interpolation property CIP; this has been proved nonconstructively by Maksimova in 1977 [9], and constructively by Baaz and Veith in 1999 [2]. In the commutative case, the CIP is equivalent to a very strong version of the RP, the superRP; in general, the superRP implies a strong version of the RP, the strongRP, and the strongRP implies the RP; the latter is equivalent, in the commutative case, to the DIP [8]. Therefore, a constructive proof of the RP (equivalently, of the DIP) for Gödel logic is implicit in the aforementioned work of Baaz and Veith.

Nevertheless the methods we adopt, based on the finite structure of the free finitely generated Gödel algebra, are naturally related to the dual space of Gödel algebras [5], and are suitable for investigating consequence relations and interpolation properties of other substructural and many-valued logics [1]. In fact, an important motivation for us to investigate the combinatorics of the DIP in Gödel logic is related to Hájek's Basic logic BL, as we now explain. [1]

BL fails the CIP but, as Montagna proved nonconstructively in recent work, enjoys the DIP [10]. A natural development of the latter result, in fact asked for

[1] Hájek's Basic logic, BL, can be equivalently introduced as the logic of all continuous triangular norms and their residua (emphasizing its foundational rôle with respect to many-valued logics), or the logic of commutative bounded integral divisible prelinear residuated lattices, namely BL-algebras (emphasizing its pretty high positioning in the lattice of substructural logics). For background, we refer the reader to [7, 6].

by Montagna, is to provide an explicit construction of deductive interpolants in BL, together with an analysis of the computational complexity of the construction. Now, exploiting a nontrivial combination of geometrical techniques from Łukasiewicz logic and combinatorial techniques from Gödel logic, we recently obtained a concrete representation of the free finitely generated BL-algebra in terms of suitable real functions [3]. In our opinion, this semantics setting is a natural framework for the investigation of the consequence relation in BL, and in particular for attaining a constructive, direct proof of the DIP (together with a complexity bound). Since, in this perspective, the constructive DIP is obtained from the fusion of the geometry of Łukasiewicz logic and the combinatorics of Gödel logic, the present work complements the aforementioned work of Busaniche and Mundici, and prepares future work on BL.

2 Contribution

Gödel algebras are Heyting algebras satisfying the prelinearity equation $(x \to y) \vee (y \to x) = \top$; they form the algebraic semantics of Gödel logic (a.k.a. Dummet logic). Gödel logic can be regarded from an intuitionistic perspective, as the intermediate logic complete with respect to linear Kripke frames, or from a many-valued perspective, as the fuzzy logic complete with respect to the algebra $([0,1], \wedge, \to, \bot)$ of type $(2,2,0)$, where $x \wedge y = \min(x,y)$, $x \to y$ equals 1 if $x \leq y$ and y otherwise, and $\bot = 0$.

By universal algebraic facts, for every finite set X of variables, the free X-generated Gödel algebra G_X is isomorphic to the clone of term operations from $[0,1]^X$ to $[0,1]$ in the algebra $([0,1], \wedge, \to, \bot)$ above, equipped with the basic operations defined pointwise. The algebra G_X has the following, nice combinatorial description [1].

First, build the labelled forest S, as follows. [2] Step 0: For each possible ordered partition (B_1, B_2) of the set $X \cup \{0,1\}$ into two nonempty blocks, such that $0 \in B_1$ and $1 \in B_2$, create a node r with label (B_1, B_2). Step $1 \leq i \leq n+1$: Let v be a leaf of S at step $i-1$, labelled by $(B_1, \dots, B_m)$. For each possible ordered partition (B'_m, B''_m) of the block B_m into two nonempty blocks, such that $1 \in B''_m$, create a node v' with label $(B_1, \dots, B_{m-1}, B'_m, B''_m)$, and add an edge (v, v'). [3] Second, given S, build the forest S_X, whose domain is a multiset of subsets of $X \cup \{0,1\}$, as follows: Iterate over the leaves of S in some arbitrary but fixed order; let $(B_1, \dots, B_{j-1}, B_j, B_{j+1}, \dots, B_m)$ be the leaf of S addressed by the ith iteration; if $0 \leq j \leq m$ is the greatest index such that edges $(B_1, B_2), \dots, (B_{j-1}, B_j)$ are in S_X after the $(i-1)$th iteration, take new copies of $B_{j+1}, \dots, B_{m-1}, B_m$ and put edges $(B_j, B_{j+1}), \dots, (B_{m-1}, B_m)$ in S_X. See Figure 1 in Appendix A.

Let $\mathbf{A}_X$ denote the maximal antichains in S_X, and let $\mathbf{C}_X$ denote the maximal chains in S_X. The natural (chainwise) order over $\mathbf{A}_X$ yields a bounded lattice, with bottom $\bot_X = \{\min(C_X)\}_{C_X \in \mathbf{C}_X}$ and top $\top_X = \{\max(C_X)\}_{C_X \in \mathbf{C}_X}$.

Fact 1. *The free X-generated Gödel algebra G_X is isomorphic to the algebra over $\mathbf{A}_X$ where $\bot = \bot_X$ and, for every $A_X, A'_X \in \mathbf{A}_X$ and every $C_X \in \mathbf{C}_X$:*

[2] For background on posets, we refer the reader to any standard reference.

[3] The poset structure of S, far from being artificial, is extremely natural. It is related to the category that is dual to finite Gödel algebras and their homomorphisms, namely, forests and open order preserving maps [5].

$(A_X \wedge A'_X) \cap C_X = \min(A_X \cap C_X, A' \cap C_X)$; $(A_X \to A'_X) \cap C_X$ *equals* $\max(C_X)$ *if* $A_X \cap C_X \leq A'_X \cap C_X$ *and* $A'_X \cap C_X$ *otherwise.*

In particular, the variety of Gödel algebras is locally finite. The above combinatorial representation of free finitely generated Gödel algebras allows for a constructive proof of the CIP of Gödel logic.

We prepare the notation. Let $X \subseteq Y \subseteq Z$ be finite sets of variables. Given a maximal chain $C_Y \in \mathbf{C}_Y$, or a maximal antichain $A_Y \in \mathbf{A}_Y$, we want to identify the *projection* of C_Y, or A_Y, with respect to the variables in $Y \setminus X$, and the *cylindrification* of C_Y, or A_Y, with respect to the variables in $Z \setminus Y$. The formalism follows.

Notation 1. *Fix* $C_Y \in \mathbf{C}_Y$. *We write* C_X *for the maximal chain in* $\mathbf{C}_X$ *such that,* $P < P'$ *is in* C_X *iff there exist* $Q < Q'$ *in* C_Y *such that* $P = Q \cap X$ *and* $P' = Q' \cap X$. *We write* C_Z *for* any *maximal chain in* $\mathbf{C}_Z$ *such that, if* $P < P'$ *is in* C_Z, $P \cap Y \neq \emptyset$, *and* $P' \cap Y \neq \emptyset$, *then* $P \cap Y < P' \cap Y$ *is in* C_Y.

Fix $A_Y \in \mathbf{A}_Y$. *We write* A_X *for the maximal antichain in* $\mathbf{A}_X$ *such that, for every maximal chain* $C_X \in \mathbf{C}_X$ *and every cylindrification* C_Y *of* C_X *over* $Y \setminus X$, *it holds that* $A_X \cap C_X = (A_Y \cap C_Y) \cap X$. *We stipulate that* A_X *exists iff, for every fixed* $C_X \in \mathbf{C}_X$, *it holds that* $|\{(A_Y \cap C_Y) \cap X : C_Y \text{ cylindrification of } C_X \text{ over } Y \setminus X\}| = 1$. *We write* A_Z *for the maximal antichain in* $\mathbf{A}_Z$ *such that, for every maximal chain* $C_Y \in \mathbf{C}_Y$, *and every maximal chain* $C'_Z \in \mathbf{C}_Z$ *such that* $C'_Y = C_Y$, $(A_Z \cap C'_Z) \cap Y = A_Y \cap C_Y$. *This notation extends to subsets of* $\mathbf{A}_Y$, *as follows. Let* $H_Y \subseteq \mathbf{A}_Y$. *Then,* $H_X = \{A_X \in \mathbf{A}_X : A_Y \in H_Y\}$, *and* $H_Z = \{A_Z \in \mathbf{A}_Z : A_Y \in H_Y\}$.

Definition 1 (CIP)**.** *Gödel logic has the CIP iff, for every pair of finite sets* X *and* Y *of variables, with* $Z = X \cap Y$ *and* $W = X \cup Y$, *every* $A'_X \in \mathbf{A}_X$, *and every* $A''_Y \in \mathbf{A}_Y$, *if* $A'_W \to A''_W = \top_W$, *then there exists* $A_Z \in \mathbf{A}_Z$ *such that* $A'_W \to A_W = A_W \to A''_W = \top_W$.

In the combinatorial setting of Fact 1, we can readily construct the *strongest* interpolant A_Z to A'_X and A''_Y, that is, an interpolant A_Z such that, if $B_Z \leq A_Z \in \mathbf{A}_Z$ interpolates A'_X and A''_Y, then $B_Z = A_Z$. For a pictorial intutition of the construction, see Figure 2 in Appendix A.

Proposition 1. *Gödel logic has the CIP.*

The motivation previously discussed leads us to now consider a weaker property than the CIP, namely, the RP. We recall some terminology first.

A *(proper) filter* H_X in the free X-generated Gödel algebra is a (proper) subset of $\mathbf{A}_X$ such that $\top_X$ is in H_X and, if both A_X and $A_X \to A'_X$ are in H_X, then A'_X is in H_X. A filter H_X is *prime* if it is proper and, for every pair $A_X, A'_X \in \mathbf{A}_X$, either H_X contains $A_X \to A'_X$ or H_X contains $A'_X \to A_X$. By Fact 1, filters of free Gödel algebras have the following combinatorial structure.

Fact 2. *Let* $H_X \subseteq \mathbf{A}_X$. *(i)* H_X *is a filter iff there is* $A_X \in \mathbf{A}_X$ *such that* $H_X = \{A'_X \mid A_X \leq A'_X\}$. *Call* A_X *the* generator *of* H_X, *and write* $H_X = \langle A_X \rangle$. *(ii)* $H_X = \langle A_X \rangle$ *is prime iff, for a suitable choice of* $C_X \in \mathbf{C}_X$ *and* $B > \min(C_X)$ *in* C_X, A_X *is the lowest element in* $\mathbf{A}_X$ *such that* $A_X \cap C_X = B$. *Call* $B \in C_X$ *the* pivot *of* H_X. [4]

[4] Given a prime filter generator A_X, we can uniquely determine *the* pivot of A_X by imposing a linear order over subsets of $X \cup \{0, 1\}$.

Note that every filter H_X is trivially generated by $\bigwedge_{A_X \in H_X} A_X$. More intrinsically, every filter can be displayed as the intersection of finitely many prime filters, namely,

Fact 3. *For every filter H_X in G_X, there exist prime filters $\langle A_1 \rangle, \ldots, \langle A_k \rangle$ in G_X such that $H_X = \langle \bigvee_{i \in [k]} A_i \rangle$.*

We follow [4] to recast in Gödel logic the classical definition of the RP [8]. Below, X and Y are finite sets of variables, with $Z = X \cap Y$ and $W = X \cup Y$.

Definition 2 (prime RP, constructive prime RP)**.** *Gödel logic has the* (constructive) prime RP *iff, for every every pair H_X and I_Y of prime filters in G_X and G_Y respectively, if $H_Z = I_Z$, there is (a construction of) a prime filter J_W in G_W such that $J_X = H_X$ and $J_Y = I_Y$.*

Theorem 1. *Gödel logic has the constructive prime RP.*

For a pictorial intuition on the construction, see Figure 3 in Appendix A. Now the classical RP, that is, the property that for every pair $H_X = \langle A'_X \rangle$ and $I_Y = \langle A''_Y \rangle$ of filters in G_X and G_Y respectively, if $H_Z = I_Z$, then the filter $\langle A'_W \wedge A''_W \rangle$ in G_W is such that $\langle A'_W \wedge A''_W \rangle_X = H_X$ and $\langle A'_W \wedge A''_W \rangle_Y = I_Y$, follows as a corollary, exploiting Fact 3.

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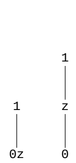
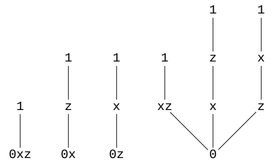
A Figures

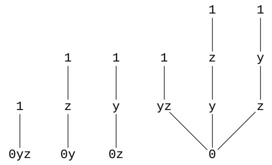
Figure 1: The labelled forests S_Z, S_X, S_Y, where $X = \{x, z\}$, $Y = \{y, z\}$, $Z = X \cap Y$, $W = X \cup Y$. Compare page 2.



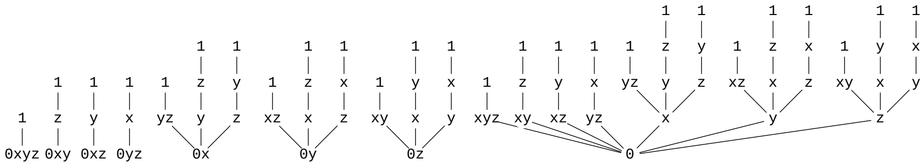
(a) S_Z.



(b) S_X.

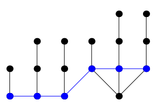


(c) S_Y.

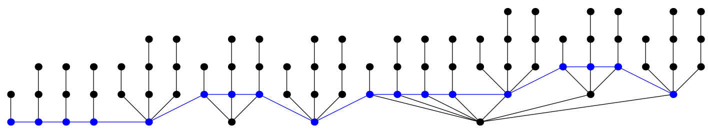


(d) S_W.

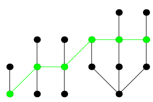
Figure 2: Sampling the CIP (Definition 1 and Proposition 1) with $X = \{x, z\}$ and $Y = \{y, z\}$. The labels of S_X, S_Y, S_Z, and S_W, here omitted, are as in Figure 1.



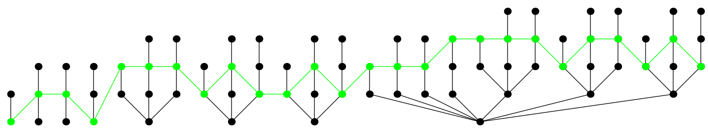
(a) $A'_X \in \mathbf{A}_X$.



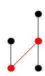
(b) $A'_W \in \mathbf{A}_W$.



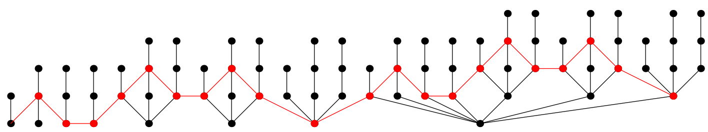
(c) $A''_Y \in \mathbf{A}_Y$.



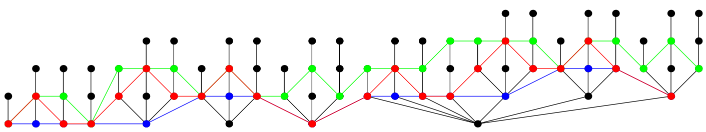
(d) $A''_W \in \mathbf{A}_W$.



(e) $A_Z \in \mathbf{A}_Z$.

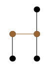


(f) $A_W \in \mathbf{A}_W$.

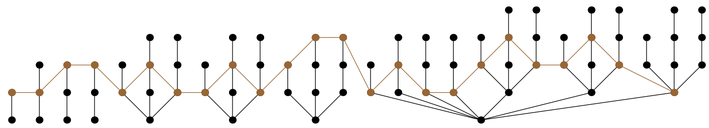


(g) $A'_W \to A''_W = \top_W$. A_Z interpolates A'_X and A''_Y.

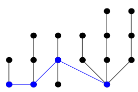
Figure 3: Sampling the constructive prime RP (Definition 2 and Theorem 1). The labels of S_X, S_Y, S_Z, and S_W are as in Figure 1. H_X and I_Y are the prime filters in G_X and G_Y respectively, generated by A'_X and A''_Y in (c) and (e). Note that $H_Z = I_Z = \langle A_Z \rangle$ for the A_Z in (a). The generator B_W of prime filter J_W in G_W such that $J_X = H_X$ and $J_Y = I_Y$ is the red antichain depicted in (g).



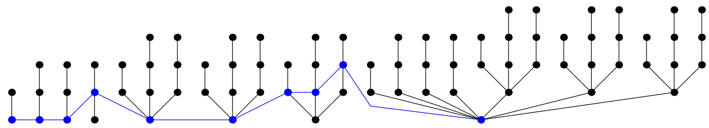
(a) $A_Z \in \mathbf{A}_Z$.



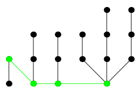
(b) $A_W \in \mathbf{A}_W$.



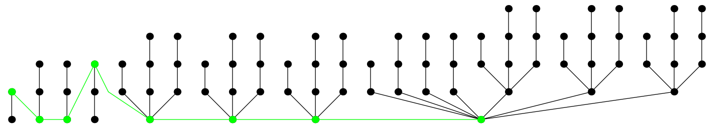
(c) $A'_X \in \mathbf{A}_X$.



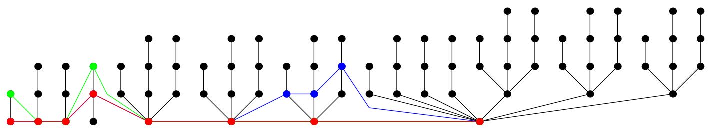
(d) $A'_W \in \mathbf{A}_W$.



(e) $A''_Y \in \mathbf{A}_Y$.



(f) $A''_W \in \mathbf{A}_W$.



(g) $B_W \in \mathbf{A}_W$, the red antichain, generates the desired J_W in G_W.