Applications of Finite Duality to Locally Finite Varieties of BL-algebras

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Abstract. We are concerned with the subvariety of commutative, bounded, and integral residuated lattices, satisfying divisibility and prelinearity, namely, BL-algebras. We give an explicit combinatorial description of the category that is dual to finite BL-algebras. Building on this, we obtain detailed structural information on the locally finite subvarieties of BL-algebras that are analogous to Grigolia's subvarieties of finite-valued MV-algebras. As an illustration of the power of the finite duality presented here, we give an exact recursive formula for the cardinality of free finitely generated algebras in such varieties.

Key words: BL-algebras, prime filters, dualities, free BL-algebras, subvarieties of BL-algebras, locally finite varieties.

1 Introduction

Hájek's Basic Logic BL [8] is the logic of all continuous triangular norms and their residua [4]. It is a fundamental object of study in the area of mathematical fuzzy logic, whose aim is to develop formal systems to make inferences in the presence of vagueness or uncertainty. The Lindenbaum-Tarski algebraic semantics of BL is given by the variety of BL-algebras, that is, commutative, bounded, integral residuated lattices satisfying divisibility and prelinearity.

To use such a tool as BL in practice, one needs to be able to manipulate BL-algebras effectively. In this direction, combinatorial representations of BL-algebras are of the foremost importance. In this paper we show that combinatorial representations are available for finite BL-algebras and locally finite[4] subvarieties of BL-algebras. Towards this aim, we shall introduce a full-fledged spectral duality for finite BL-algebras.

[4] A variety of algebras is locally finite if each finitely generated member of the variety is finite; equivalently, if finitely generated free algebras are finite.

It turns out that dual objects are finite *weighted* forests, that is, forests labeled with natural numbers. We define morphisms of weighted forests so as to provide a natural categorical equivalence with the opposite of the category of finite BL-algebras and their homomorphisms. Thus, any finite BL-algebra arises as the algebra of parts of a weighted forest, in an appropriate sense.

The combinatorial structure of forests allows us to effectively compute products and coproducts in the dual category, and then the duality affords a translation back to finite BL-algebras. This provides us with a powerful tool to extract structural information from finite BL-algebras. As an example of this machinery, we study the BL-algebraic analogous of Grigolia's subvarieties of finite-valued MV-algebras, obtaining the exact structure of their dual weighted forests, together with an exact recursive formula for the cardinality of free finitely generated algebras in such varieties.

2 Preliminaries

We write $\mathbb{N} = \{1, 2, \ldots\}$, $\setminus$ for set-theoretic difference, $a|b$ if $a, b \in \mathbb{N}$ and a divides b, and $|A|$ for the cardinality of the set A.

A *basic hoop* is an algebra $(A, \odot, \to, \wedge, \vee, \top)$ of type $(2, 2, 2, 2, 0)$ such that $(A, \odot, \top)$ is a commutative monoid, $(A, \wedge, \vee)$ is a lattice, and the following properties hold:

(*residuation*)	$x \odot y \leq z$ if and only if $x \leq y \to z$,
(*integrality*)	$x \wedge \top = x$,
(*divisibility*)	$x \wedge y = y \odot (y \to x)$,
(*prelinearity*)	$(x \to y) \vee (y \to x) = \top$;

equivalently, a basic hoop is a commutative, integral, divisible residuated lattice satisfying prelinearity. Note that basic hoops form a variety since residuation can be formulated by identities; see [8, 2.3.10]. A *Wajsberg hoop* is a basic hoop satisfying $\neg\neg x = x$. (Throughout, we use $\neg x$ as an abbreviation for $x \to \bot$.)

A *BL-algebra* is an algebra $(A, \odot, \to, \wedge, \vee, \bot, \top)$ of type $(2, 2, 2, 2, 0, 0)$ such that $(A, \odot, \to, \wedge, \vee, \top)$ is a *bounded* basic hoop, that is,

$$x \wedge \bot = \bot$$

holds. In each BL-algebra, the operations $\wedge$, $\vee$, and $\top$ are definable from the other operations, as follows: $x \wedge y = x \odot (x \to y)$, $x \vee y = ((x \to y) \to y) \wedge ((y \to x) \to x)$, and $\top = \bot \to \bot$; see [8, 2.1.10]. In the sequel, we shall therefore feel free to use the shorter signature $(A, \odot, \to, \bot)$ instead of the complete one, whenever convenient.

A BL-algebra is called: a *BL-chain*, if the reduct $(A, \wedge, \vee, \bot, \top)$ is totally ordered; a *Gödel algebra*, if it satisfies $x \odot x = x$ (elements of a BL-algebra satisfying $x \odot x = x$ are said to be *idempotent*); and an *MV-algebra* [3] if it satisfies $\neg\neg x = x$.

We shall make use of the *ordinal sum* construction. Let $\{(A_i, \odot_i, \to_i, \top_i)\}_{i \in I}$, for I a linearly ordered set with minimum 0, be a family of totally ordered

Wajsberg hoops such that $A_i \cap A_j = \{\top_i\} = \{\top_j\}$ for all $i, j \in I$, with A_0 bounded. Then the ordinal sum of the family $\{A_i\}_{i \in I}$ is the BL-algebra $(\bigcup_{i \in I} A_i, \odot, \to, \bot_0)$, where:

$$x \odot y = \begin{cases} x & \text{if } x \in A_i, y \in A_j, i < j \\ x \odot_i y & \text{if } x, y \in A_i \\ y & \text{if } x \in A_i, y \in A_j, j < i \end{cases}$$

$$x \to y = \begin{cases} \top_0 & \text{if } x \in A_i, y \in A_j, i < j \\ x \to_i y & \text{if } x, y \in A_i \\ y & \text{if } x \in A_i, y \in A_j, j < i \end{cases}$$

3 Spectral Duality for Finite BL-algebras

Fix a finite BL-algebra A. Recall that a *filter* of A is a nonempty upper set[5] $F \subseteq A$ that is closed under $\odot$. Further, a filter F of A is *prime* if $x \to y \in F$ or $y \to x \in F$ for each $x, y \in F$. We write $\operatorname{Spec} A$ for the (*prime*) *spectrum* of A, the set of prime filters of A partially ordered by reverse inclusion. Congruences θ of A are in bijection with filters F of A via $F = \{x \in A \mid (x, \top) \in \theta\}$. Prime filters precisely correspond to those congruences θ on A such that A/θ is a BL-chain. See [8, 2.3.14] for details.

For any finite BL-chain C we define the *top part* of C to be

$$T(C) = \{x \in C \mid x > c,\ c \text{ the largest idempotent below } \top\} .$$

The *weighted spectrum* of A is the function $\operatorname{wSpec} A \colon \operatorname{Spec} A \to \mathbb{N}$ such that

$$\mathfrak{p} \mapsto |T(A/\mathfrak{p})| ,$$

for every prime filter $\mathfrak{p} \in \operatorname{Spec} A$.

Throughout, *poset* means *finite* partially ordered set (with the partial order relation usually denoted by $\leq$). A *forest* is a poset such that the collection of lower bounds of any given element is totally ordered. A *weighted forest* is a function $w \colon F \to \mathbb{N}$, where F is a forest. Consider two weighted forests $w \colon F \to \mathbb{N}$, $w' \colon F' \to \mathbb{N}$. By a *morphism* $g \colon w \to w'$ we mean an order-preserving map $g \colon F \to F'$ that is

(M1) *open* (or is a *p-morphism*), i.e. whenever $x' \leq g(x)$ for $x' \in F'$ and $x \in F$, then there is $y \leq x$ in F such that $g(y) = x'$, and
(M2) *respects weights*, meaning that for each $x \in F$, there exists $y \leq x$ in F such that $g(y) = g(x)$ and $w'(g(y))$ divides $w(y)$.

[5] A *lower set* of a poset P is a subset D such that $x \in D$ and $y \leq x \in P$ imply $y \in D$. The smallest lower set containing a subset $S \subseteq P$ is denoted $\downarrow S$. *Upper sets* and the notation $\uparrow S$ are defined analogously.

Contemplation of these definitions shows that weighted forests and their morphisms form a category. Let us write WF for the latter category, and FBL for the category of finite BL-algebras and their homomorphisms.

It is possible to prove that $\mathrm{wSpec}\,A$ is a weighted *forest* for any finite BL-algebra A. In fact, wSpec can be turned into a contravariant functor from FBL to WF, as follows. Given a homomorphism $h\colon A \to B$ of finite BL-algebras, one proves that there is a function

$$\mathrm{Spec}\,h\colon \mathrm{Spec}\,B \to \mathrm{Spec}\,A \tag{1}$$

defined by

$$\mathfrak{p} \in \mathrm{Spec}\,B \mapsto h^{-1}(\mathfrak{p}) \in \mathrm{Spec}\,A \ . \tag{2}$$

Moreover, one checks that $\mathrm{Spec}\,h$ is an open order-preserving map from the forest $\mathrm{Spec}\,B$ to the forest $\mathrm{Spec}\,A$, and that it respects the weights of $\mathrm{wSpec}\,B$ and $\mathrm{wSpec}\,A$. Hence, $\mathrm{Spec}\,h$ defines a morphism

$$\mathrm{wSpec}\,h\colon \mathrm{wSpec}\,B \to \mathrm{wSpec}\,A$$

of weighted forests. Direct inspection now shows that wSpec sends identity maps to identity maps, and preserves composition. To sum up, wSpec is a contravariant functor from FBL to WF.

Conversely, we next construct a contravariant functor from weighted forests to finite BL-algebras. If F is a forest, a *subforest* of F is any lower set of F. If $w\colon F \to \mathbb{N}$ is a weighted forest, a *weighted subforest* of w is defined as any $w'\colon F' \to \mathbb{N}$ with F' a subforest of F such that $w'(x) \leq w(x)$ for all $x \in \max F'$, and $w'(x) = w(x)$ otherwise. We write $\mathrm{Sub}\,w$ for the set of all weighted subforests of w.

It turns out that $\mathrm{Sub}\,w$ carries a natural structure of BL-algebra, as follows. To begin with, writing $\emptyset\colon \emptyset \to \mathbb{N}$ for the unique empty weighted forest, we set $\bot = \emptyset$, and $\top = w$. To define $\odot$, consider subforests $u\colon U \to \mathbb{N}$ and $v\colon V \to \mathbb{N}$ of w. Define a function $a\colon U \cap V \to \mathbb{N} \cup \{0\}$ by

$$a(x) = \begin{cases} \max(0, u(x) + v(x) - w(x)) & \text{if } x \in \max U \cap \max V \\ u(x) & \text{if } x \in \max U \text{ and } x \notin \max V \\ v(x) & \text{if } x \notin \max U \text{ and } x \in \max V \\ w(x) & \text{otherwise,} \end{cases}$$

for each $x \in U \cap V$. Let $E = \{x \in U \cap V \mid a(x) > 0\}$, and define $u \odot v\colon E \to \mathbb{N}$ by the restriction $u \odot v = a \restriction E$.

Turning to implication, we define $u \to v$. First, we set

$$\begin{aligned} A &= F \setminus \uparrow (U \setminus V) \ , \\ B &= \{x \mid x \in \max U \cap \max V \ \text{ and } \ u(x) > v(x)\} \ . \end{aligned}$$

Then we set $E = (A \setminus \uparrow B) \cup B$. We define $(u \to v)\colon E \to \mathbb{N}$ by

$$(u \to v)(x) = \begin{cases} v(x) + w(x) - u(x) & \text{if } x \in B \\ v(x) & \text{if } x \in (U \cap V) \setminus (\max U \cap \max V) \\ w(x) - u(x) & \text{if } x \in \min U \setminus V \\ w(x) & \text{otherwise,} \end{cases}$$

for each $x \in E$.

Lattice operations $u \vee v\colon U \cup V \to \mathbb{N}$ and $u \wedge v\colon U \cap V \to \mathbb{N}$ turn out to be as follows:

$$(u \vee v)(x) = \begin{cases} \max(u(x), v(x)) & \text{if } x \in U \text{ and } x \in V \\ u(x) & \text{if } x \in U \text{ and } x \notin V \\ v(x) & \text{if } x \notin U \text{ and } x \in V \end{cases}$$

for each $x \in U \cup V$, and

$$(u \wedge v)(x) = \begin{cases} \min(u(x), v(x)) & \text{if } x \in \max U \cap \max V \\ u(x) & \text{if } x \in \max U \text{ and } x \notin \max V \\ v(x) & \text{if } x \notin \max U \text{ and } x \in \max V \\ w(x) & \text{otherwise,} \end{cases}$$

for each $x \in U \cap V$, respectively.

It can now be proved that for any weighted forest $w\colon F \to \mathbb{N}$, the algebraic structure

$$(\operatorname{Sub} w, \odot, \to, \wedge, \vee, \bot, \top)$$

is a (finite) BL-algebra. To turn Sub into a contravariant functor from WF to FBL, we take inverse images again. Namely, if $g\colon w \to w'$ is a morphism between the weighted forests $w\colon F \to \mathbb{N}$ and $w'\colon F' \to \mathbb{N}$, we define $\operatorname{Sub} g\colon \operatorname{Sub} F' \to \operatorname{Sub} F$ by

$$U \in \operatorname{Sub} F' \mapsto g^{-1}(U) \in \operatorname{Sub} F \ .$$

One can prove that $\operatorname{Sub} g$ so defined is a homomorphism of BL-algebras. To sum up, Sub is a contravariant functor from WF to FBL.

Finally, one can prove that wSpec and Sub yield a duality. Here we omit the proof for space constraints.

Theorem 1 (Finite Duality). *The category of finite BL-algebras and their homomorphisms is dually equivalent to the category of weighted forests and their morphisms. That is, the composite functors* wSpec $\circ$ Sub *and* Sub $\circ$ wSpec *are naturally isomorphic to the identity functors on* WF *and* FBL*, respectively.*

In particular, by [10, Thm. IV.4.1] the functor wSpec is essentially surjective, and this yields the following representation theorem for finite BL-algebras.

Corollary 1. *Any finite BL-algebra is isomorphic to* $(\mathrm{Sub}\, w, \odot, \to, \wedge, \vee, \bot, \top)$*, for a weighted forest* $w\colon F \to \mathbb{N}$ *that is unique to within an isomorphism of weighted forests.*

While the previous corollary has already been proved in [6, §5] and, as a special case of a more general construction, in [9, §6], the finite duality theorem is a novelty. In the rest of the paper, we illustrate the potential of this duality for the investigation, and possibly the classification, of locally finite subvarieties of BL-algebras.

4 Grigolia's Subvarieties of BL-algebras

In the variety of BL-algebras, we adopt the abbreviation $x \oplus y$ for the binary term operation

$$((x \to (x \odot y)) \to y) \vee ((y \to (y \odot x)) \to x)\,.$$

In each MV-algebra, one has $((x \to (x \odot y)) \to y) \vee ((y \to (y \odot x)) \to x) = \neg(\neg x \odot \neg y)$, that is, $\oplus$ coincides with the Łukasiewicz sum. Thus, our usage of $\oplus$ is consistent with standard MV-algebraic notation. Further, it is an exercise to check that, in every BL-algebra, the operation $\oplus$ is commutative, associative, and satisfies $x \oplus \top = \top$; cf. [2, Definition 2.2]. Thus, we can consistently shorten $x \oplus x \oplus \cdots \oplus x$ to hx, and similarly $x \odot x \odot \cdots \odot x$ to x^h, where in both cases x occurs h many times, for $h > 0$ an integer. Finally, we set $x^0 = \top$ and $0x = \bot$.

In [7, pag. 81–82], Grigolia axiomatized the variety $\mathbb{MV}_k$ generated by the k-element MV-chain Ł_k, for each integer $k \geq 2$, extending the axioms for MV-algebras by the following axiom schemata.[6]

(G1) $x^k = x^{k-1}$,
(Gh) $k(x^h) = (h(x^{h-1}))^k$, for every integer $2 \leq h \leq k-2$ that does not divide $k-1$.

For a given $k \geq 2$, we define $\mathbb{BL}_k$ to be the variety of BL-algebras satisfying (G1–Gh), for all integers h such that $2 \leq h \leq k-2$, and such that h is not a divisor of $k-1$.

Note that $\mathbb{BL}_k$ contains the variety of Gödel algebras. Indeed, one checks that each Gödel algebra satisfies $x^h = x^k$ and $hx = kx$ for every $h, k > 0$, so that axioms (G1) and (Gh) boil down to $x = x$.

For $k, l \in \mathbb{N}$, we write B_k^l to denote the ordinal sum of l copies of Ł_k.

Lemma 1. *Fix* $k \geq 2$.

(1) *The variety* $\mathbb{BL}_k$ *is generated by* $\{B_k^l \mid l \in \mathbb{N}\}$.
(2) *The variety* $\mathbb{BL}_k$ *is locally finite.*
(3) *For a finite BL-algebra* A*, the following are equivalent.*
 (i) $A \in \mathbb{BL}_k$.

[6] Here, Grigolia's axioms are presented in the version adopted in [3, 8.5.1].

(ii) $\mathrm{wSpec}\, A$ *has range included in the set of divisors of* $k-1$.

Proof. (1) Suppose a term $\tau(X_1, \ldots, X_n)$ in the language of BL-algebras fails — i.e., evaluates to an element $\neq \top$ — in a BL-algebra B lying in $\mathbb{BL}_k$. Since each BL-algebra is a subdirect product of BL-chains ([8, 2.3.16]), it is safe to assume that B is a chain. By [1, Theorem 3.7], B is an ordinal sum of totally ordered Wajsberg hoops, the first of which is bounded (equivalently, is an MV-chain). Moreover, τ fails in a finitely generated subalgebra of B. Since a finitely generated subalgebra of an ordinal sum of Wajsberg hoops is an ordinal sum of finitely many components, we may assume that B is a finite ordinal sum of Wajsberg hoops W_i, $i = 1, \ldots, l$, with W_1 an MV-chain. Either by direct inspection, or by the argument in [11, Theorem 1], one sees it is safe to assume $l = n+1$. Since B satisfies (G1), if $x \in W_i \setminus \{\top\}$ then x^{k-1} is idempotent. But then x^{k-1} must be the bottom of W_i. Indeed, it is well-known that the only idempotents of linearly ordered Wajsberg hoops are the top and (when it exists) the bottom element. Therefore, each W_i is an MV-chain. Since by hypothesis this MV-chain satisfies (Gh) and (Gh) is $\perp$-free, $h \geq 1$, it follows that each W_i lies in $\mathbb{MV}_k$. In conclusion, since $\mathbb{MV}_k$ is generated by $Ł_k$, τ fails in the ordinal sum of $n+1$ copies of $Ł_k$, as was to be shown.

(2) This is an immediate consequence of the fact that $\mathbb{MV}_k$ is locally finite by [3, 8.6.1], along with the observation that an n-generated BL-chain is the ordinal sum of at most $n+1$ summands.

(3) (i) $\Rightarrow$ (ii). If $\mathfrak{p}$ is a prime filter of A, the top part $T(A/\mathfrak{p})$ of the BL-chain $A/\mathfrak{p}$ can be made into an MV-chain lying in $\mathbb{MV}_k$ by adding a bottom element to it, and extending the operations in the only possible way. If such an MV-chain has cardinality c, then $c-1$ divides $k-1$, hence (ii) follows.
(ii) $\Rightarrow$ (i). Suppose $A \notin \mathbb{BL}_k$. Then there is a prime filter $\mathfrak{p}$ of A such that the BL-chain $A/\mathfrak{p}$ does not lie in $\mathbb{BL}_k$. Equivalently, $A/\mathfrak{p}$ is an ordinal sum of finitely many finite MV-chains $Ł_{c_i}$, $i = 1, \ldots, u$, and there exists $j \in \{1, \ldots, u\}$ such that $Ł_{c_j}$ does not lie in $\mathbb{MV}_k$. The latter condition means that $c_j - 1$ does not divide $k-1$. Let $\mathfrak{q}$ be the prime filter of $A/\mathfrak{p}$ generated by the bottom of $Ł_{c_{j+1}}$, if $j < u$; otherwise, let $\mathfrak{q}$ be the trivial filter $\{\top\}$ of $A/\mathfrak{p}$. Now $|T((A/\mathfrak{p})/\mathfrak{q})|$ does not divide $k-1$ by construction, and $|T((A/\mathfrak{p})/\mathfrak{q})|$ is in the range of $\mathrm{wSpec}\, A$ by the isomorphism theorems.

5 The Weighted Spectrum of Free Algebras in $\mathbb{BL}_k$

We write $\mathrm{Free}_{n,k}$ for the free n-generated algebra in $\mathbb{BL}_k$, for $k \geq 2$ and $n \geq 0$ an integer. By [3, 8.6.1], the free n-generated MV-algebra in $\mathbb{MV}_k$ is given by the direct product

$$\mathrm{FreeMV}_{n,k} = \prod_{d \,|\, (k-1)} Ł_{d+1}^{\alpha(n,d)} \;, \tag{3}$$

where $d \in \mathbb{N}$, $\alpha(0,d)$ is 0 if $d > 1$ and 1 if $d = 1$, and, for $n \geq 1$,

$$\alpha(n,d) = (d+1)^n + \sum_{\emptyset \neq X \subseteq \mathrm{PrDiv}\,(d)} (-1)^{|X|} (\gcd X + 1)^n \;, \tag{4}$$

where $\mathrm{PrDiv}\,(d)$ is the set of coatoms in the lattice of divisors of d. Geometrically, $\alpha(n,d)$ counts the number of points in $[0,1]^n$ whose denominator is d.[7] We now define a variant of α. We let $\beta(0,d)=0$ if $d>1$ and 1 if $d=1$, and, for $n\geq 1$,

$$\beta(n,d)=d^n+\sum_{\emptyset\neq X\subseteq \mathrm{PrDiv}\,(d)}(-1)^{|X|}(\gcd X)^n\ . \tag{5}$$

Geometrically, $\beta(n,d)$ counts the number of points in $[0,1)^n$ whose denominator is d — in other words, β does not take into account those points of $[0,1]^n$ having at least one coordinate set to 1. Compare Figure 1 for an example. Notice that letting $L(d)=\{\,(e_1,e_2)\mid \mathrm{lcm}\,(e_1,e_2)=d\,\}$, where $\mathrm{lcm}\,(a,b)$ denotes the least common multiple of integers a and b, we have

$$\sum_{(e_1,e_2)\in L(d)}\beta(h,e_1)\beta(k,e_2)=\beta(h+k,d)\,. \tag{6}$$

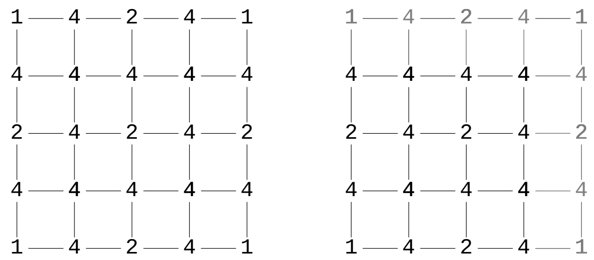

Fig. 1. $\alpha(1,4)=4$, $\alpha(2,4)=5$, $\alpha(4,4)=16$ (left), and $\beta(1,4)=1$, $\beta(2,4)=3$, $\beta(4,4)=12$ (right).

We next use (5) to define a family of weighted forests that shall be proved dual to free algebras in $\mathbb{BL}_k$. To this purpose, we introduce some additional notation. If U is a forest we write $U_\perp$ for the forest obtained from U by adding a new bottom element $\perp$. Further, if $u\colon U\to\mathbb{N}$ is a weighted forest, we write $u_\perp$ for the weighted forest having $U_\perp$ as domain, and such that $u_\perp$ agrees with u over U, and $u_\perp(\perp)=1$. It is a standard fact that varieties of algebras are both complete and cocomplete. If, moreover, $\mathbb{V}$ is a subvariety of the variety $\mathbb{W}$, then a product of $\mathbb{V}$-algebras computed in $\mathbb{V}$ coincides with the same product computed in $\mathbb{W}$. Then Theorem 1 implies at once that WF has all finite coproducts. We write $u+v$ for the coproduct of the weighted forests $u\colon U\to\mathbb{N}$ and $v\colon V\to\mathbb{N}$. Clearly, since products in varieties of algebras are Cartesian, we have $u+v\colon U+V\to\mathbb{N}$, where

[7] The *denominator* of a rational point $(r_1/s_1,\ldots,r_n/s_n)$, where $r_i,s_i\geq 0$ are integers such that $s_i\neq 0$ and r_i and s_i are relatively prime, is the least common multiple of the set s_i's.

$U+V$ is the disjoint union of U and V, and $u+v$ agrees with u on U, and with v on V. Thus, up to an isomorphism, any weighted forest can be written as a coproduct $\sum_{i=1}^{u} T_i$ of weighted trees in essentially just one way. Here, as usual, a *tree* is a forest with a minimum element.

For coproducts in varieties and subvarieties the situation is generally not as simple. In our case, however, we have the following properties (proofs of the following two lemmas are omitted for space constraints).

Lemma 2. (i) *The category* WF *has all finite products.* (ii) *If* u, v *are weighted forests, and* $u \times v$ *is their product with projections* $\pi_u \colon u \times v \to u$ *and* $\pi_v \colon u \times v \to v$*, then* $\mathrm{Sub}\, u \times v$*, along with the injections* $\mathrm{Sub}\, \pi_u$*,* $\mathrm{Sub}\, \pi_v$*, is the coproduct of* $\mathrm{Sub}\, u$ *and* $\mathrm{Sub}\, v$ *in the category of all BL-algebras.* (iii) *For each integer* $k \geq 2$*, coproducts computed in* $\mathbb{BL}_k$ *coincide with coproducts computed in the category of all BL-algebras.* (iv) *For any three weighted forests* u, v, w*, we have* $u \times (v+w) \cong (u \times v) + (u \times w)$.

Note that, using (iv) in Lemma 2, we obtain the binomial expansion

$$(u+v)^m \cong \sum_{i=0}^{m} \binom{m}{i} u^i v^{m-i} \ ,$$

for any two weighted forests u, v. Here and in the sequel, in the expressions involving products and coproducts we adopt the standard notation of elementary arithmetic.

We now describe the finite products in the category WF. Let $v \colon F_v \to \mathbb{N}$ and $w \colon F_w \to \mathbb{N}$ be two weighted forests. By $F_v \times F_w$ we mean the product of the underlying forests as described in [5]. We further denote by $\pi_v \colon (F_v \times F_w) \to F_v$ and $\pi_w \colon (F_v \times F_w) \to F_w$ the associated projections. The product of v and w in WF is the function $(v \times w) \colon (F_v \times F_w) \to \mathbb{N}$ together with the projections π_v and π_w, defined as follows.

Pick $p \in F_v$ and $q \in F_w$. In case $p \notin \min F_v$ and $q \notin \min F_w$ there are exactly three disjoint classes of points in $F_v \times F_w$, denoted by $(p|q)$, (p,q), $(q|p)$, such that all points in them project through π_v to p and through π_w to q. Further, any point in $F_v \times F_w$ that projects to p and q, respectively, falls into one of these classes. In particular, by [5], each point in $(p|q)$ is such that its predecessor in $F_v \times F_w$ projects to p through π_v and to the predecessor of q in F_w through π_w. The case of the class of points $(q|p)$ is symmetric. Each point in (p,q) is such that its predecessor in $F_v \times F_w$ projects to the respective predecessors through both projections.

If exactly one point in $\{p, q\}$ is minimal in the forest it belongs, say $p \in \min F_v$, then there is exactly one point in $F_v \times F_w$, precisely in $(p|q)$, such that its predecessor projects to p through π_v and to the predecessor of q in F_w through π_w. If $p \in \min F_v$ and $q \in \min F_w$ then there is exactly one point in $F_v \times F_w$, classed in (p,q), such that $\pi_v(p,q) = p$ and $\pi_w(p,q) = q$. Moreover, $(p,q) \in \min(F_v \times F_w)$.

Let $r \in F_v \times F_w$ be such that $\pi_v(r) = p$ and $\pi_w(r) = q$. Then

$$(v \times w)(r) = \begin{cases} w(q) & \text{if } r \in (p|q) \\ v(p) & \text{if } r \in (q|p) \\ \mathrm{lcm}\,(v(p), w(q))) & \text{if } r \in (p, q) \end{cases}$$

where $\mathrm{lcm}\,(a, b)$ denotes the least common multiple of $a, b \in \mathbb{N}$.

Throughout, let P_d denote the weighted tree consisting of just one point having weight d, for $d \in \mathbb{N}$. The description of finite products in WF given above allows to prove the following properties.

Lemma 3. *Let $u_\perp$ and $v_\perp$ be two trees in WF. If $u \neq \emptyset \neq v$ then*

$$u_\perp \times v_\perp \cong (u \times v_\perp + u \times v + u_\perp \times v)_\perp \, .$$

Further, $P_d P_e \cong P_{\mathrm{lcm}\,(d,e)}$, $P_d u_\perp \times P_e v_\perp \cong P_{\mathrm{lcm}\,(d,e)}(u_\perp \times v_\perp)$, $P_1 u_\perp \cong u_\perp$.

For $k \geq 2$, we introduce the following definitions:

- $M_k^0 = P_1$.
- For each integer $n \geq 1$,

$$M_k^n = \sum_{d|(k-1)} \beta(n, d) P_d \, .$$

Note in particular that if $k-1$ is prime, or equal to 1, then $M_k^1 = P_1 + (k-2)P_{k-1}$.

Lemma 4. *Fix $k \geq 2$. Set $F_k^1 = M_k^1 + (M_k^1)_\perp$. Then*

$$\mathrm{wSpec}\,\mathrm{Free}_{1,k} \cong F_k^1 \, .$$

Proof. If S is any set and B is a BL-algebra, let us write B^S for the BL-algebra of all functions $S \to B$ endowed with the operations inherited pointwise from B. By the argument proving (1) in Lemma 1, we know that if a term $\tau(X_1)$ in the language of BL-algebras fails in some BL-algebra lying in $\mathbb{BL}_k$, then it must fail in B_k^2. Hence, by standard universal-algebraic considerations, $\mathrm{Free}_{1,k}$ is (isomorphic to) the subalgebra of $(B_k^2)^{B_k^2}$ that is generated by the identity function, the latter being a free generator. Let us write $C \cong \text{Ł}_k$ and $D \cong \text{Ł}_k$ for the first and second summand of B_k^2, respectively, and b for the bottom element of D (i.e., the unique idempotent of B_k^2 besides top and bottom). A trivial structural induction shows that any element $f \in \mathrm{Free}_{1,k}$ is such that (i) $f(p/q) = r/q$ for p/q an irreducible fraction in $[0, 2]$ such that q divides $k-1$; (ii) $f(c) \in C$ for any $c \in C$; (iii) $f(\top) = \perp$ implies $f(d) = \perp$, while $f(\top) = \top$ implies $f(d) \geq b$, for any $d \in D$. Conversely, let $f \in (B_k^2)^{B_k^2}$ satisfy (i–iii) above. A straightforward adaptation of [11, Thm. 2] shows that $f \in \mathrm{Free}_{1,k}$. As an immediate consequence of this representation of $\mathrm{Free}_{1,k}$ it follows that $\mathrm{Spec}\,\mathrm{Free}_{1,k}$ is isomorphic to the underlying forest of F_k^1. A further computation confirms that $\mathrm{wSpec}\,\mathrm{Free}_{1,k}$ is isomorphic to F_k^1. As a matter of fact, each prime filter of $\mathrm{Free}_{1,k}$ is singly generated by a function $f \in \mathrm{Free}_{1,k}$ of one of the following three types: (1) there exists $c \in C \setminus \{\top\}$ such that $f(c) = \top$ and $f(a) = 0$ for all $c \neq a \in B_k^2$; (2) there exists $d \in D \setminus \{\top\}$ such that $f(d) = f(\top) = \top$, while $f(e) = b$ for all $e \in D \setminus \{d, \top\}$, and $f(c) = 0$ for all $c \in C \setminus \{\top\}$; (3) $f(\top) = \top$, $f(d) = b$ for all $d \in D \setminus \{\top\}$, and $f(c) = 0$ for all $c \in C \setminus \{\top\}$. Notice that the only filter of type (3) includes all filters of type (2) and no other inclusions hold in $\mathrm{wSpec}\,\mathrm{Free}_{1,k}$.

Example 1. Figure 2 displays the weighted forest F_3^1 (to the left), and the weighted forest F_7^1 (to the right).

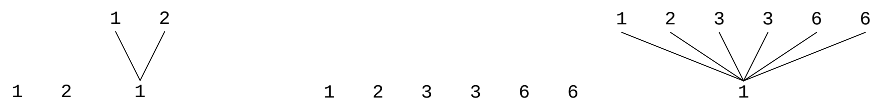

Fig. 2. F_3^1 (left) and F_7^1 (right).

For each $k \geq 2$ and each integer $n \geq 1$, let us define

$$F_k^n = \underbrace{F_k^1 \times \cdots \times F_k^1}_{n \text{ times}} = (F_k^1)^n \ .$$

Lemma 5. *Fix $k \geq 2$, and for each integer $n \geq 0$,*

$$F_k^n \cong \mathrm{wSpec}\,\mathrm{Free}_{n,k} \ .$$

Proof. By standard universal algebra, in any variety the free algebra on κ free generators, for κ a cardinal, is isomorphic to the copower of κ-many copies of the free algebra on one generator. Thus, $\mathrm{Free}_{n,k} \cong \sum_{i=1}^{n} \mathrm{Free}_{1,k}$, where the right-hand side coproduct is computed in $\mathbb{BL}_k$. Using Lemma 2 we have $\mathrm{wSpec}\,\mathrm{Free}_{n,k} \cong \mathrm{wSpec}\,(\sum_{i=1}^{n} \mathrm{Free}_{1,k}) \cong \prod_{i=1}^{k} \mathrm{wSpec}\,\mathrm{Free}_{1,k}$. By Lemma 4, $\prod_{i=1}^{k} \mathrm{wSpec}\,\mathrm{Free}_{1,k} \cong \prod_{i=1}^{k} F_k^1 = F_k^n$, and the lemma is proved.

Our next objective is to obtain an explicit description of F_k^n for any n and k. To this aim, we define the following weighted trees. Fix $k \geq 2$, and an integer $d \geq 1$:

- $T_{k,d}^0 = P_d$.
- For each integer $n \geq 1$,

$$T_{k,d}^n = P_d \left(\sum_{i=1}^{n} \sum_{e \mid k-1} \binom{n}{i} \beta(i,e) T_{k,e}^{n-i} \right)_\perp \ .$$

Lemma 6. *Fix integers $k \geq 2$, $d \geq 1$, and $m \geq 1$.*

(1) $T^1_{k,1} \cong (M^1_k)_\perp$.
(2) $T^m_{k,d} \cong P_d T^m_{k,1}$.
(3) $M^m_k \cong (M^1_k)^m$.
(4) $T^m_{k,d} \cong (T^1_{k,d})^m$.

Proof. (1) follows immediately from the definition of $T^1_{k,1}$, $T^0_{k,1}$ and M^1_k. (2) follows from Lemma 3 and the definition of $T^1_{k,1}$ and $T^m_{k,1}$.

(3) By induction on m. The base case is trivial. Write $(M^1_k)^m$ as $M^1_k \times (M^1_k)^{m-1}$. By induction, $M^{m-1}_k \cong (M^1_k)^{m-1}$. By Lemma 3, since products distribute over coproducts,

$$\begin{aligned} M^1_k \times M^{m-1}_k &\cong \sum_{d|k-1} \beta(1,d) P_d \times \sum_{e|k-1} \beta(m-1,e) P_e \\ &\cong \sum_{d|k-1} \sum_{e|k-1} \beta(1,d)\beta(m-1,e) P_{\mathrm{lcm}\,(d,e)} \\ &\cong \sum_{d|k-1} \beta(m,d) P_d \cong M^m_k \qquad \text{(by (6))}. \end{aligned}$$

(4) By induction on m. The base case is trivial. Write $(T^1_{k,d})^m$ as $T^1_{k,d} \times (T^1_{k,d})^{m-1}$. By induction $T^{m-1}_{k,d} \cong (T^1_{k,d})^{m-1}$. Let

$$V = \sum_{i=1}^{m-1} \sum_{e|k-1} \binom{m-1}{i} \beta(i,e) T^{m-1-i}_{k,e}\,,$$

so $P_d V_\perp \cong T^{m-1}_{k,d}$. By Lemma 3,

$$T^1_{k,d} \times T^{m-1}_{k,d} \cong P_d\left((M^1_k)_\perp \times V_\perp\right) \cong P_d\left((M^1_k \times V_\perp) + (M^1_k \times V) + (T^1_{k,1} \times V)\right)_\perp.$$

By distributivity and Lemma 3, since M^1_k is a forest of one-point trees:

$$M^1_k \times V_\perp \cong \sum_{e|k-1} \beta(1,e) P_e V_\perp \cong \sum_{e|k-1} \beta(1,e) T^{m-1}_{k,e}\,;$$

analogously,

$$\begin{aligned} M^1_k \times V &\cong \left(\sum_{e|k-1} \beta(1,e) P_e\right) \times \left(\sum_{i=1}^{m-1} \sum_{e|k-1} \binom{m-1}{i} \beta(i,e) T^{m-1-i}_{k,e}\right) \\ &\cong \sum_{i=1}^{m-1} \sum_{e_1|k-1} \sum_{e_2|k-1} \binom{m-1}{i} \beta(1,e_1)\beta(i,e_2) T^{m-(i+1)}_{k,\mathrm{lcm}\,(e_1,e_2)} \\ &\cong \sum_{i=2}^{m-1} \sum_{e|k-1} \binom{m-1}{i-1} \beta(i,e) T^{m-i}_{k,e} \qquad \text{(by (6))}\,; \end{aligned}$$

finally,

$$\begin{aligned}
T^1_{k,1} \times V &\cong \sum_{i=1}^{m-1} \sum_{e|k-1} \binom{m-1}{i} \beta(i,e)(T^1_{k,1} \times T^{m-1-i}_{k,e}) \\
&\cong \sum_{i=1}^{m-1} \sum_{e|k-1} \binom{m-1}{i} \beta(i,e) T^{m-i}_{k,e} \qquad \text{(Induction Hypothesis)} \\
&\cong \sum_{i=2}^{m-1} \sum_{e|k-1} \binom{m-1}{i} \beta(i,e) T^{m-i}_{k,e} + \sum_{e|k-1} (m-1)\beta(1,e) T^{m-1}_{k,e} \,.
\end{aligned}$$

Summing up we have

$$\begin{aligned}
T^1_{k,d} \times T^{m-1}_{k,d} &\cong P_d \left((M^1_k \times V_\perp) + (M^1_k \times V) + (T^1_{k,1} \times V) \right)_\perp \\
&\cong P_d \left(\sum_{i=1}^{m} \sum_{e|k-1} \binom{m}{i} \beta(i,e) T^{m-i}_{k,e} \right)_\perp \,,
\end{aligned}$$

as was to be proved.

Example 2. The rightmost trees in Figure 2 are $T^1_{3,1}$ (left) and $T^1_{7,1}$ (right). The rightmost tree in Figure 3 is $T^2_{3,1}$.

We are finally in a position to exhibit the promised explicit description of $\mathrm{wSpec}\,\mathrm{Free}_{n,k}$.

Theorem 2. *For each $k \geq 2$ and each integer $n \geq 0$,*

$$F^n_k \cong \sum_{i=0}^{n} \sum_{d|k-1} \beta(i,d) \binom{n}{i} T^{n-i}_{k,d} \,.$$

Proof. By definition,

$$F^n_k = (F^1_k)^n = (M^1_k + (M^1_k)_\perp)^n \cong (M^1_k + T^1_{k,1})^n \,. \tag{7}$$

Since products distribute over coproducts, (7) yields

$$(M^1_k + T^1_{k,1})^n \cong \sum_{i=0}^{n} \binom{n}{i} (M^1_k)^i (T^1_{k,1})^{n-i} \,. \tag{8}$$

By Lemma 6 along with the definition of M^i_k, from (8) we deduce

$$(M^1_k + T^1_{k,1})^n \cong \sum_{i=0}^{n} \binom{n}{i} \sum_{d|k-1} \beta(i,d) P_d T^{n-i}_{k,1} \,. \tag{9}$$

As by Lemma 6.(2), $P_d T^{n-i}_{k,1} \cong T^{n-i}_{k,d}$, the lemma follows from (9) at once.

Example 3. Figure 3 displays the weighted forest F^2_3.

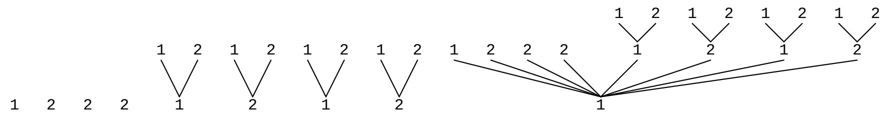

Fig. 3. $F_3^2 = (F_3^1)^2$.

6 The Cardinality of Free Algebras in $\mathbb{BL}_k$

In this final section, we use Theorems 1 and 2 to obtain the cardinality of $\mathrm{Free}_{n,k}$. We shall write $t(k,n,d)$ for the cardinality of the BL-algebra $\mathrm{Sub}\, T^n_{k,d}$, where $k \geq 2$, $d \in \mathbb{N}$, and $n \geq 0$ an integer.

Lemma 7. *Fix $k \geq 2$, and an integer $n \geq 0$. Then $t(k,0,d) = d+1$, and*

$$t(k,n,d) = d + \prod_{i=1}^{n} \prod_{e|k-1} t(k, n-i, e)^{\binom{n}{i}\beta(i,e)} .$$

Proof. Follows immediately from Lemma 6.(4).

Theorem 3. *Fix $k \geq 2$, and an integer $n \geq 0$. Then:*

$$|\mathrm{Free}_{n,k}| = \prod_{i=0}^{n} \prod_{d|k-1} t(k, n-i, d)^{\binom{n}{i}\beta(i,d)} .$$

Proof. Follows immediately from Theorem 2.

To conclude, in the following table we report the cardinalities of $\mathrm{Free}_{n,k}$ for some values of n and k, computed using Theorem 3. Approximations are from below.

	$n=1$	$n=2$	$n=3$
$k=2$	6	342	137186159382
$k=3$	42	28677559680	$\sim 2.255534588 \cdot 10^{91}$
$k=4$	1056	$\sim 4.587963634 \cdot 10^{28}$	$\sim 1.230577614 \cdot 10^{373}$
$k=5$	22650	$\sim 1.525862962 \cdot 10^{55}$	$\sim 4.141165490 \cdot 10^{957}$
$k=6$	6721056	$\sim 1.738126059 \cdot 10^{106}$	$\sim 2.246803010 \cdot 10^{2299}$

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