Schauder Hats for the 2-variable Fragment of BL

Stefano Aguzzoli
D.S.I., Università di Milano
Milano, Italy
Email: aguzzoli@dsi.unimi.it

Simone Bova
Department of Mathematics, Vanderbilt University
Nashville TN, USA
Email: simone.bova@vanderbilt.edu

***Abstract*—The theory of Schauder hats is a beautiful and powerful tool for investigating, under several respects, the algebraic semantics of Łukasiewicz infinite-valued logic [CDM99], [MMM07], [Mun94], [P95]. As a notably application of the theory, the elements of the free n-generated MV-algebra, that constitutes the algebraic semantics of the n-variate fragment of Łukasiewicz logic, are obtained as (t-conorm) monoidal combination of finitely many hats, which are in turn obtained through finitely many applications of an operation called *starring*, starting from a finite family of *primitive* hats.**

The aim of this paper is to extend this portion of the Schauder hats theory to the two-variable fragment of Hájek's Basic logic. This step represents a non-trivial generalization of the one-variable case studied in [AG05], [Mon00], and provides sufficient insight to capture the behaviour of the n-variable case for $n \geq 1$.

I. Introduction

For background notions and facts on Łukasiewicz and Basic logic (in short, BL), and their algebraic semantics, respectively the varieties of MV-algebras and BL-algebras, we refer the reader to [CDM99], [Háj98], [CEGT00], [AM03]. We only mention that the free BL-algebra over n-many generators, in symbols $\mathbb{BL}_n$, is the subalgebra of the BL-algebra of all functions from $((n+1)[0,1])^n$ to $(n+1)[0,1]$ generated by the projections, where $(n+1)[0,1]$ is the ordinal sum of $n+1$ many copies of the generic MV-algebra $[0,1]$. The generic MV-algebra $[0,1]$ is (term equivalent to) the algebra given by the interval $[0,1]$, equipped with the constant $\bot = 0$, and the operations $x \odot y = \max\{0, x+y-1\}$ and $x \to y = \min\{1, 1-x+y\}$. We define $\neg x = x \to \bot$, $\top = \neg\bot$, $x \oplus y = \neg x \to y$, $x \ominus y = x \odot \neg y$, $x \wedge y = x \odot (x \to y)$, and $x \vee y = ((x \to y) \to y) \wedge ((y \to x) \to x)$. Moreover, for any integer $m > 0$, we denote $m\varphi$ and φ^m the $\oplus$-disjunction and the $\odot$-conjunction, respectively, of m occurrences of φ.

We shall develop a notion of *BL-Schauder hat* (for short, *BL-hat*) such that the following two facts hold: (i) each element of $\mathbb{BL}_n$ is a t-norm monoidal combination of finitely many BL-hats. (ii) each BL-hat in the above combination is constructed, as a BL-formula, via a *refinement* procedure consisting in a BL-combination of a finite set of *primitive* BL-hats.

The key ingredients of the construction are presented in the general case $n \geq 1$ in [AB09], [Bov08], where the free n-generated BL-algebra is characterized as a BL-algebra of geometric-combinatorial objects called *encodings*. In this paper, in the interest of intuition and readability, we avoid the technicalities involved in the general case, and we study directly the two-variable case. Indeed, the two-variable case is complex enough to enlighten the construction in the general case, and allows for a neat geometrical intuition of the behaviour of BL-hats in the refinement procedure.

II. Free MV-algebras and free Wajsberg hoops

We collect from the literaure the following representations of the free n-generated MV-algebra, $\mathbb{MV}_n$, and the free n-generated Wajsberg hoop, $\mathbb{WH}_n$, in terms of n-ary McNaughton functions. Recall that a *Wajsberg hoop* is the $\{\odot, \to, \top\}$-subreduct of an MV-algebra, and a continuous function $f : [0,1]^n \to [0,1]$ is a *McNaughton function* if and only if there are finitely many linear polynomials with integer coefficients, $p_1, \ldots, p_k$, such that, for every $\mathbf{x} \in [0,1]^n$, there is $j \in \{1, \ldots, k\}$ such that $f(\mathbf{x}) = p_j(\mathbf{x})$.

Theorem 1 ([McN51], [AP02])**.** *$\mathbb{MV}_n$ is (isomorphic to) the algebra of n-ary McNaughton functions, where $\bot$ is realized by the constant 0, and $\odot$ and $\to$ are realized by the operations pointwise defined by the corresponding operations of the generic MV-algebra $[0,1]$.*

$\mathbb{WH}_n$ is (isomorphic to) the algebra of n-ary McNaughton functions f such that $f(1,1,\ldots,1) = 1$, where $\odot$, $\to$ and $\top$ are realized by the operations pointwise defined by the corresponding operations of the generic MV-algebra $[0,1]$.

III. The Free 2-Generated BL-Algebra

In this section, we introduce the notion of *(binary) encoding*, and we describe the free 2-generated BL-algebra, $\mathbb{BL}_2$, in terms of (binary) encodings, as in [AB09].

Given a subset $K = \{j_1, j_2, \ldots, j_k\}$ of $\{1, \ldots, n\}$ we denote π_K the *projection* over K, that is, $\pi_K(t_1, \ldots, t_n) = (t_{j_1}, \ldots, t_{j_k})$.

By a *(rational) prism* we mean a set $P \subseteq [0,1]^2$ either of the form $[0,1] \times Q$ or of the form $Q \times [0,1]$ for $Q \subseteq [0,1)$ being either (a singleton containing) a rational point or an open interval with rational endpoints. The set Q is called the *base* of P and is denoted $B(P)$.

Definition 1. Let $K \subseteq \{1,2\}$. A function f is *essentially K-ary prismwise Wajsberg* if the following holds.

Case $K = \emptyset$: In this case, $f = \emptyset$, the empty function (the only function with empty domain).

Case $K = \{1\}$ or $K = \{2\}$: In this case, the following holds. (i) $\mathrm{dom}(f)$ is the union of as set Δ of finitely many prisms $P \subseteq [0,1]^2$, of the first form if K ={1}, or of the second if $K = \{2\}$. (ii) For each $P \in \Delta$ there is $g \in \mathbb{WH}_1$ such that $f(x_1, x_2) = g(\pi_K(x_1, x_2))$ for all $(x_1, x_2) \in P$.

Let $Q = B(P)$. We denote the restriction of f to P by $g|Q$. If $\mathrm{dom}(f) = \{P\}$, then we denote f simply by $g|Q$. If $\bigcup_{P\in\Delta} B(P) = [0,1)$ then we say f is *total*.

Case $K = \{1,2\}$: In this case, $f \in \mathbb{WH}_2$.

We let $\mathbb{PW}_2$ denote the set of all essentially K-ary prism-wise Wajsberg functions, for all $K \in 2^{\{1,2\}}$.

For each function $f\colon [0,1]^2 \to [0,1]$, each $b \in \{0,1\}$ and each $i \in \{1,2\}$, we let $\mathbf{b}_i(f) = \pi_{\{1,2\}\setminus\{i\}}(f^{-1}(b) \cap \{(x_1,x_2) \mid \pi_{\{i\}}(x_1,x_2) = 1\}) \setminus \{1\}$.

Definition 2 ((Binary) Encoding)**.** A *(binary) encoding* is a 6-tuple,

$$f = \langle f_{00}, f_{01}, f_{02}, f_{10}, f_{11}, f_{12}\rangle,$$

satisfying the following properties:

1) $f_{ij} \in \mathbb{PW}_2$ for all $(i,j) \in \{0,1\} \times \{0,1,2\}$.
2) $f_{00} \in \mathbb{WH}_2$, and, either $f_{10} \in \mathbb{WH}_2$ or $f_{10} = \emptyset$.
3) Let $b \in \{0,1\}$ such that $b = 0$ if and only if $f_{10} = \emptyset$, let $i \in \{0,1\}$, and let $j \in \{1,2\}$. Then,

$$\mathrm{dom}(f_{ij}) = \{(x_1,x_2) \mid x_{3-j} \in \mathbf{b}_j(f_{i0})\}.$$

We let A_2 denote the set of all binary encodings.

It follows that $f_{10} = \emptyset$ implies $f_{11} = f_{12} = \emptyset$.

For any pair (f,g) where f is an encoding and g either an encoding or an encoding component, we set $\nu_f(g) = g$ if $f_{10} \neq \emptyset$, $\nu_f(g) = \neg g$ if $f_{10} = \emptyset$.

Theorem 2. *The free 2-generated BL-algebra, $\mathbb{BL}_2$, is (isomorphic to) the BL-algebra,*

$$\mathbb{BL}_2 = \langle A_2, \odot, \to, \bot\rangle,$$

obtained by equipping the binary encodings with the following constant and operations. Let $f, g \in A_2$. Then,

- $\bot = \langle \top, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset\rangle$.
- $f \odot g = e$, *where $e \in A_2$ is defined as follows.* $\mathrm{dom}(e_{ij}) = \mathrm{dom}(f_{ij}) \cap \mathrm{dom}(g_{ij})$, *for each* $(i,j) \in \{0,1\} \times \{0,1,2\}$*; for all* $(x_1,x_2) \in \mathrm{dom}(e_{ij})$, *if* $(i,j) \in (\{0,1\} \times \{0,1,2\}) \setminus \{(0,0)\}$ *then* $e_{ij}(x_1,x_2) = f_{ij}(x_1,x_2) \odot g_{ij}(x_1,x_2)$, *while*

$$e_{00} = \begin{cases} f_{00} \oplus g_{00} & \text{if } f_{10} = \emptyset \text{ and } g_{10} = \emptyset,\\ g_{00} \to f_{00} & \text{if } f_{10} = \emptyset \text{ and } g_{10} \neq \emptyset,\\ f_{00} \to g_{00} & \text{if } f_{10} \neq \emptyset \text{ and } g_{10} = \emptyset,\\ f_{00} \odot g_{00} & \text{if } f_{10} \neq \emptyset \text{ and } g_{10} \neq \emptyset.\end{cases}$$

- $f \to g = e$, *where $e \in A_2$ is defined as follows.*

$$e_{00} = \begin{cases} g_{00} \to f_{00} & \text{if } f_{10} = \emptyset \text{ and } g_{10} = \emptyset,\\ f_{00} \oplus g_{00} & \text{if } f_{10} = \emptyset \text{ and } g_{10} \neq \emptyset,\\ f_{00} \odot g_{00} & \text{if } f_{10} \neq \emptyset \text{ and } g_{10} = \emptyset,\\ f_{00} \to g_{00} & \text{if } f_{10} \neq \emptyset \text{ and } g_{10} \neq \emptyset.\end{cases}$$

If $f_{10} \neq \emptyset$ then $e_{10} = f_{10} \to g_{10}$ and for each $(i,j) \in \{0,1\} \times \{1,2\}$, $\mathrm{dom}(e_{ij}) = \mathrm{dom}(g_{ij}) \cup \{(x_1,x_2) \mid \nu_f(f_{ij}(y_1,y_2)) \leq \nu_g(g_{ij}(y_1,y_2)),\ y_j = 1, y_{3-j} = x_{3-j}\}$ *and* $e_{ij}(x_1,x_2) = (f_{ij} \to g_{ij})(x_1,x_2)$ *if* $(x_1,x_2) \in \mathrm{dom}(f_{ij}) \cap \mathrm{dom}(g_{ij})$, $e_{ij}(x_1,x_2) = 1$ *otherwise. If $f_{10} = \emptyset$ then e_{0j} is defined as above for each $j \in \{1,2\}$, while e_{1j} is total and coinciding with $\top$ for all $j \in \{0,1,2\}$.*

The two generators are $x_1^{\mathbb{BL}_2} = \langle x_1, x_1, \emptyset, x_1, x_1, \emptyset\rangle$, *and* $x_2^{\mathbb{BL}_2} = \langle x_2, \emptyset, x_2, x_2, \emptyset, x_2\rangle$.

The interpretation $\varphi^{\mathbb{BL}_2}$ of a formula φ in the two-variable fragment of BL is the image $\iota(\varphi)$ of φ under the $\{\odot, \to, \bot\}$-homomorphism ι from the algebra of all two-variable formulas of BL to $\mathbb{BL}_2$, uniquely determined by $\iota(x_i) = x_i^{\mathbb{BL}_2}$.

IV. Convex Geometry Background

To recall the notion of Schauder hat and define $\mathbb{BL}_2$-hats we need to introduce some notions of convex geometry (see [Ewa96], for further background).

An *n-simplex* $S \subseteq \mathbb{R}^m$ (for $m \geq n$) is the convex hull of $n+1$ many affinely independent points of $\mathbb{R}^m$, called the *vertices* of S. That is, a 0-simplex is a (set containing exactly one) point, a 1-simplex is a line segment, a 2-simplex is a triangle, etc. By *rational* n-simplex in $\mathbb{R}^m$ we mean an n-simplex S whose vertices $\mathbf{v}_1, \ldots, \mathbf{v}_{n+1}$ are rational points in $[0,1]^m$, that is each component of each $\mathbf{v}_i$ is a rational number δ, $0 \leq \delta \leq 1$. In the following we shall consider only rational n-simplices, which we will call simply "n-simplices", or even "simplices" when the dimension does not need to be specified. A k-dimensional *face* of a n-simplex S, for $-1 \leq k \leq n$ is the convex hull of $k+1$ vertices of S. An *open* simplex is the relative interior of a simplex (note that vertices, that is, 0-faces of simplices, are both 0-simplices and open 0-simplices; the empty set is the only (-1)-dimensional face of any simplex).

The *denominator* $\mathrm{den}(\mathbf{v})$ of a rational point $\mathbf{v} \in ([0,1] \cap \mathbb{Q})^m$ is the least common denominator $\mathrm{den}(\mathbf{v})$ of the coordinates of $\mathbf{v}$. The *homogeneous expression* of $\mathbf{v}$ is $\mathrm{den}(\mathbf{v})(\mathbf{v},1) \in \mathbb{Z}^{m+1}$. The *Farey mediant* of a finite set of rational points $\{\mathbf{v}_j\}_{j\in J} \subset ([0,1] \cap \mathbb{Q})^m$ is the point $(\sum_{j\in J} \mathrm{den}(\mathbf{v}_j)\mathbf{v}_j)/(\sum_{j\in J} \mathrm{den}(\mathbf{v}_j))$. A rational m-simplex $S \subseteq \mathbb{R}^m$ is *unimodular* if 1 is the absolute value of the determinant of the matrix whose rows are the homogeneous expressions of the vertices of S. A rational n-simplex $F \subseteq \mathbb{R}^m$, with $n \leq m$ is unimodular if it is a face of a unimodular m-simplex. Note that a rational 0-simplex (a *vertex*) is always unimodular.

A *unimodular triangulation* of $[0,1]^m$ is a finite collection U of n-simplices, for all $-1 \leq n \leq m$, such that $\bigcup\{S \in U\} = [0,1]^m$, the intersection of any two members S_1, S_2 of U is a common face of both S_1 and S_2, and U is closed under taking faces. We say that an open simplex S *belongs* to U (in symbols, $S \in U$) if there is $T \in U$ such that S is the relative interior of T.

V. Schauder Hats

In this section we collect basic notions and results about Schauder hats that we shall be using in the paper (see [CDM99], [Mun94], [P95]).

Definition 3. Let U be a unimodular triangulation of $[0,1]^n$ and let S be a k-simplex of U. Then the *starring* of U at S, in symbols $U * S$, is the set of simplices obtained as follows.

1) Put in $U * S$ all simplices of U not containing S.
2) Display $\mathbf{v}_1, \ldots, \mathbf{v}_k$ the vertices of S. Then, for each $d \in \{1, \ldots, k-1\}$ and each d-dimensional face T of S, displaying $\{\mathbf{w}_1, \ldots, \mathbf{w}_{d+1}\}$ the vertices of T, replace each simplex $T \subseteq R \in U$ with the collection $\{R_1, \ldots, R_{d+1}\}$, where R_i is the simplex whose vertices are those of R with $\mathbf{w}_i$ replaced by the Farey mediant $\mathbf{v}_S$ of $\mathbf{v}_1, \ldots, \mathbf{v}_k$.

Note that $U*S$ is again a unimodular triangulation of $[0,1]^n$. If S is a 1-simplex, the starring $U*S$ is called an *edge* starring.

Definition 4. Given a unimodular triangulation U of $[0,1]^n$ and a vertex (0-simplex) $\mathbf{v}$ of U, the *Schauder hat* with *apex* $\mathbf{v}$ in U is the continuous function $h_{\mathbf{v},U} \colon [0,1]^n \to [0,1]$ determined by the following conditions:

1) $h_{\mathbf{v},U}(\mathbf{v}) = 1/\mathrm{den}(\mathbf{v})$.
2) $h_{\mathbf{v},U}(\mathbf{u}) = 0$ for all vertices $\mathbf{u} \neq \mathbf{v}$ of U.
3) $h_{\mathbf{v},U}$ is linear over each simplex of U.

The *star* of $\mathbf{v}$ in U is the set of all simplices of U having $\mathbf{v}$ among their vertices. The *Schauder set* H_U associated with a unimodular triangulation U is the set of all hats of the form $h_{\mathbf{v},U}$ for $\mathbf{v}$ a vertex of U.

Let $\mathbf{v}_1, \ldots \mathbf{v}_k$ be the vertices of a simplex $S \in U$. Then the *star refinement* $H_U * S$ of H_U at S is obtained as follows: Let h_i be the hat in H_U with apex $\mathbf{v}_i$, and let $h_S = \bigwedge_{i=1}^q h_i$. Then put in $H_U * S$ the function h_S together with all hats of H_U distinct from any h_i and replace h_j by $h_j \ominus h_S$, for each $j \in \{1, \ldots, k\}$.

Lemma 1. *$H_U * S$ is a Schauder set. In particular $H_U * S = H_{U*S}$ and the apex of h_S is the Farey mediant of the vertices of S.*

Definition 5. Let T be the n-simplex whose set of vertices is $\{\mathbf{v}_j\}_{j=0}^n$, for $\pi_{\{i\}}(\mathbf{v}_j) = 0$ if $i + j \leq n$, $\pi_{\{i\}}(\mathbf{v}_j) = 1$, otherwise. Let Sym_n be the group of all permutations of the set $\{1, 2, \ldots, n\}$. For each $\sigma \in \mathrm{Sym}_n$ let T_σ be the simplex whose ith vertex is such that its jth component is $\pi_{\{\sigma(j)\}}(\mathbf{v}_i)$. Let F_σ be the set of all faces of T_σ. Then

$$U_0^n = \bigcup_{\sigma \in \mathrm{Sym_n}} F_\sigma$$

is a unimodular triangulation of $[0,1]^n$, called the *fundamental partition* of $[0,1]^n$.

Example 1. $U_0^1 = \{\{0\}, [0,1], \{1\}\}$*, and* $H_{U_0^1} = \{x_1, \neg x_1\}$*. The* 2*-simplices of* U_0^2 *are* $\{(t_1, t_2) \mid 0 \leq t_1 \leq t_2 \leq 1\}$ *and* $\{(t_1, t_2) \mid 0 \leq t_2 \leq t_1 \leq 1\}$*. Moreover,* $H_{U_0^2} = \{x_1 \wedge x_2, x_1 \ominus x_2, x_2 \ominus x_1, \neg x_1 \wedge \neg x_2\}$.

Before stating the normal form theorem for $\mathbb{MV}_n$ we collect for later use a fundamental technical result on unimodular triangulations.

Lemma 2. *Let S be either a rational 0-simplex or a 1-simplex lying on an edge of the hypercube $[0,1]^n$. Then there is a unimodular triangulation U of $[0,1]^n$ such that $S \in U$. Moreover, U is obtained via finitely many edge starrings from U_0^n.*

Lemma 3. *For each McNaughton function $f \colon [0,1]^n \to [0,1]$ there is a unimodular triangulation U_f of $[0,1]^n$ such that f is linear over each simplex $S \in U_f$. Moreover, U_f is obtained via finitely many edge starrings from U_0^n.*

Proof: This is one of the main arguments in Panti's geometric proof of the completeness of Łukasiewicz infinite-valued logic, see [P95, Lemma 2.2]. ■

Theorem 3. *For each element $f \in \mathbb{MV}_n$ there is a Schauder set $H_f = \{h_i\}_{i \in I}$ and nonnegative integers $\{m_i\}_{i \in I}$ such that*

$$f = \bigoplus_{i \in I} m_i h_i \,.$$

Proof: One takes $H_f = H_{U_f}$, and $m_i = f(\mathbf{v}_i)\mathrm{den}(\mathbf{v}_i)$, for $\mathbf{v}_i$ the apex of h_i, for each $i \in I$. ■

VI. $\mathbb{BL}_2$-HATS

As is well known, each MV-algebra $A = \langle A, \oplus, \neg, 0\rangle$ is isomorphic to its order-dual $A^\partial = \langle A, \odot, \neg, 1\rangle$ via the map $\cdot^\partial \colon a \mapsto \neg a$. Note that $(a \ominus b)^\partial = b^\partial \to a^\partial$, and clearly, $(a \vee b)^\partial = a^\partial \wedge b^\partial$ and $(a \wedge b)^\partial = a^\partial \vee b^\partial$. We call *Schauder co-hat* any function of the form h^∂ for h a Schauder hat. Let U be a unimodular triangulation of $[0,1]^n$, for some n. The apex and the star of a co-hat k^∂ in U are the apex and the star of the hat k in U, respectively. The co-Schauder set associated with U is the set of all co-hats of the Schauder set of U. The star refinement of the co-Schauder set H_U at a simplex $S \in U$ is defined as for Schauder sets, replacing by duality $\bigwedge h_i$ with $\bigvee h_i^\partial$ and $h_j \ominus h_S$ with $h_S^\partial \to h_j^\partial$.

Definition 6. A Schauder co-hat $k \colon [0,1]^n \to [0,1]$ is *virtual* iff its apex is $(1, 1, \ldots, 1)$; k is *actual* iff it is not virtual.

Note that a Schauder co-hat $h \colon [0,1]^n \to [0,1]$ is an element of $\mathbb{WH}_n$ iff it is actual.

Theorem 4. *For each element $f \in \mathbb{WH}_n$ there is a co-Schauder set $H_f = \{h_i\}_{i \in I}$ and nonnegative integers $\{m_i\}_{i \in I}$ such that*

$$f = \bigodot_{i \in I} h_i^{m_i} \,,$$

where $m_i = 0$ if h_i is virtual.

Proof: Immediate from Theorem 1 and Theorem 3. ■

A *primitive* Schauder co-hat is a function h^∂ for $h \in H_{U_0^n}$.

Example 2. *The set of primitive Schauder co-hats for* $\mathbb{MV}_1$ *is* $H_0^1 = \{x_1, \neg x_1\}$*. The set of* primitive *Schauder co-hats for* $\mathbb{MV}_2$ *is* $H_0^2 = \{x_1 \vee x_2, x_2 \to x_1, x_1 \to x_2, \neg x_1 \vee \neg x_2\}$.

Definition 7. A $\mathbb{BL}_2$*-hat* is a 6-tuple of functions $h = \langle h_{00}, h_{01}, h_{02}, h_{10}, h_{11}, h_{12}\rangle$ belonging to one of the following kinds:

k1: Either $h = \langle k, \top|\mathbf{1}_1(k), \top|\mathbf{1}_2(k), \top, \top, \top\rangle$ or $h = \langle \top, \top, \top, k, \top|\mathbf{1}_1(k), \top|\mathbf{1}_2(k)\rangle$, and k is a Schauder co-hat.

k2: There is a pair $(i,j) \in \{0,1\} \times \{1,2\}$, a unimodular triangulation U of $[0,1]$ and an open unimodular simplex $Q \in U$ such that $h_{i'j'} = \top$ for all $(i',j') \in (\{0,1\} \times \{0,1,2\}) \setminus \{(i,j)\}$, and $h_{ij} = \top|Q'$ for every open simplex $Q' \neq Q$ in U, while $h_{ij} = k|Q$ for k a Schauder co-hat in one variable.

We say k is the Schauder co-hat *associated* with h. The *star* and the *apex* of a $\mathbb{BL}_2$-hat h are the star and the apex of the associated Schauder co-hat. A $\mathbb{BL}_2$-hat h is *actual* (resp. *virtual*) if so is its associated Schauder co-hat. A $\mathbb{BL}_2$-hat h is *total* if it belongs to kind k1 or $Q \in \{\{0\}, (0,1)\}$.

Lemma 4. *Let h be a $\mathbb{BL}_2$-hat. Then $h \in \mathbb{BL}_2$ iff h is actual.*

VII. Refinement Process

Let U be a unimodular triangulation of $[0,1]^2$. Then a *relevant face* of U is an open k-simplex F of U, for $k \in \{0,1\}$, such that $F \subseteq \{1\} \times [0,1)$ or $F \subseteq [0,1) \times \{1\}$. We denote F^1_U the set of relevant faces of U of the first form, and F^2_U the set of relevant faces of U of the second form.

Definition 8. A $\mathbb{BL}_2$-*triangulation* is a 6-tuple $\langle U_{00}, U_{01}, U_{02}, U_{10}, U_{11}, U_{12}\rangle$ such that U_{j0} is a unimodular triangulation of $[0,1]^2$ for each $j \in \{0,1\}$, and U_{ji} is a map that associates with each relevant face in $F^i_{U_{j0}}$ a unimodular triangulation of $[0,1]$, for each $j \in \{0,1\}$ and each $i \in \{1,2\}$.

We say that a k-simplex S is a simplex *of* U if either $S \in U_{i0}$ for some $i \in \{0,1\}$ or there is $(i,j) \in \{0,1\} \times \{1,2\}$, and a simplex $R \in \mathrm{dom}(U_{ij})$ such that $S \in U_{ij}(R)$.

The $\mathbb{BL}_2$-*fundamental partition* is

$$B = \langle U_0^2, V, V, U_0^2, V, V\rangle\,,$$

where V is the following map: $\{0\} \mapsto U_0^1$, $(0,1) \mapsto U_0^1$ (recall from Definition 5 that U_0^n is the fundamental partition of $[0,1]^n$).

The $\mathbb{BL}_2$-set H_U associated with a $\mathbb{BL}_2$-triangulation U is a 6-tuple $\langle H_{00}, H_{01}, H_{02}, H_{10}, H_{11}, H_{12}\rangle$ such that, for each $i \in \{0,1\}$, H_{i0} is the set of k1 $\mathbb{BL}_2$-hats such that their associated co-hats form the co-Schauder set for U_{i0}; for each $j \in \{1,2\}$, H_{ij} is the map with the same domain as U_{ij} defined as follows. For each $S \in \mathrm{dom}(H_{ij})$, $H_{ij}(S)$ is the set of total k2 $\mathbb{BL}_2$-hats such that their associated co-hats form the co-Schauder set for $U_{ij}(S)$.

Note that each hat of H_U is linear over each simplex of U.

Definition 9. Let

$$\begin{aligned}
p^0_{00} &= \langle x_1 \vee x_2, \top, \top, \top, \top, \top\rangle,\\
p^1_{00} &= \langle x_1 \to x_2, \emptyset, \top, \top, \top, \top\rangle,\\
p^2_{00} &= \langle x_2 \to x_1, \top, \emptyset, \top, \top, \top\rangle,\\
\hat{p}_{00} &= \langle \neg x_1 \vee \neg x_2, \top|\{0\}, \top|\{0\}, \top, \top, \top\rangle,\\
p^0_{10} &= \langle \top, \top, \top, x_1 \vee x_2, \top, \top\rangle,\\
p^1_{10} &= \langle \top, \top, \top, x_1 \to x_2, \emptyset, \top\rangle,\\
p^2_{10} &= \langle \top, \top, \top, x_2 \to x_1, \top, \emptyset\rangle,\\
\hat{p}_{10} &= \langle \top, \top, \top, \neg x_1 \vee \neg x_2, \top|\{0\}, \top|\{0\}\rangle,\\
p_{01} &= \langle \top, x_1, \top, \top, \top, \top\rangle,\\
\hat{p}_{01} &= \langle \top, \neg x_1, \top, \top, \top, \top\rangle,\\
p_{02} &= \langle \top, \top, x_2, \top, \top, \top\rangle,\\
\hat{p}_{02} &= \langle \top, \top, \neg x_2, \top, \top, \top\rangle,\\
p_{11} &= \langle \top, \top, \top, \top, x_1, \top\rangle,\\
\hat{p}_{11} &= \langle \top, \top, \top, \top, \neg x_1, \top\rangle,\\
p_{12} &= \langle \top, \top, \top, \top, \top, x_2\rangle,\\
\hat{p}_{12} &= \langle \top, \top, \top, \top, \top, \neg x_2\rangle.
\end{aligned}$$

Let further, for $j \neq 0$, $p^0_{ij} = (p^j_{i0} \to (p^j_{i0} \odot p^j_{i0})) \to p_{ij}$, $p^1_{ij} = p^0_{ij} \to p_{ij}$, and $\hat{p}^0_{ij} = (p^j_{i0} \to (p^j_{i0} \odot p^j_{i0})) \to \hat{p}_{ij}$, $\hat{p}^1_{ij} = \hat{p}^0_{ij} \to \hat{p}_{ij}$. Then the set P of *primitive* $\mathbb{BL}_2$-hats is the 6-tuple $P = \langle P_{00}, P_{01}, P_{02}, P_{10}, P_{11}, P_{12}\rangle$, where $P_{i0} = \{p^0_{i0}, p^1_{i0}, p^2_{i0}, \hat{p}_{i0}\}$ for $i \in \{0,1\}$, P_{ij} is the map $\{0\} \mapsto \{p^0_{ij}, \hat{p}^0_{ij}\}$, $(0,1) \mapsto \{p^1_{ij}, \hat{p}^1_{ij}\}$, for $(i,j) \in \{0,1\} \times \{1,2\}$. Note that the hats of the form $\hat{p}_{ij}$, $\hat{p}^b_{ij}$ are virtual, and all other hats are actual.

Proposition 1. *P is the $\mathbb{BL}_2$-set associated with the $\mathbb{BL}_2$-fundamental partition B.*

We now adapt the definition of starring of triangulations (Def. 3) and star refinements of Schauder sets (Def. 4) to our current $\mathbb{BL}_2$ setting.

Let U be a $\mathbb{BL}_2$-triangulation, and let S be a 1-simplex of U. Let $\mathbf{v}_S$ be the Farey mediant of the vertices $\mathbf{v}_1$ and $\mathbf{v}_2$ of S, and let S_1, S_2 be the 1-simplices obtained by replacing $\mathbf{v}_1$ and $\mathbf{v}_2$ by $\mathbf{v}_S$, respectively. Let $S_3 = \{\mathbf{v}_S\}$. Then the *starring* of U at S, in symbols $U * S$ is the 6-tuple $\langle U'_{00}, U'_{01}, U'_{02}, U'_{10}, U'_{11}, U'_{12}\rangle$ defined as follows:

- If $S \in U_{i0}$ for some $i \in \{0,1\}$ then:
  - $U'_{i'j'} = U_{i'j'}$ for $i' = 1 - i$ and $j' \in \{0,1,2\}$;
  - $U'_{i0} = U_{i0} * S$;
  - If $S \subseteq F^j_{U_{i0}}$, for one $j \in \{1,2\}$, then the map U'_{ij} has domain $(\mathrm{dom}(U_{ij}) \setminus \{S\}) \cup \{S_1, S_2, S_3\}$, and $U'_{ij}(S_k) = U_{ij}(S)$ for each $k \in \{1,2,3\}$; otherwise $U'_{ij} = U_{ij}$.
- If there is (i,j) and R such that $S \in U_{ij}(R)$, then:
  - $U_{i'j'} = U_{ij}$ for all $i \in \{0,1\}$ and $j \neq j' \in \{0,1,2\}$;
  - $\mathrm{dom}(U'_{ij}) = \mathrm{dom}(U_{ij})$ and $U'_{ij}(R') = U_{ij}(R')$ for all $R \neq R' \in \mathrm{dom}(U_{ij})$, while $U'_{ij}(R) = (U_{ij}(R) \setminus \{S\}) \cup \{S_1, S_2, S_3\}$.

Let U be a $\mathbb{BL}_2$-triangulation, and let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_S$ be the vertices of a 1-simplex S of U and their Farey mediant, respectively. Then the *star refinement* $H_U * S$ of H_U at S is the 6-tuple K obtained by one of the following processes:

k1 -refinement: $S \in U_{i0}$ for some $i \in \{0,1\}$. Then let h_i be the $\mathbb{BL}_2$-hat with apex $\mathbf{v}_i$ and let $h_S = h_1 \vee h_2$. Set $K_{i0} = ((H_U)_{i0} \setminus \{h_1, h_2\}) \cup \{h_S, h_S \to h_1, h_S \to h_2\}$; moreover, if $S \in F^j_{U_{i0}}$ for one $j \in \{1,2\}$, then $\mathrm{dom}(K_{ij}) = (\mathrm{dom}((H_U)_{ij}) \setminus \{S\}) \cup \{S_1, S_2, S_3\}$, for S_1, S_2 being the 1-simplices obtained by starring S at $\mathbf{v}_S$, $S_3 = \{\mathbf{v}_S\}$, and $K_{ij}(S_k) = (H_U)_{ij}(S)$ for all $k \in \{1,2,3\}$, while for all other $R \in \mathrm{dom}(K_{ij})$, $K_{ij}(R) = (H_U)_{ij}(R)$. If $S \notin F^j_{U_{i1}} \cup F^j_{U_{i2}}$, set $K_{ij} = (H_U)_{ij}$ for all $j \in \{1,2\}$. Set $K_{i'j'} = (H_U)_{i'j'}$ for $i' = 1 - i$ and $j \in \{0,1,2\}$.

k2 -refinement: There is (i,j) and R such that $S \in U_{ij}(R)$. Then let h_i be the $\mathbb{BL}_2$-hat with apex $\mathbf{v}_i$ and let $h_S = h_1 \vee h_2$. Set $K_{ij}(R) = ((H_U)_{ij}(R) \setminus \{h_1, h_2\}) \cup \{h_S, h_S \to h_1, h_S \to h_2\}$. Set $K_{ij}(R') = (H_U)_{ij}(R')$ for all $R \neq R' \in \mathrm{dom}((H_U)_{ij})$. Set $K_{i',j'} = (H_U)_{i'j'}$ for all $(i',j') \in (\{0,1\} \times \{0,1,2\}) \setminus \{(i,j)\}$.

Proposition 2. *$H_U * S$ is the $\mathbb{BL}_2$-set associated with $U * S$.*

We now single out some families of functions in $\mathbb{BL}_2$. Each function in $\mathbb{BL}_2$ will turn out to be a combination of suitably chosen functions in these special families. In turn, we shall represent any function belonging to one of these families as a combination of $\mathbb{BL}_2$-hats in the same family.

Lemma 5. *Let $f \in \mathbb{BL}_2$ be either of the form $\langle g, \top|\mathbf{1}_1(g), \top|\mathbf{1}_2(g), \top, \top, \top\rangle$, or of the form $\langle \top, \top, \top, g, \top|\mathbf{1}_1(g), \top|\mathbf{1}_2(g)\rangle$. Then f is a finite $\odot$-combination of actual $\mathbb{BL}_2$-hats obtained by finitely many* k1*-refinements from the set of primitive hats P_{00}, or the set P_{10}, respectively.*

Proof: Consider first $f = \langle g, \top|\mathbf{1}_1(g), \top|\mathbf{1}_2(g), \top, \top, \top\rangle$. Since $g \in WH_2$, by Lemma 3, $g = \bigodot_{i\in I} k_i^{m_i}$ for suitable integers $\{m_i\}_{i\in I}$ and a finite set of actual Schauder co-hats $\{k_i\}_{i\in I}$ obtained from U_0^2 by finitely many edge star refinements. That is $\{k_i\}_{i\in I} = H_{U_u}$ for a unimodular triangulation U_u of $[0,1]^2$, and there exist 1-simplices $S_1, S_2, \ldots, S_u \subseteq [0,1]^2$ such that $U_u = U_0^2 * S_1 * S_2 * \cdots * S_u$. As each S_i is either a simplex of the fundamental partition of $[0,1]^2$ or it is obtained by a finite sequence of edge starrings from the fundamental partition, then we can form the $\mathbb{BL}_2$-triangulation $B_u = B * S_1 * \cdots * S_u$. By Proposition 1 and Proposition 2, $P_u = P * S_1 * \cdots * S_u$ is the $\mathbb{BL}_2$-set of B_u. In particular $(P_u)_{00} = \{h_i\}_{i\in I}$, where each h_i is a $\mathbb{BL}_2$-hat of kind k1 whose associated Schauder co-hat is k_i. Since $\top \vee \top = \top \to \top = \top \odot \top = \top$, then for each $(j,l) \in (\{0,1\} \times \{0,1,2\}) \setminus \{0,0\}$, it holds that $(\bigodot_{i\in I} h_i^{m_i})_{jl}$ is constantly $\top$ over its domain. Then

$$\bigodot_{i\in I} h_i^{m_i} = \langle g, \top|\mathbf{1}_1(g), \top|\mathbf{1}_2(g), \top, \top, \top\rangle .$$

The case $f = \langle \top, \top, \top, g, \top|\mathbf{1}_1(g), \top|\mathbf{1}_2(g)\rangle$ is dealt with analogously. ∎

Lemma 6. *Let $f \in \mathbb{BL}_2$ be of the form $\langle g, \top|\mathbf{0}_1(g), \top|\mathbf{0}_2(g), \emptyset, \emptyset, \emptyset\rangle$, Then f is the negation of a finite $\odot$-combination of actual $\mathbb{BL}_2$-hats obtained by finitely many* k1*-refinements from the set of primitive hats P_{00}.*

Proof: By Theorem 2, f is such that $f = \neg\neg f$. Now, $\neg f = \langle g, \top|\mathbf{1}_1(g), \top|\mathbf{1}_2(g), \top, \top, \top\rangle$, and by Lemma 5, there is a finite set $\{h_i\}_{i\in I}$ of actual $\mathbb{BL}_2$-hats obtained by finitely many k1-refinements from the set of primitive hats P_{00} and suitable positive integers $\{m_i\}_{i\in I}$ such that $\neg f = \bigodot_{i\in I} h_i^{m_i}$. Hence $f = \neg\neg f = \neg \bigodot_{i\in I} h_i^{m_i}$. ∎

Lemma 7. *Each total $\mathbb{BL}_2$-hat h belonging to kind* k2 *is obtained by finitely many* k2*-refinements from the set of primitive hats $P_{ij}(\{0\})$ or $P_{ij}((0,1))$, for some $(i,j) \in \{0,1\} \times \{1,2\}$.*

Proof: Each Schauder co-hat k in one variable is obtained by finitely many star refinements from the set of primitive co-hats $\{x_1, \neg x_1\}$. Since $\top \vee \top = \top \to \top = \top$, we immediately conclude that h is obtained by finitely many k2-refinements from the set $P_{ij}(\{0\})$ or $P_{ij}((0,1))$. ∎

There remains to deal with $\mathbb{BL}_2$-hats that are not total.

Definition 10. Let U be a $\mathbb{BL}_2$-triangulation. Fix $(i,j) \in \{0,1\} \times \{1,2\}$, and let $\hat{h}$ be the only virtual $\mathbb{BL}_2$-hat in $(H_U)_{i0}$. Pick $k \in (H_U)_{ij}(S)$ (then k is a total k2-hat). Let further h', h'' be actual k1-hats in $(H_U)_{i0}$. Denote $H^\circ = (H_U)_{i0} \setminus \{h', h'', \hat{h}\}$, $H(h') = H^\circ \cup \{h''\}$ and $H(h'') = H^\circ \cup \{h'\}$. Then the function

$$\Uparrow (h', k) = (\bigodot_{h\in H(h')} h) \to k$$

is the *vertical refinement* of the pair (h', k). The function

$$\Uparrow (h', h'', k) = (\Uparrow (h', k) \odot \Uparrow (h'', k)) \to ((\bigodot_{h\in H^\circ} h) \to k)$$

is the *vertical refinement* of the triple (h', h'', k).

Lemma 8. *Each non-total $\mathbb{BL}_2$-hat h belonging to kind* k2 *is obtained by vertical refinement from a set of hats obtained by finitely many steps of* k1-k2*-refinement from P.*

Proof: Consider a non-total k2 $\mathbb{BL}_2$-hat h, and let $h_{ij} = k|Q$ as in Definition 7, for some $(i,j) \in \{0,1\} \times \{1,2\}$. Let g be the total hat of kind k2 such that $g_{ij} = k|(0,1)$. By Lemma 7, g is obtained by k2-refinement from P. Consider first the case $(i,j) = (0,1)$ and suppose $Q = \{v\}$ for some $v \in (0,1) \cap \mathbb{Q}$. Then let U be any $\mathbb{BL}_2$-triangulation, obtained via finitely many starrings from B, such that $\mathbf{w} \in U$ for the point defined by $\pi_1(\mathbf{w}) = v$ and $\pi_2(\mathbf{w}) = 1$. Such U exists by Lemma 2. Let $K = H_U$. Then K_{00} contains an actual $\mathbb{BL}_2$-hat f with apex $\mathbf{w}$. Let $K' = K_{00} \setminus \{f, \hat{e}\}$, for $\hat{e}$ the unique virtual hat of K_{00}. Then $\bigodot_{e\in K'} e$ is an element of $\mathbb{BL}_2$ of the form $\langle f', \top|\{v\}, \top, \top, \top, \top\rangle$ for some $f' \in \mathbb{WH}_2$. Direct computation using the operations defined in Theorem 2 shows the vertical refinement $\Uparrow (f, g)$ is h. Now suppose $Q = (v_1, v_2) \subset (0,1)$ is an open unimodular segment with rational endpoints. Let U be any $\mathbb{BL}_2$-triangulation, obtained via finitely many starrings from B, such that $[\mathbf{w}_1, \mathbf{w}_2] \in U$ for points $\mathbf{w}_l$ defined by $\pi_1(\mathbf{w}_l) = v_l$ and $\pi_2(\mathbf{w}_l) = 1$, for $l \in \{1,2\}$. The existence of such U is granted by Lemma 2, again. Let f_1, f_2 be $\mathbb{BL}_2$-hats with apices $\mathbf{w}_1, \mathbf{w}_2$ in K_{00}. Direct computation now shows $\bigodot_{e\in K_{00}\setminus\{f_1,f_2,\hat{e}\}} e$ is the function $\langle f', \top|[v_1, v_2], \top, \top, \top, \top\rangle$ for some $f' \in \mathbb{WH}_2$, and hence $\Uparrow (f_1, f_2, g) = h$. The cases $(i,j) \in (\{0,1\} \times \{1,2\}) \setminus \{(0,1)\}$ are dealt with analogously. ∎

Theorem 5 (Normal Form). *Each $f \in \mathbb{BL}_2$ can be expressed as*

$$f = \nu_f(\bigodot_{j\in J_0} h_{0,j}^{m_{0,j}}) \odot \bigodot_{j\in J_1} h_{1,j}^{m_{1,j}} ,$$

where J_0 and J_1 are finite index sets, and for each $i \in \{0,1\}$, and $j \in J_i$, the exponent $m_{i,j}$ is a nonnegative integer and

$h_{i,j}$ is an actual $\mathbb{BL}_2$-hat obtained by a finite process of k1, k2*, vertical refinements from the set of primitive hats P.*

Proof: If $f_{10} = \emptyset$ then we use Lemma 6 to obtain a finite family of $\mathbb{BL}_2$-hats $\{h_{0,j}\}_{j\in J_0}$ and integers $\{m_{0,j}\}_{j\in J_0}$ such that, setting $g^{00} = \neg \bigodot_{j\in J_0} h_{0,j}^{m_{0,j}}$, we have $g^{00} = \langle f_{00}, \top \mid \mathbf{0}_1(f_{00}), \top \mid \mathbf{0}_2(f_{00}), \emptyset, \emptyset, \emptyset\rangle$. Let U be a $\mathbb{BL}_2$-triangulation such that f is linear over each simplex of U. Such U exists by Definition 8 and Lemma 3. Note that for each open simplex $S \in F_U^j$, $j \in \{1,2\}$, either f_{0j} is not defined over $[0,1]\times\pi_{3-j}(S)$ (if $j=1$) or $\pi_{3-j}(S)\times[0,1]$ (if $j=2$), or, in the notation of Definition 2, $f_{0j} = g|\pi_{3-j}(S)$ for some $g \in \mathbb{WH}_1$. In the latter case, use Theorem 4 to express g as $\bigodot_{i\in J_S} k_i^{m_i}$ for some finite set J_S, and then use Lemma 8 to build $\bigodot_{i\in J_S} h(k_i)^{m_i}$, where $h(k_i)$ is the k2 non-total hat such that $(h(k_i))_{0j} = k_i|\pi_{3-j}(S)$. Then $\bigodot_{i\in J_S} h(k_i)^{m_i}$ is $\top$ everywhere but on $[0,1]\times\pi_{3-j}(S)$ (if $j=1$) or $\pi_{3-j}(S)\times[0,1]$ (if $j=2$) where it coincides with f_{0j}. Let J_1 be the disjoint union of all sets J_S such that $S \in F_U^1 \cup F_U^2$ and $f_{01} = g|\pi_{3-j}(S)$ for some $g \in \mathbb{WH}_1$. Then $g^{00} \odot \bigodot_{i\in J_1} h(k_i)^{m_i}$ is the desired normal form for f.

In case $f_{10} \neq \emptyset$ we reason analogously, using Lemma 5 instead of Lemma 6, to obtain functions g^{00} and g^{10} such that $g^{00}_{00} = f_{00}$ and $g^{10}_{10} = f_{10}$. We then use Lemma 8 as before to obtain all non-total k2 $\mathbb{BL}_2$-hats needed. ∎

We remark that Theorem 5 cannot be strengthened by omitting virtual hats from P: the minimal set of *actual* $\mathbb{BL}_2$-hats allowing to express all elements of $\mathbb{BL}_2$ with a finite normal form is not finite.

The refinement procedure provides an explicit construction of the BL-terms whose interpretation in $\mathbb{BL}_2$ correspond to $\mathbb{BL}_2$-hats. First, we provide BL-terms whose interpretation in $\mathbb{BL}_2$ correspond to the actual primitive $\mathbb{BL}_2$-hats. We define,

$$x \lhd y = (x \to y) \odot ((y \to x) \to x),$$
$$x \diamond y = ((x \lhd y) \to y) \wedge ((y \lhd x) \to x),$$

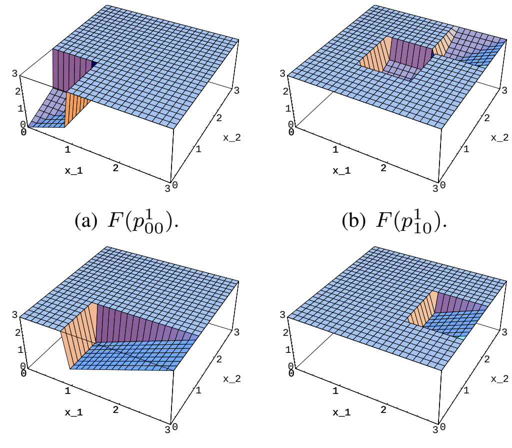
and we prepare ($i = 1, 2$),

$$x_{i00} = ((\bot \diamond x_i) \wedge (\bot \diamond x_{3-i}) \wedge (x_i \diamond x_{3-i})) \to x_i,$$
$$x_{i01} = ((\bot \diamond x_{3-i}) \wedge (x_{3-i} \lhd x_i)) \to x_i,$$
$$x_{i10} = ((\bot \lhd x_i) \wedge (\bot \lhd x_{3-i}) \wedge (x_i \diamond x_{3-i})) \to x_i,$$
$$x_{i11} = ((\bot \lhd x_i) \wedge (\bot \lhd x_{3-i}) \wedge (x_{3-i} \lhd x_i)) \to x_i.$$

Proposition 3. *The following hold:*
$(x_{100} \vee x_{200})^{\mathbb{BL}_2} = p^0_{00}$*;* $(x_{100} \to x_{200})^{\mathbb{BL}_2} = p^1_{00}$*;*
$(x_{200} \to x_{100})^{\mathbb{BL}_2} = p^2_{00}$*;* $(x_{110} \vee x_{210})^{\mathbb{BL}_2} = p^0_{10}$*;*
$(x_{110} \to x_{210})^{\mathbb{BL}_2} = p^1_{10}$*;* $(x_{210} \to x_{110})^{\mathbb{BL}_2} = p^2_{10}$*;*
$(x_{101})^{\mathbb{BL}_2} = p_{01}$*;* $(x_{202})^{\mathbb{BL}_2} = p_{02}$*;*
$(x_{111})^{\mathbb{BL}_2} = p_{11}$*;* $(x_{212})^{\mathbb{BL}_2} = p_{12}$*.*

Proof: Direct computation. ∎

Given the BL-terms for primitive hats, it is possible to iterate through the refinement process to construct BL-terms for all the actual $\mathbb{BL}_2$-hats. We provide an example of such construction (compare Lemma 7, see also [AG05] for the virtual-hat elimination algorithm for the one-variable case).



(a) $F(p^1_{00})$. (b) $F(p^1_{10})$. (c) $F(p^1_{01})$. (d) $F(p^1_{11})$.

Fig. 1. Sampling the functional representation of some primitive $\mathbb{BL}_2$-hats. In [AB09] we define an isomorphism of BL-algebras F from the BL-algebra of encodings $\mathbb{BL}_n$ to the BL-algebra of real functions from $[0, n+1]^n$ to $[0, n+1]$ generated by the projections $x_i(t_1, \ldots, t_n) = t_i$ (see [AM03]). As an example of this functional representation in the 2-variable case, we depict here the graph of the functions corresponding to some primitive $\mathbb{BL}_2$-hats.

Example 3. *We construct the BL-term whose interpretation in $\mathbb{BL}_2$ corresponds to $f = \langle \top, x_1 \vee \neg x_1, \top, \top, \top, \top\rangle$. The encoding f is obtained in a single step of* k2*-refinement from the set $\{p_{01}, \hat{p}_{01}\}$. The BL-term corresponding to f is obtained as follows: eliminate the negations from the Schauder co-hat $x_1 \vee \neg x_1$ (maintaining equivalence, compare* [Bov08] *for details), obtaining the term $x_1 \to x_1^2$. Then substitute x_1 by x_{101}. We have $(x_{101} \to x_{101}^2)^{\mathbb{BL}_2} = f$. The total $\mathbb{BL}_2$-hat h such that $h_{01} = (x_1 \vee \neg x_1)|\{0\}$ and $h_{01} = \top|(0,1)$ is obtained from f by substituting x_{101} with $(p^1_{00} \to (p^1_{00} \odot p^1_{00})) \to x_{101}$.*

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