Schauder Hats for the Two-Variable Fragment of Hájek's Basic Logic

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Outline

Background

- Free MV-Algebra
- Schauder Hats

Contribution

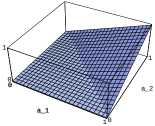
- Free BL-Algebra
- BL-Hats

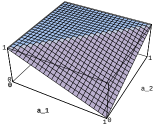
Free n-Generated MV-Algebra

Definition

$$[0,1] = ([0,1], \vee^{[0,1]}, \wedge^{[0,1]}, \odot^{[0,1]}, \rightarrow^{[0,1]}, \neg^{[0,1]}, \top^{[0,1]}, \bot^{[0,1]})$$

$$= ([0,1], \max, \min, \max\{0, x+y-1\}, \min\{1, y+1-x\}, 1-x, 1, 0)$$

$= ([0,1], \max, \min,$ 

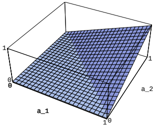
, 

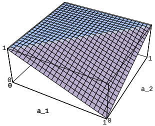
$, 1-x, 1, 0)$

Free n-Generated MV-Algebra

Definition

$$
\begin{aligned}
[0,1] &= ([0,1], \vee^{[0,1]}, \wedge^{[0,1]}, \odot^{[0,1]}, \to^{[0,1]}, \neg^{[0,1]}, \top^{[0,1]}, \bot^{[0,1]}) \\
&= ([0,1], \max, \min, \max\{0, x+y-1\}, \min\{1, y+1-x\}, 1-x, 1, 0)
\end{aligned}
$$

$= ([0,1], \max, \min,$ 

, 

$, 1-x, 1, 0)$

Fact

$\mathbf{F}_{HSP([0,1])}(n)$ *is the subalgebra of* $[0,1]^{[0,1]^n}$ *generated by the projections, with operations defined pointwise.*

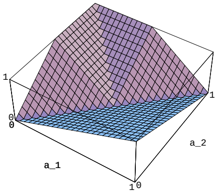
Theorem (Chang)

$[0,1]$ *generates the variety of MV-algebras (commutative bounded integral divisible prelinear involutive residuated lattices), then* $\mathbf{F}_{HSP([0,1])}(n)$ *is* $\mathbf{F}_{MV}(n)$.

McNaughton Functions

Definition (McNaughton Function)

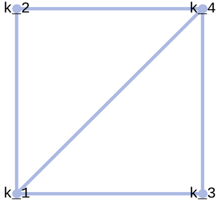
An *n-ary McNaughton function* is a continuous function $f\colon [0,1]^n \to [0,1]$ such that there exist linear polynomials with integer coefficients $p_1, \ldots, p_k\colon \mathbb{R}^n \to \mathbb{R}$ such that: for every $\mathbf{a} \in [0,1]^n$, there exists $j \in \{1, \ldots, k\}$ such that $f(\mathbf{a}) = p_j(\mathbf{a})$.

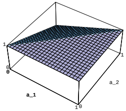


Idea

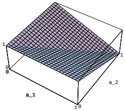
Write each McNaughton function
as a finite monoidal combination of *Schauder hats*.

Fundamental Triangulation and Primitive Hats on $[0,1]^2$

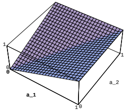




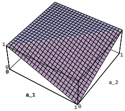
$k_2 = (y \to x)^{[0,1]} \in \mathbf{F}_{MV}(2)$



$k_4 = (\neg y \vee \neg x)^{[0,1]} \in \mathbf{F}_{MV}(2)$

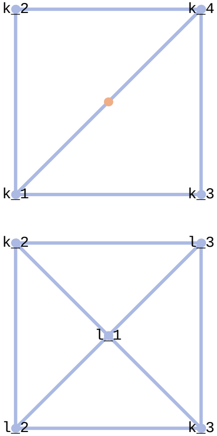


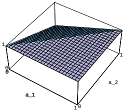
$k_1 = (x \vee y)^{[0,1]} \in \mathbf{F}_{MV}(2)$



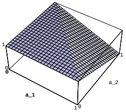
$k_3 = (x \to y)^{[0,1]} \in \mathbf{F}_{MV}(2)$

Edge Starring and Derived Hats on $[0,1]^2$

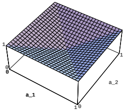




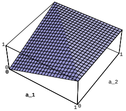
$k_2 = (y \longrightarrow x)^{[0,1]}$



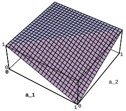
$l_3 = (l_1 \longrightarrow k_4)^{[0,1]}$



$l_1 = (k_1 \vee k_4)^{[0,1]}$

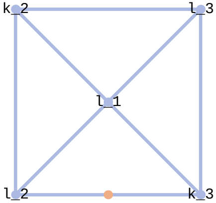


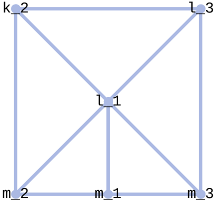
$l_2 = (l_1 \longrightarrow k_1)^{[0,1]}$

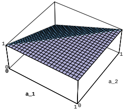


$k_3 = (x \longrightarrow y)^{[0,1]}$

Edge Starring and Derived Hats on $[0,1]^2$







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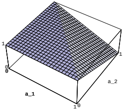

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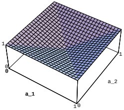
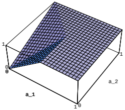
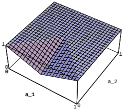


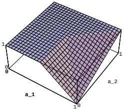
$l_1 = (k_1 \vee k_4)^{[0,1]}$



$m_2 = (m_1 \longrightarrow k_2)^{[0,1]}$



$m_1 = (k_2 \vee l_3)^{[0,1]}$



$m_3 = (m_1 \longrightarrow l_3)^{[0,1]}$

Completeness Theorem

Theorem (Panti)

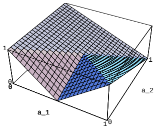
Each n-ary McNaughton function is linear on some triangulation of $[0,1]^n$ reachable from the fundamental triangulation via finitely many edge starrings.

Completeness Theorem

Corollary (McNaughton, Mundici)

1. *Each n-ary McNaughton function can be written in the form* $\bigodot_{i\in I}^{[0,1]} h_i^{m_i}$, *for some hats* h_i*'s and nonnegative integers* m_i*'s with* $i \in |I| < \omega$.

Example

Write 

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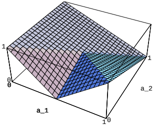
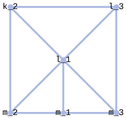
Write 

, linear on 

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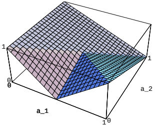
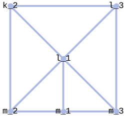
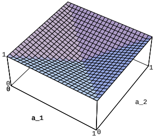
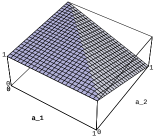
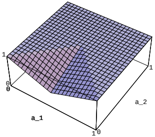
Write 

, linear on 

, as $l_1 \odot^{[0,1]} l_3 \odot^{[0,1]} m_1^2$:

() $\odot$ () $\odot$ ()^2.

Completeness Theorem

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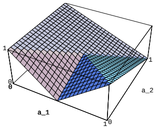
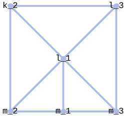
Example

Write



, linear on



, as $l_1 \odot^{[0,1]} l_3 \odot^{[0,1]} m_1^2$:

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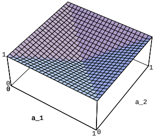
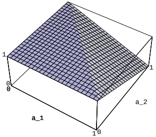
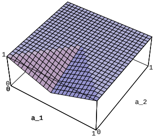


) $\odot$ (



) $\odot$ (



)^2.

2. $\mathbf{F}_{MV}(n)$ *is the algebra of n-ary McNaughton functions, with the operations defined pointwise by the operations of* $[0,1]$.

Outline

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- Free MV-Algebra
- Schauder Hats

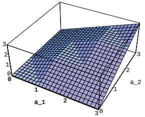
Contribution
- Free BL-Algebra
- BL-Hats

Free 2-Generated BL-Algebra

Definition

$$[0,3] = ([0,3], \vee^{[0,3]}, \wedge^{[0,3]}, \odot^{[0,3]}, \rightarrow^{[0,3]}, \neg^{[0,3]}, \top^{[0,3]}, \bot^{[0,3]})$$

$= ([0,3], \max, \min,$



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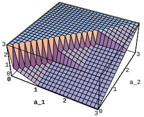

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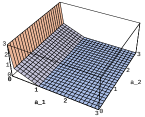
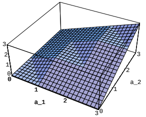


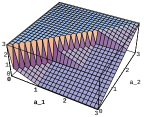
$, 3, 0)$

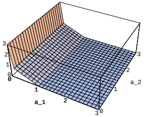
Free 2-Generated BL-Algebra

Definition

$$[0,3] = ([0,3], \vee^{[0,3]}, \wedge^{[0,3]}, \odot^{[0,3]}, \rightarrow^{[0,3]}, \neg^{[0,3]}, \top^{[0,3]}, \bot^{[0,3]})$$

$= ([0,3], \max, \min,$ 

, 

, 

$, 3, 0)$

Fact
$\mathbf{F}_{HSP([0,3])}(2)$ *is the subalgebra of* $[0,3]^{[0,3]^2}$ *generated by the projections, with operations defined pointwise.*

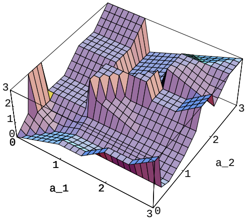
Theorem (Aglianò, Montagna)

$[0,3]$ *generates the variety* $BL(2)$ *generated by 2-generated BL-algebras (commutative bounded integral divisible prelinear residuated lattices), then* $\mathbf{F}_{HSP([0,3])}(2)$ *is* $\mathbf{F}_{BL}(2)$.

Binary BL-Functions

Roughly, $f\colon [0,3]^2 \to [0,3]$ is a BL-function iff there are McNaughton functions g_i, and maps l_i, h_i from finite *rational* partitions (blocks are rational points or open intervals with rational endpoints) of subsets of $[0,1)$ to McNaughton functions st:

$$f(\mathbf{x}) \text{ given by } \begin{cases} g_1(\mathbf{x}) & \mathbf{x} \in [0,1)^2 \\ (l_1(x_2))(x_1) & \mathbf{x} \in [1,3] \times [0,1),\, g_1(1,x_2) = 1 \\ g_1(1,x_2) & \mathbf{x} \in [1,3] \times [0,1) \\ (l_2(x_1))(x_2) & \mathbf{x} \in [0,1) \times [1,3],\, g_1(x_1,1) = 1 \\ g_1(x_1,1) & \mathbf{x} \in [0,1) \times [1,3] \\ g_2(\mathbf{x}) & \mathbf{x} \in [1,2)^2 \cup [2,3]^2 \\ (h_1(x_2))(x_1) & \mathbf{x} \in [2,3] \times [1,2),\, g_2(1,x_2) = 1 \\ g_2(1,x_2) & \mathbf{x} \in [2,3] \times [1,2) \\ (h_2(x_1))(x_2) & \mathbf{x} \in [1,2) \times [2,3],\, g_2(x_1,1) = 1 \\ g_2(x_1,1) & \mathbf{x} \in [1,2) \times [2,3] \end{cases}$$



Binary BL-Functions

Definition (Binary BL-Function)

$f\colon [0,3]^2 \longrightarrow [0,3]$ is a *binary BL-function* iff there exist McNaughton functions g_i, and maps l_i, h_i from finite rational partitions of subsets of $[0,1)$ to McNaughton functions ($i = 1, 2$), such that:

$$g_1(\mathbf{1}) = 0 \Rightarrow f(\mathbf{x}) = \begin{cases} g_1(x_1, x_2) & \mathbf{x} \in [0,1)^2 \text{ and } g_1(x_1,x_2) < 1 \\ 3 & \mathbf{x} \in [0,1)^2 \\ (l_1(x_2))(x_1 - \lfloor x_1 \rfloor) + \lfloor x_1 \rfloor & \mathbf{x} \in [1,3] \times [0,1), g_1(1,x_2) = 1, (l_1(x_2))(x_1 - \lfloor x_1 \rfloor) < 1 \\ 3 & \mathbf{x} \in [1,3] \times [0,1), g_1(1,x_2) = 1 \\ g_1(1, x_2) & \mathbf{x} \in [1,3] \times [0,1) \\ (l_2(x_1))(x_2 - \lfloor x_2 \rfloor) + \lfloor x_2 \rfloor & \mathbf{x} \in [0,1) \times [1,3], g_1(x_1,1) = 1, (l_2(x_1))(x_2 - \lfloor x_2 \rfloor) < 1 \\ 3 & \mathbf{x} \in [0,1) \times [1,3], g_1(x_1,1) = 1 \\ g_1(x_1, 1) & \mathbf{x} \in [0,1) \times [1,3] \\ 0 & \mathbf{x} \in [1,3]^2 \end{cases}$$

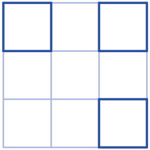
$$g_1(\mathbf{1}) = 1 \Rightarrow f(\mathbf{x}) = \begin{cases} \dots & \dots \\ g_2(x_1 - \lfloor x_1 \rfloor, x_2 - \lfloor x_2 \rfloor) + \lfloor x_1 \rfloor & \mathbf{x} \in [1,2)^2 \cup [2,3]^2 \text{ and } g_2(x_1 - \lfloor x_1 \rfloor, x_2 - \lfloor x_2 \rfloor) < 1 \\ 3 & \mathbf{x} \in [1,2)^2 \cup [2,3]^2 \\ (h_1(x_2))(x_1 - \lfloor x_1 \rfloor) + \lfloor x_1 \rfloor & \mathbf{x} \in [2,3] \times [1,2), g_2(1,x_2) = 1, (h_1(x_2))(x_1 - \lfloor x_1 \rfloor) < 1 \\ 3 & \mathbf{x} \in [2,3] \times [1,2), g_2(1,x_2) = 1 \\ g_2(1, x_2) & \mathbf{x} \in [2,3] \times [1,2) \\ (h_2(x_1))(x_2 - \lfloor x_2 \rfloor) + \lfloor x_2 \rfloor & \mathbf{x} \in [1,2) \times [2,3], g_2(x_1,1) = 1, (h_2(x_1))(x_2 - \lfloor x_2 \rfloor) < 1 \\ 3 & \mathbf{x} \in [1,2) \times [2,3], g_2(x_1,1) = 1 \\ g_2(x_1, 1) & \mathbf{x} \in [1,2) \times [2,3] \end{cases}$$

Idea

Generalize Schauder hats to BL-algebras.

Preliminaries

Fact
Blocks $[2,3]^2$, $[0,1) \times [2,3]$, $[2,3] \times [0,1)$ *are redundant.*



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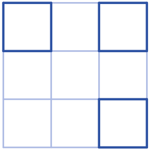

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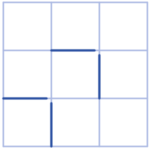
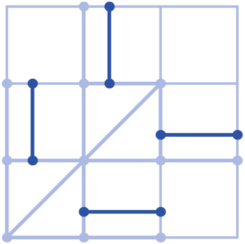


Definition

$[1,2] \times [0,1)$ *related* to $[0,1)^2$ via $\{\mathbf{x} \mid 0 \leq x_2 < 1 = x_1\}$;
$[0,1) \times [1,2]$ to $[0,1)^2$ via $\{\mathbf{x} \mid 0 \leq x_1 < 1 = x_2\}$;
$[2,3] \times [1,2)$ to $[1,2)^2$ via $\{\mathbf{x} \mid 1 \leq x_2 < 2 = x_1\}$;
$[1,2) \times [2,3]$ to $[1,2)^2$ via $\{\mathbf{x} \mid 1 \leq x_1 < 2 = x_2\}$.

Fundamental BL-Partition of $[0,3]^2$



The fundamental BL-partition of $[0,3]^2$ maps:

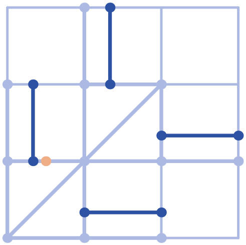
1. blocks $[0,1)^2$ and $[1,2)^2$ to fundamental triangulations of $[0,1]^2$;
2. blocks $[0,1) \times [1,2]$, $[1,2) \times [2,3]$, and $[1,2] \times [0,1)$, $[2,3] \times [1,2)$ to (pairs of) fundamental triangulations of $[0,1]$.

Edge BL-Starring in $[0, 3]^2$

Edge BL-starring acts on individual triangulations, within blocks, and affects the partitioning of related blocks.

Example

Starring at $(1/2, 1)$ the fundamental triangulation on $[0, 1)^2$,



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Edge BL-starring acts on individual triangulations, within blocks, and affects the partitioning of related blocks.

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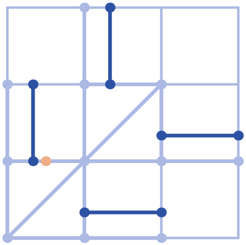

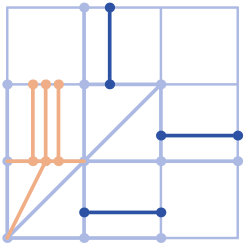

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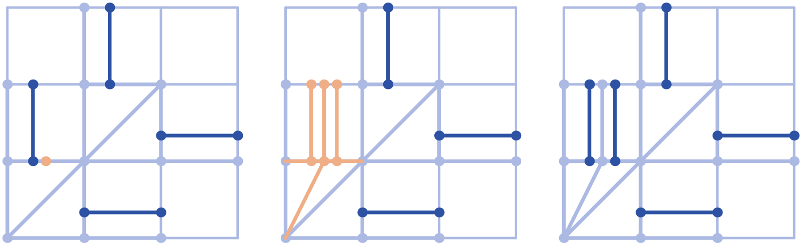

Edge BL-Starring in $[0,3]^2$

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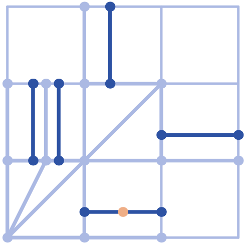
Starring a fundamental triangulation within $[1,2] \times [0,1)$ at $(3/2, \epsilon)$:



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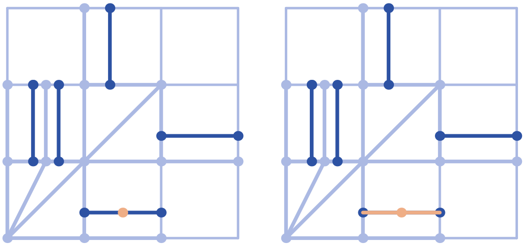

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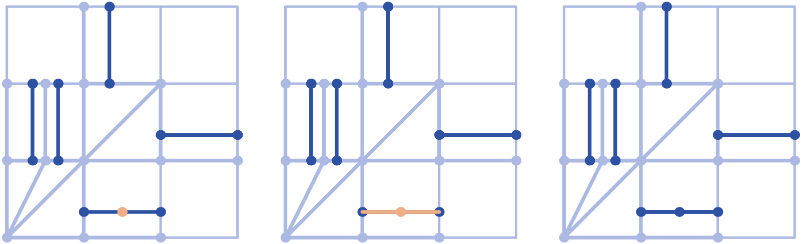
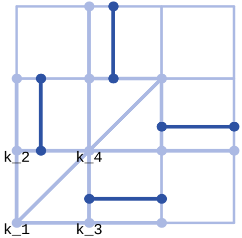
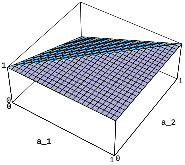
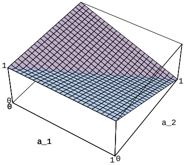


Primitive BL-Hats (Sample)

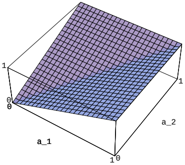




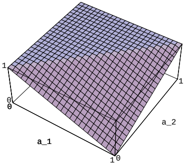
$k_2' \in \mathbf{F}_{MV}(2)$



$k_4' \in \mathbf{F}_{MV}(2)$



$k_1' \in \mathbf{F}_{MV}(2)$



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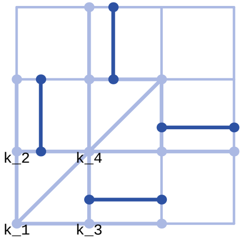
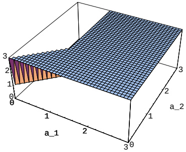
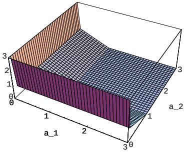




$k_2 \in \mathbf{F}_{BL}(2)$



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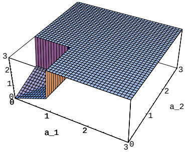
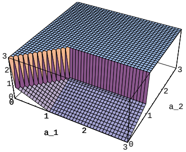


$k_1 \in \mathbf{F}_{BL}(2)$



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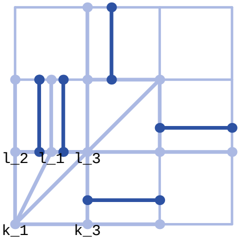
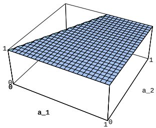
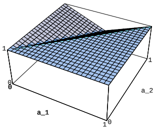
Derived BL-Hats (Sample)





$l'_2 \in \mathbf{F}_{MV}(2)$



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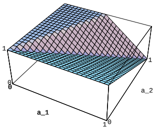

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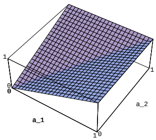

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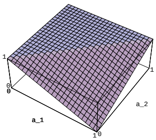

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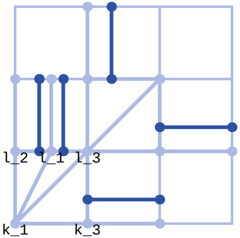

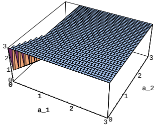

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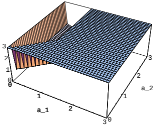

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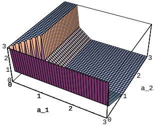

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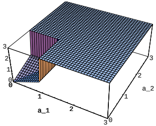

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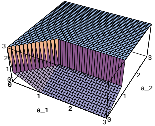

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Completeness Theorem ($n = 2$)

Theorem

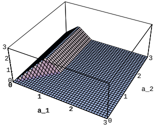
Each binary BL-function is linear on some BL-partition of $[0,3]^2$ reachable from the fundamental BL-partition via finitely many edge BL-starrings.

Completeness Theorem ($n = 2$)

Corollary

1. *Each binary BL-function can be written in the form $\bigodot_{i\in I}^{[0,3]} h_i^{m_i}$, for some BL-hats h_i's and nonnegative integers m_i's with $i \in |I| < \omega$.*

Example

Write 

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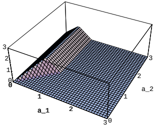
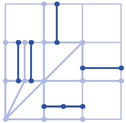
Write 

, linear on  ,

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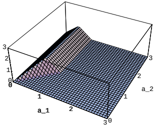
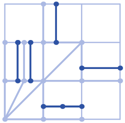
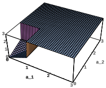
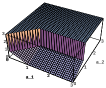
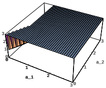
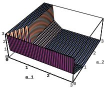
Write  , linear on  , as $\bigodot^{[0,3]}(k_1, k_3, l_2, l_3)$:

$\bigodot^{[0,3]}($  ,  ,  ,  $)$.

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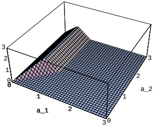
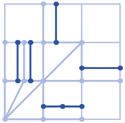
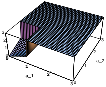
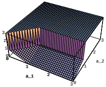
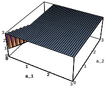
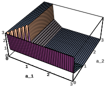
Write  , linear on  , as $\bigodot^{[0,3]}(k_1, k_3, l_2, l_3)$:

$\bigodot^{[0,3]}($  ,  ,  ,  $)$.

2. $\mathbf{F}_{BL}(2)$ *is the algebra of binary BL-functions, with the operations defined pointwise by the operations of* $[0,3]$.

Technical Remarks

1. *Virtual* BL-hats, i.e. not in $\mathbf{F}_{BL}(2)$, are necessary to give a *finite* family of primitive BL-hats.
2. Elimination of virtual BL-hats yields a construction of normal forms for $\mathbf{F}_{BL}(2)$.
3. The case $n = 2$ generalizes to the case $n < \omega$.

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Thank you!