A Bottom-Up Algorithm for t-Tautologies

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Outline

1. Motivation
   - *t*-Logic
   - Decidability of *t*-Tautologies
2. Deciding *t*-Tautologies
   - Subformulas Orders
   - Brute-Force vs. Bottom-Up
3. Conclusion
   - Open Problems
   - References

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Fuzzy Logics

Fuzzy logics are propositional logics over $\top, \bot, \odot, \rightarrow$ s.t.:

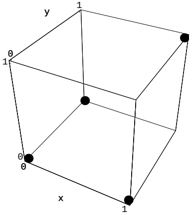
- variables $X, Y, \ldots$ are interpreted over $[0, 1]$;
- $\top$ and $\bot$ are interpreted over 1 and 0;
- $\odot$ and $\rightarrow$ are interpreted over binary functions on $[0, 1]$;
- $\neg X \rightleftharpoons X \rightarrow \bot$.

Fuzzy conjunction and implication *must* maintain:

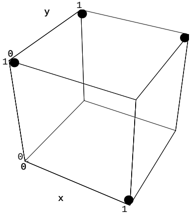
- the behavior of Boolean counterparts over $\{0, 1\}^2$;
- *intuitive* properties of Boolean counterparts over $[0, 1]^2$;
- the validity of *fuzzy modus ponens*.

Boolean Logic

Intuitive properties of Boolean conjunction and implication:



Boolean conjunction is commutative, associative, weakly increasing in both arguments, and has 1 as unit.



Boolean implication, x implies y, is 1 iff $x \leq y$, weakly decreasing in x, weakly increasing in y.

Continuous t-Norms

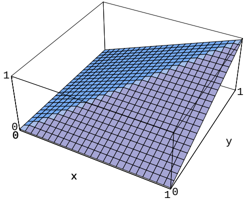
Definition (Continuous t-Norm, Residuum)

A continuous *t-norm* $\odot_*$ is a continuous binary function on $[0, 1]$ that is associative, commutative, monotone ($x \leq y$ implies $x \odot_* z \leq y \odot_* z$) and has 1 as unit ($x \odot_* 1 = x$). Given a continuous t-norm $\odot_*$, its *residuum* is the binary function $\rightarrow_*$ on $[0, 1]$ defined by $x \rightarrow_* y = max\{z : x \odot_* z \leq y\}$.

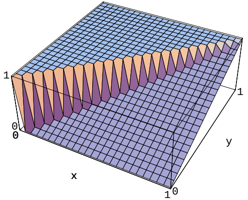
t-norms and their residua provide suitable interpretations for fuzzy conjunction and implication.

Example | Gödel Logic

$\odot_G$ and $\rightarrow_G$ over $[0,1]^2$:



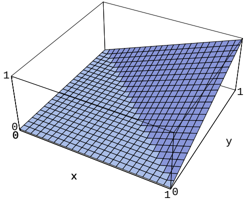
$\odot_G$ is a t-norm ...



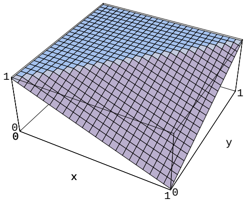
... and $\rightarrow_G$ is its residuum.

Example | Łukasiewicz Logic

$\odot_L$ and $\rightarrow_L$ over $[0,1]^2$:



$\odot_L$ is a t-norm ...



... and $\rightarrow_L$ is its residuum.

t-Tautologies

Let A be a formula over the variables $X_1, \ldots, X_m$.

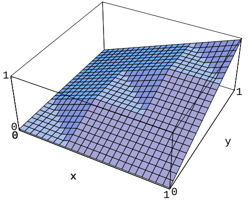
Definition (t-Tautology)

A is a *t-tautology* iff A evaluates identically to 1 for every assignment of the variables in $[0, 1]$ and every interpretation of $\odot$ over a t-norm $\odot_*$ and of $\rightarrow$ over its residuum $\rightarrow_*$.

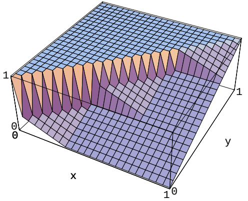
Both assignments and interpretations are infinitely many: is the t-tautology problem decidable?

t-Logic

$\odot_t$ and $\rightarrow_t$ over $[0, 1]^2$:



$\odot_t$ is a t-norm ...



... and $\rightarrow_t$ is its residuum.

Decidability of t-Tautologies

Theorem

A formula A is a t-tautology iff A evaluates identically to 1 *for every assignment of the variables in* $[0, 1]$*, interpreting* $\odot$, $\rightarrow$ *on* $\odot_t$, $\rightarrow_t$ *respectively.*

t-Logic captures all continuous t-norms and their residua.
But, is t-Logic exponential-time decidable?

Outline

Subformulas Orders | Idea

$A \in t$-TAUT iff, for every $e : \{X_1, \dots, X_m\} \to [0,1]$, $e(A) = 1$, where $\odot$, $\to$ are interpreted over $\odot_t$, $\to_t$.

Problem: The assignments are infinitely many.

Idea: For every A, there is a *finite* set $\mathcal{O}$ of *finite* objects o s.t.:

- o *covers* (possibly zero) assignments;
- the union of all o's covers all the assignments;
- o is labeled $A = \top$ iff, for every e covered by o, $e(A) = 1$;
- o is labeled $A < \top$ iff, for every e covered by o, $e(A) < 1$.

If there exist o and e such that o is labeled $A < \top$, and o covers e, then $A \notin t$-TAUT.

Subformulas Orders | Definition

Definition (Subformulas Order)

Let A be a formula of size n over m variables.
A *subformulas order* for A is a partition of the subformulas of A, $\bot$ and $\top$ into $\leq m + 2$ blocks. For $j = 0, \ldots, m + 1$, the block B_j forms a chain with least element $\bot_j = j/(m + 1)$, and holds a linear program of $O(n)$ constraints over the variables of its formulas.
The order is *semantically consistent* if and only if there exists an assignment e of the variables in $[0, 1]$ that *respects* the chains and satisfies the linear programs.

Subformulas Orders | Application

Fact

(i) *Orders are exponentially many in size(A).*

(ii) *Orders may be semantically consistent or inconsistent, and this can be decided in polynomial-time in size(A).*

(iii) *The union of consistent orders covers all the assignments.*

(iv) *$A < \top$ holds in a consistent order iff, for some e, $e(A) < 1$.*

t-TAUT $\in$ **EXPTIME**:
Search for a *consistent* order containing $A < \top$,
and output 0 if and only if such order is found.

Inconsistent orders are *useless* for deciding A.

Brute-Force vs. Bottom-Up

Instance: $A \in t$-TAUT?

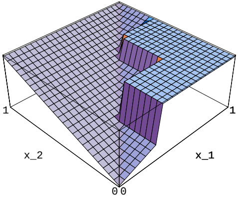
Brute-Force Version: List all the orders for A via a *purely combinatorial* procedure.

Bottom-Up Version: Build the orders for A via a *semantically oriented* procedure, avoiding a *certain* amount of useless orders.

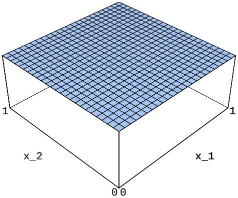
Does the bottom-up significantly shrink the search space?

Brute-Force vs. Bottom-Up

Example: $A \rightleftharpoons (((X_1 \to X_2) \to X_2) \to X_2) \to X_2$.



$= A \neq \top =$



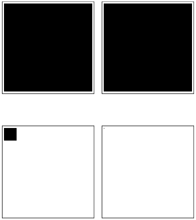
$A \notin t$-TAUT

Brute-Force vs. Bottom-Up

Search space shrinkage phenomenon:

Brute-Force:
$23,651$ orders where $\bot < X_2 < X_1 < \top$
$23,651$ orders where $\bot < X_1 < X_2 < \top$

Bottom-Up:
523 orders where $\bot < X_2 < X_1 < \top$
1 order where $\bot < X_1 < X_2 < \top$



Outline

Open Problems

- Characterize classes of easy formulas for the bottom-up method.
- Checking $2^{3n/2}$ orders suffices.
  Can the bottom-up method match this bound?

References

M. Baaz, P. Hájek, F. Montagna and H. Veith.
Complexity of t-Tautologies.
Annals of Pure and Applied Logic, 113:3–11, 2002.

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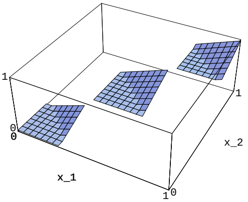
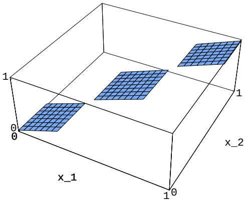
Proof Search in Hájek's Basic Logic.
To appear in *ACM Transactions on Computational Logic*.

Example ($m = 2$): If $\lfloor 3x_1 \rfloor = \lfloor 3x_2 \rfloor$, then:



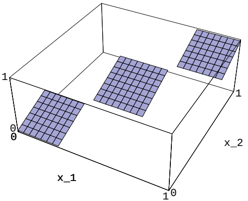
$= x_1 \odot_t x_2 < x_1 =$



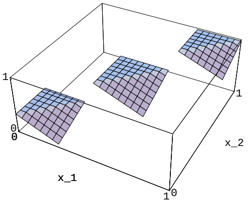
Hence, $X_1 \leq X_1 \odot X_2$ determines a *semantic inconsistency*.

Inconsistent Orders | 2

Example ($m = 2$): If $\lfloor 3x_1 \rfloor = \lfloor 3x_2 \rfloor$, then:



$= x_2 < x_1 \rightarrow_t x_2 =$



Hence, $X_1 \rightarrow X_2 \leq X_2$ determines a *semantic inconsistency*.