Type Classification of Unification Problems over Distributive Lattices and De Morgan Algebras

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Outline

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Unification Types

Let $\mathbf{P} = (P, \leq)$ be a preorder.

A *complete* set for $\mathbf{P}$ is a set $M \subseteq P$ such that:

(*i*) $x \parallel y$ for all $x, y \in M$ such that $x \neq y$;

(*ii*) for every $x \in P$ there exists $y \in M$ such that $x \leq y$.

The *type* of a preorder $\mathbf{P}$ is defined by:

$$\text{type}(\mathbf{P}) = \begin{cases} 0, & \text{if } \mathbf{P} \text{ has no complete set,} \\ \infty, & \text{if } \mathbf{P} \text{ has a complete set of infinite cardinality,} \\ p, & \text{if } \mathbf{P} \text{ has a finite complete set of cardinality } p. \end{cases}$$

Symbolic Unification

Problem SYMBEQUNIF($\mathcal{V}$)

Instance A finite set $E \subseteq \mathbb{T}_{\mathcal{V}}(n)^2$.

Solution $\zeta\colon \{x_1, \ldots, x_n\} \to \mathbb{T}_{\mathcal{V}}$ such that for all $\mathbb{A} \in \mathcal{V}$,

$$\mathbb{A} \models \bigwedge_{(s,t)\in E} s(\zeta(x_1), \ldots, \zeta(x_n)) = t(\zeta(x_1), \ldots, \zeta(x_n)).$$

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Type $\mathrm{type}_{\mathcal{V}}(E) = \mathrm{type}(\mathbf{U}_{\mathcal{V}}(E))$,
where preorder $\mathbf{U}_{\mathcal{V}}(E) = (U_{\mathcal{V}}(E), \leq)$ is defined by:

(i) $U_{\mathcal{V}}(E) = \{\zeta \mid \zeta \text{ solution to } E\}$;

(ii) $\zeta_1 \leq \zeta_2$ iff there exists $\varsigma\colon \mathbb{T}_{\mathcal{V}} \to \mathbb{T}_{\mathcal{V}}$ st for all $\mathbb{A} \in \mathcal{V}$,

$$\mathbb{A} \models \bigwedge_{i\in[n]} \zeta_1(x_i) = \varsigma \circ \zeta_2(x_i).$$

Ghilardi Algebraic Unification

Problem ALGEQUNIF($\mathcal{V}$)

Instance A finitely presented algebra $\mathbb{A} \in \mathcal{V}$.

Solution A σ-homomorphism $h \colon \mathbb{A} \to \mathbb{P}$ such that $\mathbb{P} \in \mathcal{V}$ is finitely presented projective.

Ghilardi Algebraic Unification

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(i) $U_{\mathcal{V}}(\mathbb{A}) = \{h \mid h \text{ solution to } \mathbb{A}\}$;
(ii) $h_1 \leq h_2$ iff there exists σ-hom f st $h_1 = f \circ h_2$.

Ghilardi Algebraic Unification

Theorem (Ghilardi [6])

If $E \subseteq \mathbb{T}_{\mathcal{V}}(n)^2$ *finitely presents* $\mathbb{A} \in \mathcal{V}$*, then* $\text{type}_{\mathcal{V}}(E) = \text{type}_{\mathcal{V}}(\mathbb{A})$*.*

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Proof (Idea).

Using that $\mathbb{P} \in \mathcal{V}$ is finitely presented projective iff
$\mathbb{P}$ is a retract of $\mathbb{F}_{\mathcal{V}}(n)$ for some $n < \omega$,
prove that $\mathbf{U}_{\mathcal{V}}(E)$ and $\mathbf{U}_{\mathcal{V}}(\mathbb{A})$ are equivalent categories. □

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Proof (Sketch).

For $\zeta\colon \{x_1, \ldots, x_n\} \to \mathbb{T}_{\mathcal{V}}(m)$ solution to E, define $u_\zeta\colon \mathbb{A} \to \mathbb{F}_{\mathcal{V}}(m)$ solution to $\mathbb{A}$ by $u_\zeta([t]) = [\zeta(t)]$.
(i) For $u\colon \mathbb{A} \to \mathbb{P}$ any solution to $\mathbb{A}$ with $g\colon \mathbb{P} \to \mathbb{F}_{\mathcal{V}}(l), f\colon \mathbb{F}_{\mathcal{V}}(l) \to \mathbb{P}$ st $f \circ g = \mathrm{id}_{\mathbb{P}}$, let $\zeta\colon \{x_1, \ldots, x_n\} \to \mathbb{T}_{\mathcal{V}}(m)$ be the solution to E st $g(u([x_i])) = [\zeta(x_i)]$. Prove that $u \leq u_\zeta$ and $u_\zeta \leq u$ in $\mathbf{U}_{\mathcal{V}}(\mathbb{A})$.
(ii) Prove that $\zeta_1 \leq \zeta_2$ in $\mathbf{U}_{\mathcal{V}}(E)$ iff $u_{\zeta_1} \leq u_{\zeta_2}$ in $\mathbf{U}_{\mathcal{V}}(\mathbb{A})$. □

Ghilardi Algebraic Unification

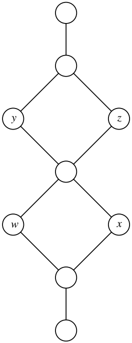

Figure: $\mathbb{L} \in \mathcal{DL}$ is finitely presented by $E = \{w \vee x = y \wedge z\}$, then $\text{type}_{\mathcal{DL}}(E) = \text{type}_{\mathcal{DL}}(\mathbb{L})$.

Ghilardi Algebraic Unification

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(i) unification is defined in terms of the categorical notions of finite presentation and projectivity, then the unification type is preserved under categorical equivalence;

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(i) unification is defined in terms of the categorical notions of finite presentation and projectivity, then the unification type is preserved under categorical equivalence;

(iii) for locally finite varieties with nice duality theorems, unification theory can be developed in the combinatorial category dual to finite algebras (in particular, the characterization of projective algebras).

Distributive Lattices | (Dual) Unification

Theorem (Birkhoff [3])

(i) *Finite bounded distributive lattices and finite posets are dually equivalent (via contravariant functors $J_{\mathcal{DL}}$ and $D_{\mathcal{DL}}$).*

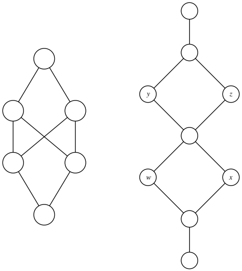

Figure: $\mathbf{P} = J_{\mathcal{DL}}(\mathbb{L})$ and $\mathbb{L} = D_{\mathcal{DL}}(\mathbf{P})$.

Distributive Lattices | (Dual) Unification

Theorem (Birkhoff [3], Balbes and Horn [1])

(i) *Finite bounded distributive lattices and finite posets are dually equivalent (via contravariant functors $J_{\mathcal{DL}}$ and $D_{\mathcal{DL}}$).*

(ii) *A finite bounded distributive lattice $\mathbb{L}$ is projective iff $J_{\mathcal{DL}}(\mathbb{L})$ is a finite nonempty lattice.*

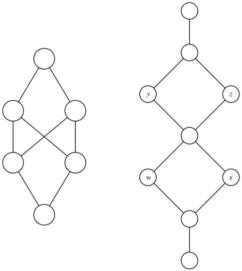

Figure: $\mathbf{P} = J_{\mathcal{DL}}(\mathbb{L})$ and $\mathbb{L} = D_{\mathcal{DL}}(\mathbf{P})$. $\mathbb{L}$ is not projective in $\mathcal{DL}$.

Distributive Lattices | (Dual) Unification

Problem EQUNIF($\mathcal{DL}$)

Instance A finite poset $\mathbf{P} = (P, \leq)$.

Solution A $\{\leq\}$-homomorphism $u\colon \mathbf{L} \to \mathbf{P}$,
where $\mathbf{L}$ is a finite nonempty lattice.

Distributive Lattices | (Dual) Unification

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Type $\mathrm{type}_{\mathcal{DL}}(\mathbf{P}) = \mathrm{type}(\mathbf{U}_{\mathcal{DL}}(\mathbf{P}))$, where preorder $\mathbf{U}_{\mathcal{DL}}(\mathbf{P}) = (U_{\mathcal{DL}}(\mathbf{P}), \leq)$ is defined by:

(i) $U_{\mathcal{DL}}(\mathbf{P}) = \{u \mid u \text{ solution to } \mathbf{P}\}$;

(ii) $u_1 \leq u_2$ iff there exists $\{\leq\}$-hom f st $u_1 = u_2 \circ f$.

Distributive Lattices | Unification Type Classification

Fact

P *is a solvable instance of* $\textsc{Unif}(\mathcal{DL})$ *iff* $P \neq \emptyset$.

Theorem ([4])

Let **P** *be a solvable instance of* $\textsc{Unif}(\mathcal{DL})$. *Then:*

$$\mathrm{type}_{\mathcal{DL}}(\mathbf{P}) = \begin{cases} p, & \textit{if every interval in } \mathbf{P} \textit{ is a lattice,} \\ & \textit{and } \mathbf{P} \textit{ has exactly } p \textit{ maximal (wrt } \subseteq\textit{) intervals;} \\ 0, & \textit{otherwise.} \end{cases}$$

Classification | Proof Idea | Type p

All intervals in **P** are lattices, **P** has p maximal intervals $\Rightarrow$ $\text{type}_{\mathcal{DL}}(\mathbf{P}) = p$:

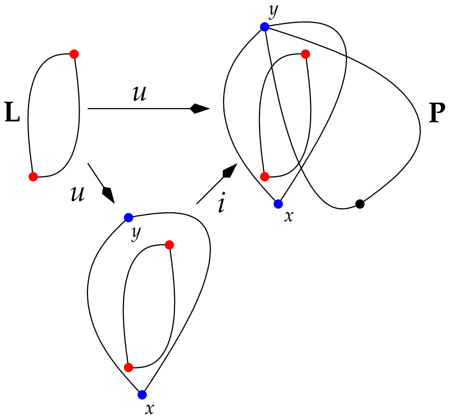


Figure: For all unifiers $u\colon \mathbf{L} \to \mathbf{P}$, there exists an inclusion map i of a maximal interval $[x, y] \subseteq P$ into **P** such that $u \leq i$ in $\mathbf{U}_{\mathcal{DL}}(\mathbf{P})$.

Classification | *Proof Idea* | *Type* 0

There exists an interval in $\mathbf{P}$ that is not a lattice $\Rightarrow$ $\text{type}_{\mathcal{DL}}(\mathbf{P}) = 0$:

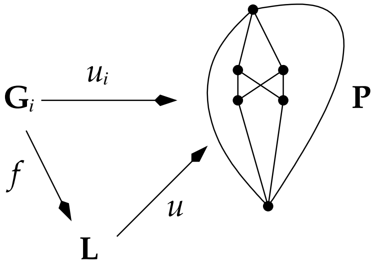


Figure: For all $i < \omega$, uniformly construct a unifier $u_i \colon \mathbf{G}_i \to \mathbf{P}$ such that, if the unifier $u \colon \mathbf{L} \to \mathbf{P}$ satisfies $u_i \leq u$, then $|L| \geq i$.

Outline

De Morgan Algebras [7]

An algebra $\mathbb{A} = (A, \wedge, \vee, ', 0, 1)$ is a *De Morgan* algebra if:

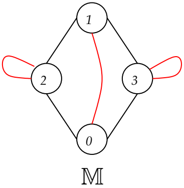
(*i*) $(A, \wedge, \vee, 0, 1)$ is a bounded distributive lattice;

(*ii*) $\mathbb{A} \models x = x''$;

(*iii*) $\mathbb{A} \models (x \wedge y)' = x' \vee y'$.

Theorem (Kalman [7])

A De Morgan algebra $\mathbb{A}$ *is subdirectly irreducible iff* $\mathbb{A} \in \{\mathbb{B}, \mathbb{K}, \mathbb{M}\}$.



De Morgan varieties ($\mathcal{B} \subset \mathcal{K} \subset \mathcal{M}$) are locally finite.

Results

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(i) Explicit characterization of injective objects in the combinatorial categories dually equivalent to finite De Morgan and Kleene algebras (key).

(ii) Complete classification of solvable instances to the (dual) De Morgan and Kleene unification problems (using the characterization),

$$\text{type}(\mathbf{Q}) = \begin{cases} 1, & \text{if the ``core'' of } \mathbf{Q} \text{ is injective;} \\ p < \omega, & \text{if the ``core'' of } \mathbf{Q} \text{ is ``almost injective'';} \\ 0, & \text{otherwise.} \end{cases}$$

Finite De Morgan Algebras | Duality

Objects Finite De Morgan algebras $\mathbb{A} = (A, \wedge, \vee, ', 0, 1)$.

Morphisms $\{\wedge, \vee, ', 0, 1\}$-homomorphisms.

Objects Finite involutive posets (fip's), that is,
finite $\{\leq, '\}$-structures $\mathbf{P} = (P, \leq, ')$ such that,
$(P, \leq)$ is a partial order,
$\mathbf{P} \models x = x''$, and $\mathbf{P} \models x \leq y$ implies $y' \leq x'$.

Morphisms $\{\leq, '\}$-homomorphisms.

Theorem (Cornish and Fowler [5])

Finite De Morgan algebras and finite involutive posets are dually equivalent (via contravariant functors $J_{\mathcal{M}}$ and $D_{\mathcal{M}}$).

Finite De Morgan Algebras | Duality

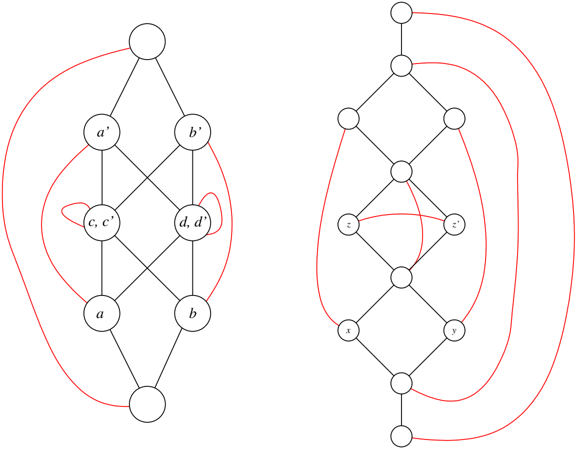

Figure: $\mathbf{P} = J_{\mathcal{M}}(\mathbb{A})$ and $\mathbb{A} = D_{\mathcal{M}}(\mathbf{P})$.

Finite De Morgan Algebras | Projective

Definition ([2])

For a cardinal κ, a poset $(Q, \leq)$ is *κ-complete* if for all $X \subseteq Q$, if all $Y \subseteq X$ such that $|Y| < \kappa$ have an upper bound, then X has a least upper bound.

Theorem ([4])

A finite De Morgan algebra $\mathbb{A}$ is projective iff $J_{\mathcal{M}}(\mathbb{A}) = (P, \leq, ')$ satisfies:

(M_1) *$(P, \leq)$ is a nonempty lattice;*

(M_2) *for all $x \in P$ st $x \leq x'$ there exists $y \in P$ st $x \leq y = y'$;*

(M_3) *$\{x \in P \mid x \leq x'\}$ with inherited order is 3-complete.*

Finite De Morgan Algebras | Projective

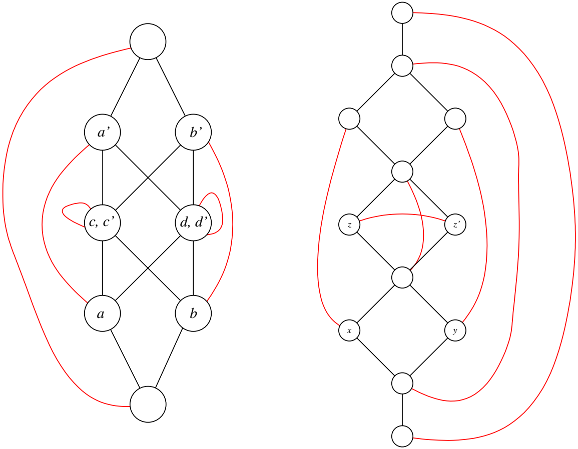

Figure: **P** fails (M_1), then $D_{\mathcal{M}}(\mathbf{P})$ is not projective.

Finite De Morgan Algebras | Projective

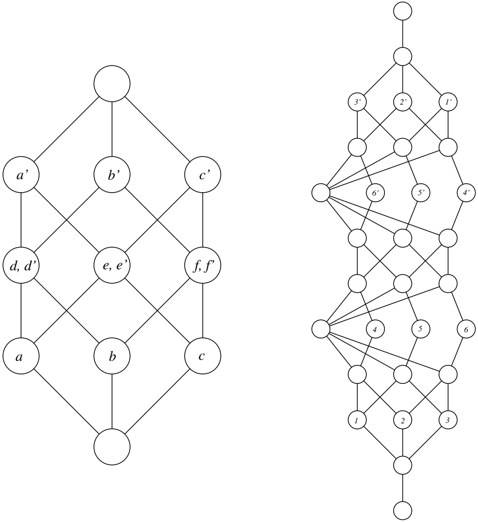

Figure: **P** fails (M_3), then $D_{\mathcal{M}}(\mathbf{P})$ is not projective.

De Morgan Algebras | Unification Core

Definition (De Morgan Unification Core)

The *De Morgan unification core* of the fip $\mathbf{Q}$ is the fip $\mathbf{Q}_m = (Q_m, \leq_m, i_m)$ st:

(*i*) $Q_m = \{x, x' \in Q \mid y \leq z, x, x' \text{ for some } y, z \in Q \text{ such that } z = z'\}$;

(*ii*) $x \leq_m y$ iff $x \leq y$ for all $x, y \in Q_m$;

(*iii*) $i_m(x) = x'$ for all $x \in Q_m$.

Lemma

If $u \colon \mathbf{P} \to \mathbf{Q}$ is a unifier for $\mathbf{Q}$, then $u(P) \subseteq Q_m$.

De Morgan Algebras | Unification Type Classification

Problem $\textsc{EqUnif}(\mathcal{M})$

Instance A finite involutive poset $\mathbf{Q} = (Q, \leq, ')$.

Solution A $\{\leq, '\}$-homomorphism $u\colon \mathbf{P} \to \mathbf{Q}$,
where $D_{\mathcal{M}}(\mathbf{P})$ is a finite projective De Morgan algebra.

Fact
$\mathbf{Q}$ *is a solvable instance of* $\textsc{EqUnif}(\mathcal{M})$ *iff* $\{x \in Q \mid x = x'\} \neq \emptyset$.

Theorem ([4])
Let $\mathbf{Q} = (Q, \leq, ')$ *be a solvable instance of* $\textsc{Unif}(\mathcal{M})$. *Then:*

$$\mathrm{type}_{\mathcal{M}}(\mathbf{Q}) = \begin{cases} p, & \text{if every interval in } \mathbf{Q}_m \text{ satisfies } (M_1), (M_2), (M_3), \\ & \text{and } \mathbf{Q}_m \text{ has exactly } p \text{ maximal intervals;} \\ 0, & \text{otherwise.} \end{cases}$$

De Morgan Classification | $\mathbf{Q} \not\models (M_1)$ *Gadget*

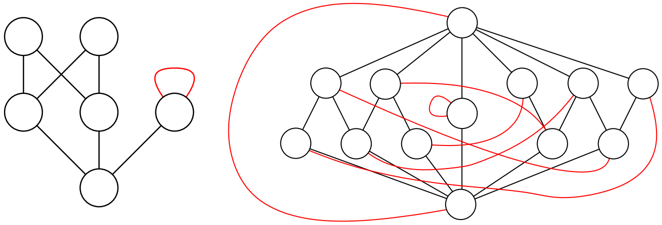

Figure: $\mathbf{Q}_m$ has poset $\mathbf{Q}_1$ (on the left). For $i < \omega$, construct $u_i \colon \mathbf{G}_i \to \mathbf{Q}_1$ such that, if the unifier $u \colon \mathbf{L} \to \mathbf{P}$ satisfies $u_i \leq u$, then $|L| \geq i$ (on the right, $\mathbf{G}_3$).

De Morgan Classification | $\mathbf{Q} \not\models (M_2)$ *Gadget*

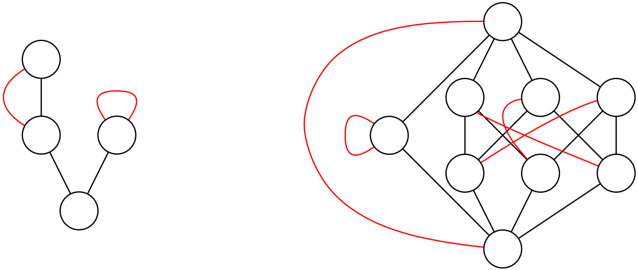

Figure: $\mathbf{Q}_m$ has poset $\mathbf{Q}_2$ (on the left). For $i < \omega$, construct $u_i \colon \mathbf{G}_i \to \mathbf{Q}_2$ such that, if the unifier $u \colon \mathbf{L} \to \mathbf{P}$ satisfies $u_i \leq u$, then $|L| \geq i$ (on the right, $\mathbf{G}_3$).

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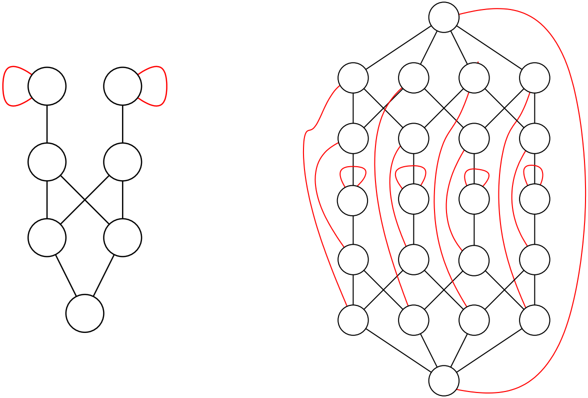

Figure: $\mathbf{Q}_m$ has poset $\mathbf{Q}_3$ (on the left). For $i < \omega$, construct $u_i \colon \mathbf{G}_i \to \mathbf{Q}_3$ such that, if the unifier $u \colon \mathbf{L} \to \mathbf{P}$ satisfies $u_i \leq u$, then $|L| \geq i$ (on the right, $\mathbf{G}_4$).

Finite Kleene Algebras | Duality and Projective

Theorem (Cornish and Fowler [5])

Finite Kleene algebras and finite involutive posets st $x \leq x'$ or $x' \leq x$ (kfip's) are dually equivalent (via contravariant functors $J_{\mathcal{K}}$ and $D_{\mathcal{K}}$).

Theorem ([4])

A finite Kleene algebra $\mathbb{A}$ is projective iff $J_{\mathcal{K}}(\mathbb{A}) = (P, \leq, ')$ satisfies:

(K_1) $\{x \in P \mid x \leq x'\}$ *with inherited order is a nonempty meet semilattice;*

(K_2) *for all* $x, y \in P$ *st* $x, y \leq y', x'$ *there exists* $z \in P$ *st* $x, y \leq z \leq z'$;

(M_2) *for all* $x \in P$ *st* $x \leq x'$ *there exists* $y \in P$ *st* $x \leq y = y'$;

(M_3) $\{x \in P \mid x \leq x'\}$ *with inherited order is 3-complete.*

Kleene Algebras | Unification Core

Definition (Kleene Unification Core)

The *Kleene unification core* of the kfip $\mathbf{Q}$ is the kfip $\mathbf{Q}_k = (Q_k, \leq_k, i_k)$ st:

(i) $Q_k = \{x, x' \in Q \mid x \leq z = z' \text{ for some } z \in Q\}$;

(ii) $x \leq_k y$ iff, $x \leq y$ and either of the following three cases occurs:

- (a) $x \leq x'$ and $y \leq y'$;
- (b) $x' \leq x$ and $y' \leq y$;
- (c) $x \leq z = z' \leq y$ for some $z \in Q$;

(iii) $i_k(x) = x'$ for all $x \in Q_k$.

Lemma

(i) *If* $u\colon \mathbf{P} \to \mathbf{Q}$ *unifies* $\mathbf{Q}$, *then* $u(P) \subseteq Q_k$ *and* $u\colon \mathbf{P} \to \mathbf{Q}_k$ *unifies* $\mathbf{Q}_k$.

(ii) $\mathbf{Q}_k$ *satisfies* (K_2) *and* (M_2).

Kleene Algebras | Unification Type Classification

Problem $\textsc{Unif}(\mathcal{K})$.

Instance A finite involutive poset $\mathbf{Q} = (Q, \leq, ')$ st $x \leq x'$ or $x' \leq x$.

Solution A homomorphism $u \colon \mathbf{P} \to \mathbf{Q}$,
where $D_{\mathcal{K}}(\mathbf{P})$ is a finite projective Kleene algebra.

Fact
$\mathbf{Q}$ *is a solvable instance of* $\textsc{Unif}(\mathcal{K})$ *iff* $\{x \in Q \mid x = x'\} \neq \emptyset$.

Theorem ([4])
Let $\mathbf{Q} = (Q, \leq, ')$ *be a solvable instance of* $\textsc{Unif}(\mathcal{K})$*. Then:*

$$\mathrm{type}_{\mathcal{K}}(\mathbf{Q}) = \begin{cases} p, & \textit{if every interval in } \mathbf{Q}_k \textit{ satisfies } (K_1) \textit{ and } (M_3), \\ & \textit{and } \mathbf{Q}_k \textit{ has exactly } p \textit{ maximal intervals;} \\ 0, & \textit{otherwise.} \end{cases}$$

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Thank you for your attention!