Finite Projective deMorgan Algebras

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honoring Jorge Martínez

Outline

Motivation

Background

Contribution

Open

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deMorgan Algebras, or Involutive Lattices [K58]

$\mathbf{A} = (A, \wedge, \vee, ', 0, 1)$ of type $(2, 2, 1, 0, 0)$. x' called *involution*.

$\mathbf{A}$ is a *deMorgan* algebra ($\mathbf{A} \in \mathcal{M}$) if:

1. $(A, \wedge, \vee, 0, 1)$ is a bounded distributive lattice;
2. $\mathbf{A} \models x = x''$ and $\mathbf{A} \models (x \wedge y)' = x' \vee y'$.

$\mathbf{A}$ is a *Kleene* algebra ($\mathbf{A} \in \mathcal{K}$) if:

1. $\mathbf{A}$ is a deMorgan algebra;
2. $\mathbf{A} \models x \wedge x' \leq y \vee y'$.

$\mathbf{A}$ is a *Boolean* algebra ($\mathbf{A} \in \mathcal{B}$) if:

1. $\mathbf{A}$ is a Kleene algebra;
2. $\mathbf{A} \models x \wedge x' = 0$.

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Remark
deMorgan algebras are finitely axiomatizable.

Projective deMorgan Algebras

Fact (Balbes and Horn [BH70], Sikorski [S51])

1. $\mathbf{A} \in \mathcal{B}$ *injective iff complete.*
2. $\mathbf{A} \in \mathcal{B}$ *projective iff countable.*

Fact (Cignoli [C75])

1. $\mathbf{A} \in \mathcal{M}$ *injective iff retract of* $\mathbf{4}^\kappa$ ($0 < \kappa$ *cardinal).*
2. $\mathbf{A} \in \mathcal{K}$ *injective iff retract of* $\mathbf{3}^\kappa$ ($0 < \kappa$ *cardinal).*

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Question

(Finite) projective Kleene and deMorgan algebras?

Applications

1. Many-Valued Logics (liar paradox)
2. Unification Theory (most general unifiers)
3. Proof Theory (rule admissibility)

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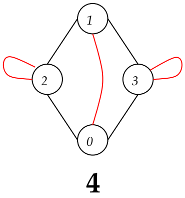
Subdirectly Irreducible deMorgan Algebras

Theorem (Kalman [K58])

$\mathbf{A} \in \mathcal{M}$ *(nontrivial) subdirectly irreducible iff* $\mathbf{A} \in \{\mathbf{2}, \mathbf{3}, \mathbf{4}\}$.



Then, (nontrivial) deMorgan varieties form a 3-element chain,

$$ISP(\mathbf{2}) = \mathcal{B} \subset ISP(\mathbf{3}) = \mathcal{K} \subset ISP(\mathbf{4}) = \mathcal{M}.$$

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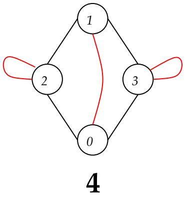

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Remark

deMorgan varieties are locally finite.

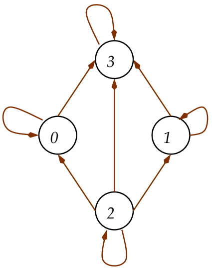
Free Finitely Generated deMorgan Algebras

Corollary

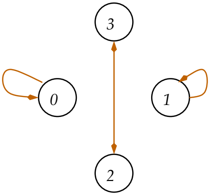
The free n-generated deMorgan algebra, $\mathbf{F}_{\mathcal{M}}(n)$*,*
is the subalgebra of $\mathbf{4}^{\mathbf{4}^n}$ *generated by the projections.*

Theorem (∼ Berman and Blok [BB01])

$\mathbf{F}_{\mathcal{M}}(n) \subseteq \{0,2,3,1\}^{\{0,2,3,1\}^n}$ *preserving* $O, D \subseteq \{0,2,3,1\}^2$. *



O



D

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*$f\colon A^n \to A$ *preserves* a relation $R \subseteq A^k$ if R is a subalgebra of $(A,f)^k$.

Graphs | Direct Products

$E \subseteq V^2$. nth *direct (or tensor) product* $E^n \subseteq (V^n)^2$ defined by, $((v_1, \ldots, v_n), (w_1, \ldots, w_n)) \in E^n$ iff $(v_i, w_i) \in E$ for all $i \in [n]$.

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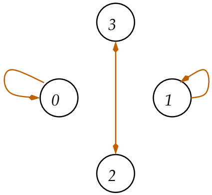

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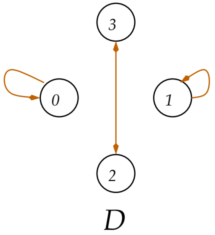

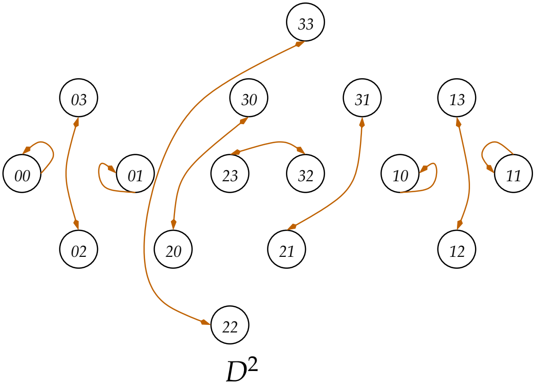

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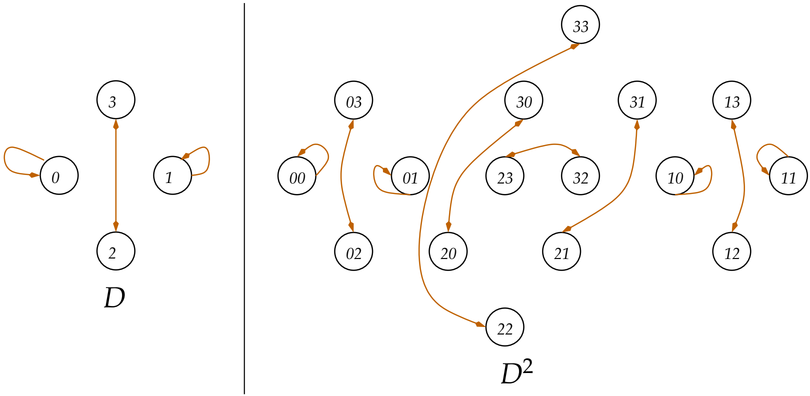

For all $x \in \{0, 2, 3, 1\}^n$, $D^n(x) = y$ iff $(x, y) \in D^n$.

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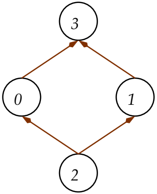

O, cover graph

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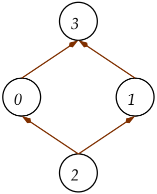

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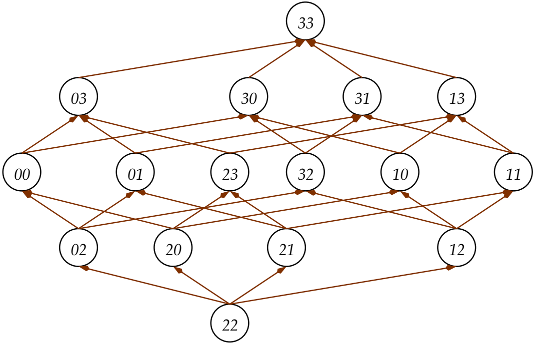

O^2, cover graph

Projective Algebras

$\mathcal{V}$ variety. $\mathbf{A}, \mathbf{B}, \mathbf{C}$ algebras in $\mathcal{V}$.

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Definition (Projective)

$\mathbf{B}$ *projective* if, for every $\mathbf{A}$, $\mathbf{C}$, $f\colon \mathbf{A} \to \mathbf{C}$ onto, $h\colon \mathbf{B} \to \mathbf{C}$, there exists $g\colon \mathbf{B} \to \mathbf{A}$ such that $f \circ g = h$.

Projective Algebras

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Definition (Retract)

B *retract* of **A** if, there are $f\colon \mathbf{A} \to \mathbf{B}$, $g\colon \mathbf{B} \to \mathbf{A}$ st $f \circ g = \mathrm{id}_B$ (f onto, g 1:1).

Theorem

$\mathbf{B}$ projective iff $\mathbf{B}$ retract of $\mathbf{F}_{\mathcal{V}}(\kappa)$ for some cardinal κ.

Projective Algebras

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$\mathbf{B}$ projective iff $\mathbf{B}$ retract of $\mathbf{F}_{\mathcal{V}}(\kappa)$ for some cardinal κ.

Corollary

$\mathcal{V}$ locally finite variety. $\mathbf{B} \in \mathcal{V}_{\mathrm{fin}}$ projective iff $\mathbf{B}$ retract of $\mathbf{F}_{\mathcal{V}}(n)$ for $n < \omega$.

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Goal

Characterize finite projective deMorgan algebras,
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Two steps:

1. instantiate Priestley duality by Cornish and Fowler [CF77] over finitely generated free deMorgan algebras (τ discrete);

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Two steps:

1. instantiate Priestley duality by Cornish and Fowler [CF77] over finitely generated free deMorgan algebras (τ discrete);
2. characterize combinatorially those objects that are dual to retracts of finitely generated free deMorgan algebras.

Finite Duality | Categories

Category **FM**, finite deMorgan algebras:

Objects: $\mathbf{A}$, finite deMorgan algebra.

Morphisms $h\colon \mathbf{A} \to \mathbf{B}$, deMorgan homomorphism.

Category **FIP**, finite involutive posets:

Objects $(\mathbf{P}, z_{\mathbf{P}})$, with $\mathbf{P} = (P, \leq_{\mathbf{P}})$ finite poset, $z_{\mathbf{P}}$ antitone bijection such that $z_{\mathbf{P}}(z_{\mathbf{P}}(x)) = x$.

Morphisms $f\colon (\mathbf{P}, z_{\mathbf{P}}) \to (\mathbf{Q}, z_{\mathbf{Q}})$, monotone map such that $f(z_{\mathbf{P}}(x)) = z_{\mathbf{Q}}(f(x))$.

Finite Duality | Contravariant Functors

Functor $J\colon \mathbf{FM} \to \mathbf{FIP}$:

Objects: $J(\mathbf{A}) = (\mathbf{P}, z_{\mathbf{P}})$, with
$\mathbf{P} = (\{[x) \mid x \text{ join irreducible in } \mathbf{A}\}, \supseteq)$,
$z_{\mathbf{P}}([x)) = A \setminus \{y' \mid y \in [x)\}$.

Morphisms $J(h\colon \mathbf{A} \to \mathbf{B}) = J(\mathbf{B}) \to J(\mathbf{A})$, where
$J(h)([x)) = h^{-1}([x))$ for all $[x) \in J(\mathbf{B})$.

Functor $D\colon \mathbf{FIP} \to \mathbf{FM}$:

Objects $D((\mathbf{P}, z_{\mathbf{P}})) = \mathbf{A} = (A, \wedge, \vee, ', 0, 1)$, with
$A = \{X \subseteq P \mid (X] = X\}$,
$X \leq Y$ iff $X \subseteq Y$, $0 = \emptyset$, $1 = P$,
$X' = P \setminus z_{\mathbf{P}}^{-1}(X)$.

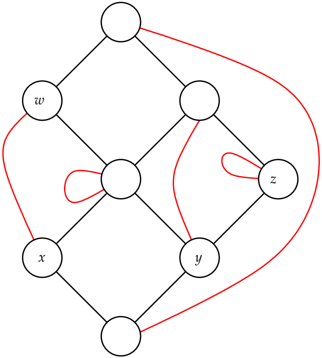
Morphisms $D(f\colon \mathbf{P} \to \mathbf{Q}) = D(\mathbf{Q}) \to D(\mathbf{P})$, where
$D(f)(X) = f^{-1}(X)$ for all $X \in D(\mathbf{Q})$.

Finite Duality | Dual Equivalence

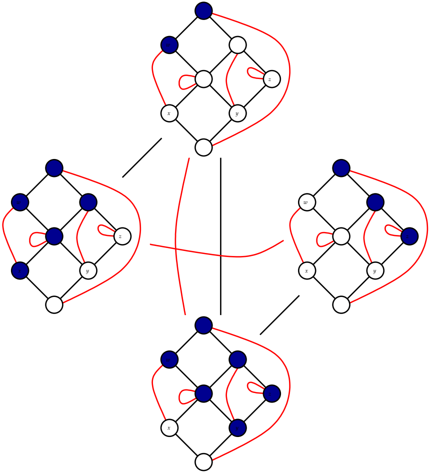
Theorem (by Cornish and Fowler [CF77])

FM *and* **FIP** *are dually equivalent via J and D.*

Example | $J(\mathbf{A})$

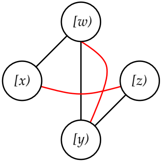


$\mathbf{A}$

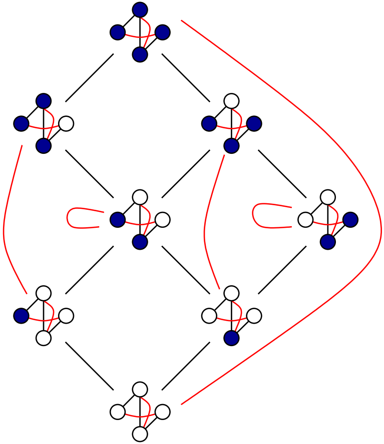


$J(\mathbf{A})$

Example | $D(J(\mathbf{A})) = \mathbf{A}$

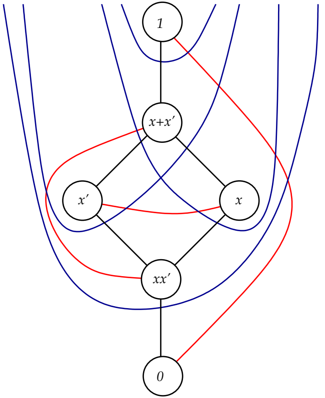


$J(\mathbf{A})$

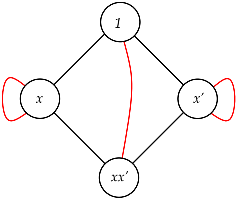


$D(J(\mathbf{A})) = \mathbf{A}$

$J(\mathbf{F}_{\mathcal{M}}(1))$



$\mathbf{F}_{\mathcal{M}}(1)$



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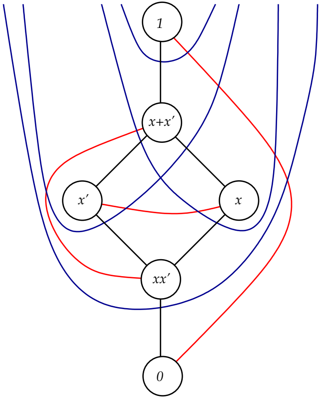
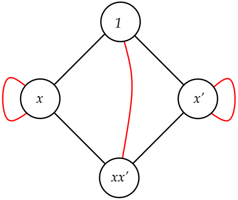


$\mathbf{F}_{\mathcal{M}}(1)$

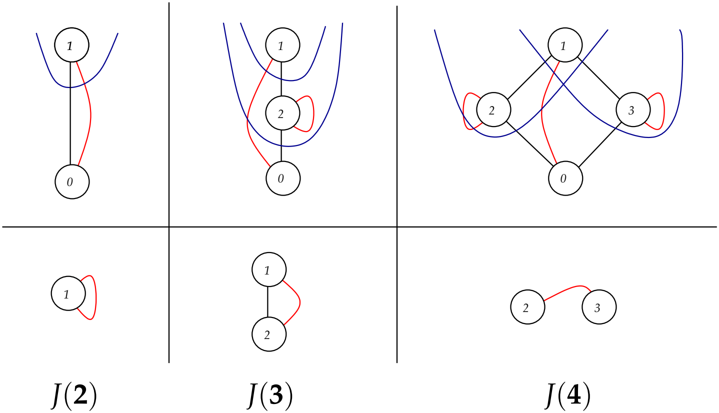


$J(\mathbf{F}_{\mathcal{M}}(1))$

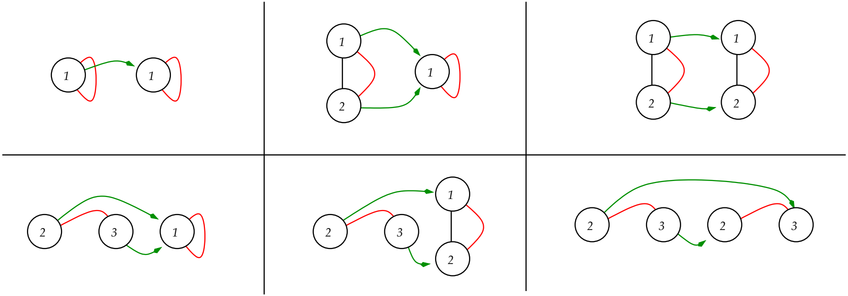
Problem

$|\mathbf{F}_{\mathcal{M}}(2)| = 168$. *Compute* $J(\mathbf{F}_{\mathcal{M}}(2))$.

$J(\mathbf{2}), J(\mathbf{3}), J(\mathbf{4})$



Morphisms over $J(\mathbf{2}), J(\mathbf{3}), J(\mathbf{4})$



† Order reflecting.

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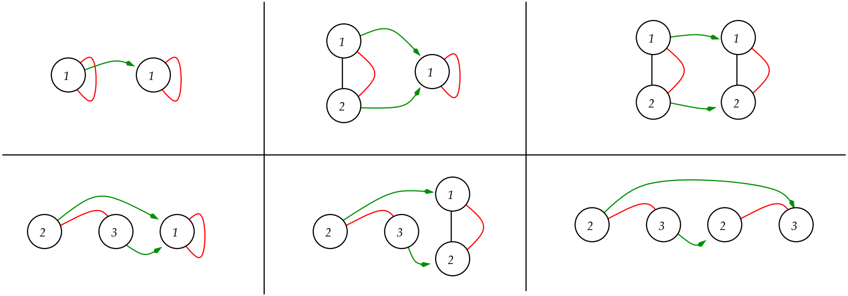

Remark

1. *Onto morphisms correspond to subalgebras (by inspection, $\mathbf{2} \leq_S \mathbf{3} \leq_S \mathbf{4}$, and $\mathbf{4} \nleq_S \mathbf{3} \nleq_S \mathbf{2}$).*

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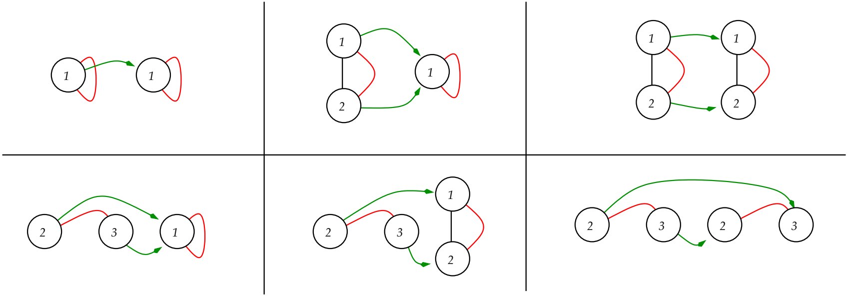
Morphisms over $J(\mathbf{2}), J(\mathbf{3}), J(\mathbf{4})$



Remark

1. *Onto morphisms correspond to subalgebras (by inspection, $\mathbf{2} \leq_S \mathbf{3} \leq_S \mathbf{4}$, and $\mathbf{4} \not\leq_S \mathbf{3} \not\leq_S \mathbf{2}$).*
2. *1:1 morphisms f st $x \leq y$ if $f(x) \leq f(y)$ † correspond to quotients (by inspection, $\mathbf{2}, \mathbf{3}, \mathbf{4}$ are simple, thus $\mathcal{M}$ is semisimple).*

† Order reflecting.

Finite Duality | Quotients

Corollary (Quotients)
$J(\mathbf{A}) = (\mathbf{P}, z_{\mathbf{P}})$. $\mathrm{Con}(\mathbf{A})$ *isomorphic to* $(\{Q \subseteq P \mid z_{\mathbf{P}}(Q) = Q\}, \supseteq)$.

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Proof (Sketch).

For 1, under Priestley (or Birkhoff) duality, onto bounded lattice homomorphisms correspond to order embeddings, thus $g \colon \mathbf{A} \to \mathbf{B}$ *onto corresponds to* $J(g) \colon J(\mathbf{B}) \to J(\mathbf{A})$ *order embedding. If* $J(\mathbf{B}) = (\mathbf{Q}, z_{\mathbf{Q}})$*, then* $J(g)(Q)$ *is essentially a subset of* P*, with inherited order and inherited involution, and by commutativity it is closed under* $z_{\mathbf{P}}$*.* □

Finite Duality | Quotients

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Remark

$\theta \in \mathrm{Con}(\mathbf{A})$ meet irreducible iff θ corresponds to some $\{x, z_{\mathbf{P}}(x)\}$ with x join irreducible in $\mathbf{A}$. For, $\{x, z_{\mathbf{P}}(x)\}$ coatom in $\mathrm{Con}(\mathbf{A})$.

Finite Duality | Projective

Corollary (Projective)

$J(\mathbf{F}_{\mathcal{M}}(n)) = (\mathbf{P}, z_{\mathbf{P}})$. $J(\mathbf{A}) = (\mathbf{Q}, z_{\mathbf{Q}})$. *Then,* $\mathbf{A}$ *projective iff,*

1. $\emptyset \neq Q \subseteq P$ *st* $z_{\mathbf{P}}(Q) = Q$;
2. *there is an involutive retraction of* P *onto* Q, *that is, a poset retraction*[‡] *such that* $z_{\mathbf{P}} \circ r = r \circ z_{\mathbf{P}}$.

[‡] $r \colon P \to Q$ onto monotone st $r(Q) = Q$.

Finite Duality | Projective

Corollary (Projective)

$J(\mathbf{F}_{\mathcal{M}}(n)) = (\mathbf{P}, z_{\mathbf{P}})$. $J(\mathbf{A}) = (\mathbf{Q}, z_{\mathbf{Q}})$. Then, $\mathbf{A}$ projective iff,

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Proof (Sketch).

$f\colon \mathbf{A} \to \mathbf{F}_{\mathcal{M}}(n)$ 1:1 and $g\colon \mathbf{F}_{\mathcal{M}}(n) \to \mathbf{A}$ onto such that $g \circ f = \mathrm{id}_{\mathbf{A}}$ iff, by duality, $J(g \circ f) = J(\mathrm{id}_{\mathbf{A}})$ iff, $J(f) \circ J(g) = \mathrm{id}_{J(\mathbf{A})}$, where $J(g)\colon J(\mathbf{A}) \to J(\mathbf{F}_{\mathcal{M}}(n))$ order embedding, and $J(f)\colon J(\mathbf{F}_{\mathcal{M}}(n)) \to J(\mathbf{A})$ onto. For 1, use the previous corollary. For 2, $J(f)$ monotone onto implies $J(f)\colon P \to Q$ poset retraction that commutes with $z_{\mathbf{P}}$. □

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$\mathbf{F}_{\mathcal{M}}(n)$ | *Dual Object*

Theorem

$J(\mathbf{F}_{\mathcal{M}}(n))$ *isomorphic to* $((\{0, 2, 3, 1\}^n, O^n), D^n)$.

$\mathbf{F}_{\mathcal{M}}(n)$ | *Dual Object*

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$J(\mathbf{F}_{\mathcal{M}}(n))$ isomorphic to $((\{0,2,3,1\}^n, O^n), D^n)$.

Proof (Sketch).

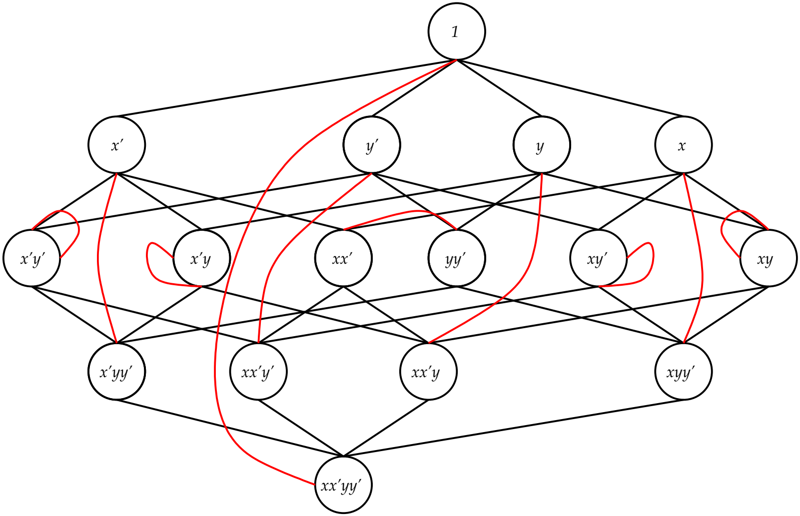
Fix antichain $X \subseteq \{0,2,3,1\}^n$ such that $(X] \cup D^n((X]) = \{0,2,3,1\}^n$ and $(X] \cap D^n((X]) = \{0,1\}^n$. Define $B = \{0,1\}^n = D^n(B)$; $B_{k,2} = (B] \setminus B = D^n(B_{k,1})$; $B_{k,1} = [B) \setminus B = D^n(B_{k,2})$; $B_{m,2} = (X] \setminus B_{k,2} = D^n(B_{m,3})$; $B_{m,3} = [X) \setminus B_{k,1} = D^n(B_{m,2})$. Then, $\{B, B_{k,2}, B_{k,1}, B_{m,2}, B_{m,3}\}$ 5-partition of $\{0,2,3,1\}^n$.

Define $M\colon \{0,2,3,1\}^n \to 2^{\{x_i, x_i' \mid i \in [n]\}}$, for $\mathbf{a} = (a_1, \dots, a_n)$, by:

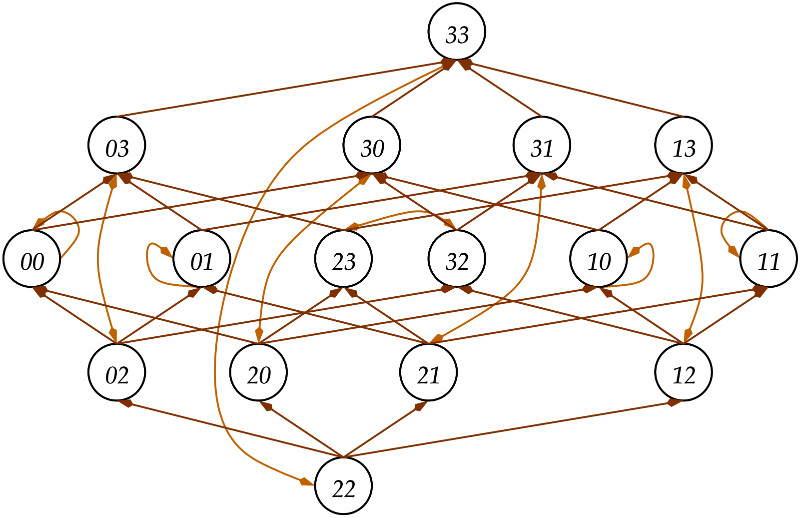
$$M(\mathbf{a}) = \begin{cases} \{x_i', x_j \mid a_i = 0, a_j = 1\}, & \text{if } \mathbf{a} \in B; \\ \{x_i', x_j, x_l', x_l \mid a_i = 0, a_j = 1, a_l = 2\}, & \text{if } \mathbf{a} \in B_{k,2}; \\ \{x_i', x_j \mid a_i = 0, a_j = 1\}, & \text{if } \mathbf{a} \in B_{k,1}; \\ \{x_i', x_j, x_l', x_l \mid a_i = 0, a_j = 1, a_l = 2\}, & \text{if } \mathbf{a} \in B_{m,2}; \\ \{x_i', x_j, x_l', x_l \mid a_i = 0, a_j = 1, a_l = 3\}, & \text{if } \mathbf{a} \in B_{m,3}. \end{cases}$$

By direct computation, M isomorphism ($\bigwedge M(\mathbf{a})$ is the minterm at $\mathbf{a}$, ie, the smallest term operation m in $\mathbf{F}_{\mathcal{M}}(n)$ st $m(\mathbf{a}) = i$, for a suitable i depending on $\mathbf{a}$). □

$J(\mathbf{F}_{\mathcal{M}}(2))$



$J(\mathbf{F}_{\mathcal{M}}(2)) = ((\{0, 2, 3, 1\}^2, O^2), D^2)$



Involutive Retractions | Problem

Instance $\emptyset \neq Q \subseteq \{0, 2, 3, 1\}^n$ st $D^n(Q) = Q$.

Involutive Retractions | *Problem*

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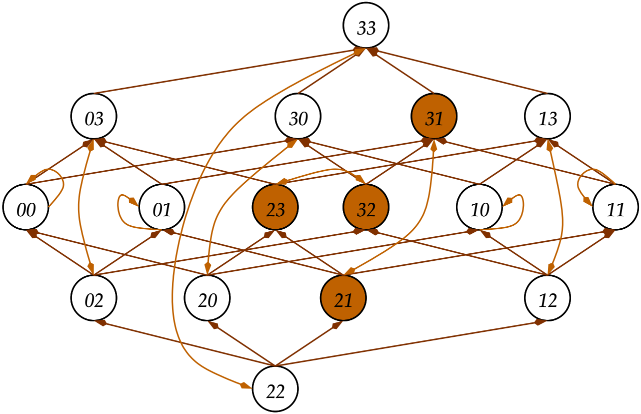

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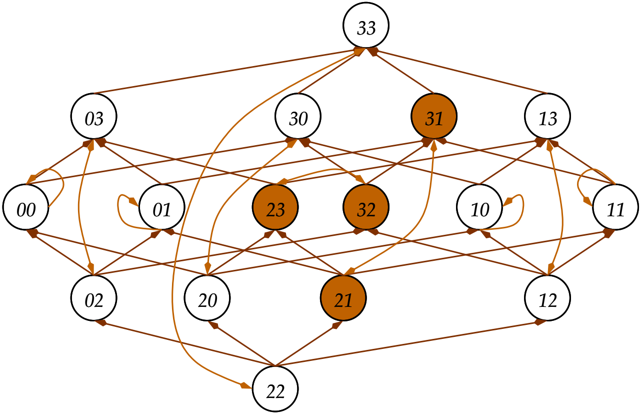


Question Is there a poset retraction r of $\{0,2,3,1\}^n$ onto Q such that $r \circ D^n = D^n \circ r$?

Involutive Retractions | Idea

(*i*) $\emptyset \neq Q \subseteq \{0,2,3,1\}^n$ st $D^n(Q) = Q$. Think $D(Q) \in H(\mathbf{F}_{\mathcal{M}}(n))$.

(*ii*) Take a morphism r in **FIP** from $\{0,2,3,1\}^n$ onto Q. r is an involutive poset retraction of $\{0,2,3,1\}^n$ onto Q that is, $r\colon \{0,2,3,1\}^n \to Q$ monotone onto st $r(Q) = Q$ and $r \circ D^n = D^n \circ r$.

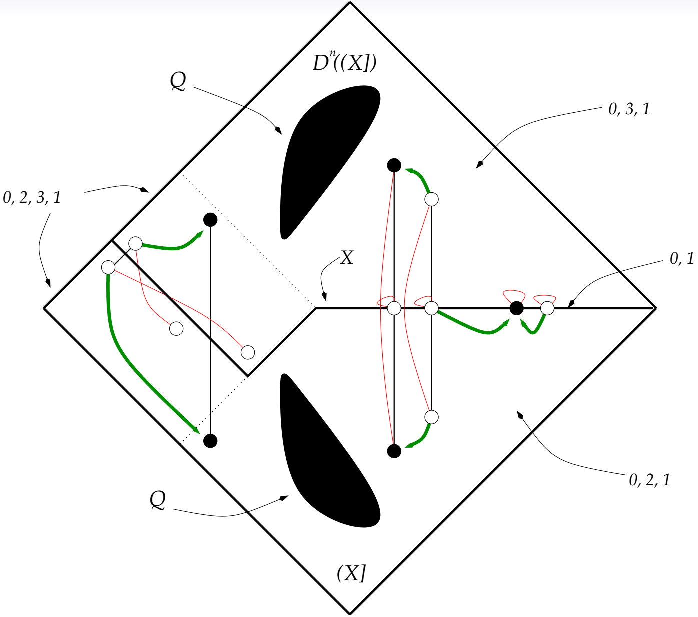
(*iii*) The goal is to collect *necessary* combinatorial conditions imposed by such an r on Q, until *sufficient* conditions arise such that conversely, $\{0,2,3,1\}^n$ admits an involutive retraction onto any Q satisfying those conditions. This suffices to characterize (duals of) finite projective deMorgan algebras.

(*iv*) $B = \{0,1\}^n$. Clearly, $r(B) = Q \cap B \neq \emptyset$ (trivial, enough for Boolean case) and $r((B]) = Q \cap (B]$.

(*v*) The key insight is that r determines in a natural (nontrivial) way a partition of $\{0,2,3,1\}^n$ into two blocks, say $(X]$ and $D^n((X])$ for some antichain $X \subseteq \{0,2,3,1\}^n$, such that $(X] \cap D^n((X]) = B$, and $(X], D^n((X])$, as well as $Q \cap (X], Q \cap D^n((X])$, are dual isomorphic via D^n (the latter since $D^n(Q) = Q$). The Kleene case reduces to the particular case where $X = B$.

(*vi*) The first (easy, enough for Kleene case) observation is that therefore, the behaviour of r over $D^n((X])$ is encoded by $r|_{(X]}$, since $r \circ D^n = D^n \circ r$.

(*vii*) The second (tricky, necessary for $\mathcal{M}$) observation is that moreover, X must satisfy a certain combinatorial property such that, if $x \leq y$ with $x \in (X]$ and $y \in D^n((X])$, then $r(x) \leq r(y)$.


$D^n((X])$
Q
0, 3, 1
0, 2, 3, 1
X
0, 1
0, 2, 1
Q
(X]

Main Result

$\emptyset \neq Q \subseteq \{0, 2, 3, 1\}^n$ st $D^n(Q) = Q$.

Definition (Interface)

$X \subseteq \{0, 2, 3, 1\}^n$ is an *interface (for Q)* if X is an antichain such that:

(I1) $(X] \cup D^n((X]) = \{0, 2, 3, 1\}^n$ and $(X] \cap D^n((X]) = B$;

(I2) for all $x \in (X]$ and $y \in D^n((X])$, if $x \leq y$ then,

$$\bigvee_{Q \cap (X]} \{z \in Q \cap (X] \mid z \leq x\} \leq \bigwedge_{Q \cap D^n((X])} \{w \in Q \cap D^n((X]) \mid w \geq y\}.$$

Definition (Better Embedded)

Q is *better embedded* in $\{0, 2, 3, 1\}^n$ if:

(E1) There exists an interface X for Q such that $Q \cap (X]$ is a meet semilattice well embedded[§] in $(X]$, with $Q \cap (B]$ well embedded in $(B]$.

(E2) Every $x \in Q \cap (B] \setminus B$ is comparable to some $y \in Q \cap B$.

[§] S poset. $R \subseteq S$ is *well embedded* in S if every $X \subseteq R$ with an upper bound in S has an upper bound in R [BB89].

Main Result

Theorem (Finite Projective deMorgan Algebras)

Let $\mathbf{A} = D(Q)$. $\mathbf{A}$ *projective iff,*
Q *is better embedded in* $\{0, 2, 3, 1\}^n$ *for some* $n < \omega$.

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Proof ($\Rightarrow$).

Given retraction r of $\{0, 2, 3, 1\}^n$ onto Q st $r(D^n(x)) = D^n(r(x))$. Let $B_d \subseteq (\{x \mid \text{level}(x) = n\}]$ st $B_d \cup D^n(B_d) = B$. $r^{-1}(Q \cap B_d)$ meet semilattice dual isomorphic to $\{0, 2, 3, 1\} \setminus r^{-1}(Q \cap B_d)$ via D^n since $Q = D^n(Q)$. Define $X \subseteq \{0, 2, 3, 1\}$ antichain by $X = \{x \mid x \text{ maximal in } r^{-1}(Q \cap B_d)\}$. Notice $(X] = r^{-1}(Q \cap B_d)$. Check (I1)-(I2). By construction, $r|_{(X]}$ retraction of $(X]$ onto $Q \cap B_d = Q \cap (X]$, then by [BB89, Lemma 2.4], since $(X]$ is a (finite, so complete) meet semilattice, $Q \cap (X]$ is a meet semilattice well embedded in $(X]$. Check better embedding. Let $X \subseteq Q \cap (B]$ with an upper bound b in $(B]$; then $r(b) = c$ for some upper bound c of $X \subseteq Q \cap (B]$, ie, $Q \cap (B]$ is well embedded in $(B]$. (E2) is easily seen necessary (ow, there is $x \in Q \cap B_{k,2}$ incomparable with every $y \in Q \cap B$, then there is $v \in B \setminus Q$ st $x \leq v$ but $r(v) \parallel y$, contradiction). □

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Proof ($\Leftarrow$).

Let X be an interface st $Q \cap (X]$ is a (finite, hence complete) meet semilattice well embedded in $(X]$. Then by [BB89, Theorem 2.7], $r(x) = \bigvee_{Q \cap (X]}\{y \in Q \cap (X] \mid y \leq x\}$ for all $x \in (X]$ retraction of $(X]$ onto $Q \cap (X]$. Since $Q \cap (B]$ well embedded in $(B]$ by (E2), the construction yields $r((B]) = Q \cap (B]$. Possibly fix $r(x) \in (B] \setminus B$ using (E3). Extend r to $\{0, 2, 3, 1\}^n$ by $r(D^n(x)) = D^n(r(x))$, sound since for all $x \in B$, we have $x = D^n(x) \in B$, but $r(x) = r(D^n(x))$. Sufficient to check r retraction (onto Q is clear). If $x \leq y \in (X]$, then $r(x) \leq r(y)$. If $x \leq y \in [D^n(X))$, then $D^n(y) \leq D^n(x) \in (X]$, then $r(D^n(y)) \leq r(D^n(x))$, then $D^n(r(y)) \leq D^n(r(x))$, then $r(x) \leq r(y)$. Case $y < x$ with $x \in (X]$, $y \in [D^n(X))$ impossible. Case $x < y$ with $x \in (X]$, $y \in [D^n(X))$, by (I2), $r(x) \leq r(y)$. □

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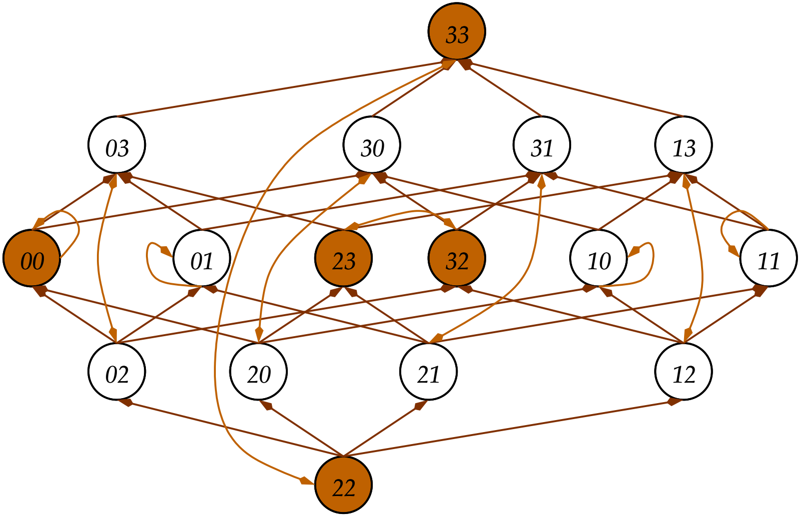
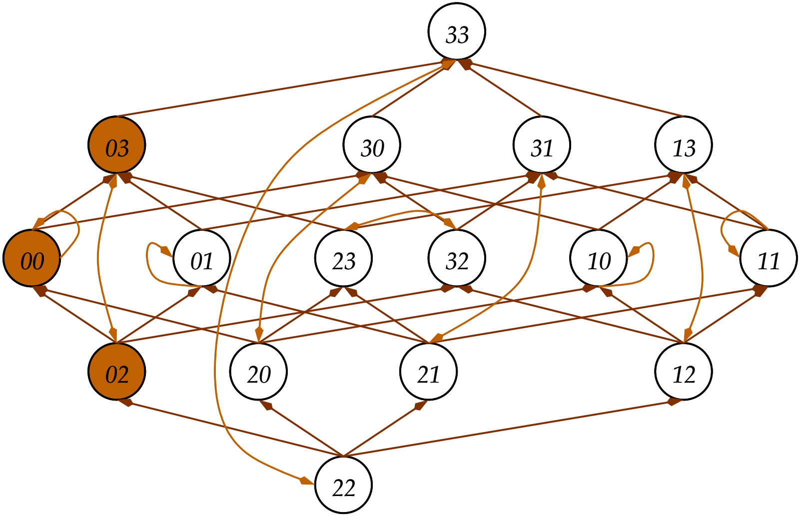
Problem

Projective deMorgan algebras?

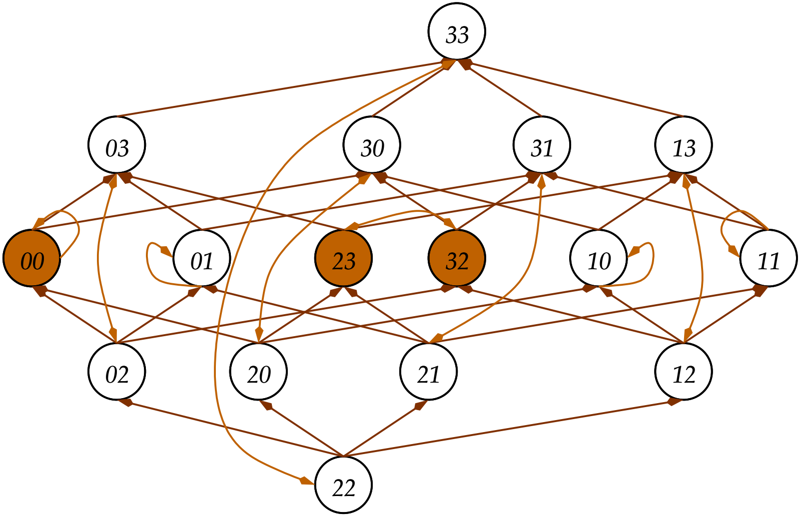
Example 1 | Projective



Example 2 | Projective



Example 3 | Not Projective



Outline

Motivation

Background

Contribution

Open

Complexity

Tabular presentation of $\mathbf{A} \in \mathcal{M}$ finite has size $\leq 5|A|^2(2\lg|A|) = o(|A|^3)$.
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Evidence.

Heyting finitary but Heyting plus $(x \wedge y)' = x' \vee y'$ *unitary,*
and finite bounded distributive lattices finitary. □

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Does it give any nontrivial insight on the lattice of clones of Kleene operations? Very interesting.

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Thank you!