Lewis Dichotomies in Many-Valued Logics

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Boolean Satisfiability

x	y	$\bot^2$	$\wedge^2$	$\oplus^2$	$\vee^2$	$\neg^2$	$\top^2$
0	0	0	0	0	0	1	1
0	1	0	0	1	1	0	1
1	0	0	0	1	1	1	1
1	1	0	1	0	1	0	1

Instance A term $t(x_1, \dots, x_n)$ on signature $(\wedge, \neg)$.

Question $(\{0,1\}, \wedge^2, \neg^2, \top^2) \models \exists x_1 \dots \exists x_n (t = \top)$?

Computationally tractable (in P) or intractable (NP-complete)?
Intractable, it is necessary to check all $\{0,1\}^{\{x_1,\dots,x_n\}}$.

Boolean Satisfiability

x	y	$\bot^{\mathbf{2}}$	$\wedge^{\mathbf{2}}$	$\oplus^{\mathbf{2}}$	$\vee^{\mathbf{2}}$	$\neg^{\mathbf{2}}$	$\top^{\mathbf{2}}$
0	0	0	0	0	0	1	1
0	1	0	0	1	1	0	1
1	0	0	0	1	1	1	1
1	1	0	1	0	1	0	1

Instance A term $t(x_1, \ldots, x_n)$ on signature $(\bot, \wedge, \vee, \top)$.

Question $(\{0,1\}, \{\bot^{\mathbf{2}}, \wedge^{\mathbf{2}}, \vee^{\mathbf{2}}, \top^{\mathbf{2}}\}) \models \exists x_1 \ldots \exists x_n (t = \top)$?

Tractable or intractable?
Tractable, it is sufficient to check $\{1\}^{\{x_1,\ldots,x_n\}}$.

Outline

Outline

Clones Lattice

A nonempty set, $0 \leq n < \omega$. $\mathbf{O}_n = A^{A^n}$, n-ary operations on A.
$\mathbf{O} = \mathbf{O}_A = \bigcup_n \mathbf{O}_n$, the finitary operations on A.

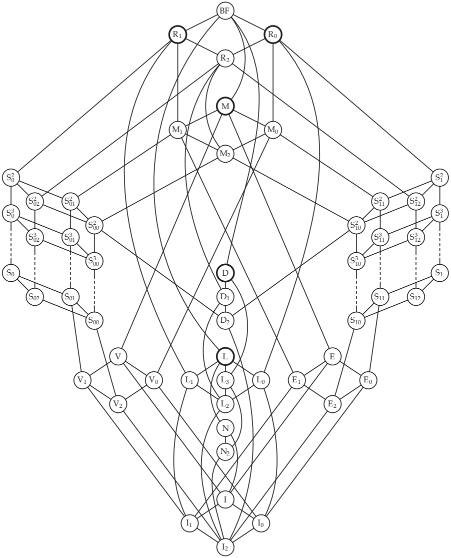
Definition

A *clone* on A is a subset of $\mathbf{O}$ containing the projection operations and closed under compositions.
$\mathrm{Cl}(A) = \{C \mid C \text{ clone on } A\}$. C_n, n-ary operations in $C \in \mathrm{Cl}(A)$.

Fact

$\mathbf{Cl}(A) = (\mathrm{Cl}(A), \subseteq)$ *is a bounded algebraic lattice.*

Clones Lattice | Boolean Case



Coclones Lattice

A nonempty set, $1 \leq n < \omega$. $\mathbf{R}_n = 2^{A^n}$, n-ary relations on A.
$\mathbf{R} = \bigcup_n \mathbf{R}_n$, the finitary relations on A.

Definition

A *coclone* on A is a subset of $\mathbf{R}$ containing the diagonal relation and closed under Cartesian products, identification of coordinates, and projection of coordinates.
$\mathrm{Co}(A) = \{S \mid S \text{ coclone on } A\}$.

Fact

$\mathbf{Co}(A) = (\mathrm{Co}(A), \subseteq)$ *is a bounded algebraic lattice.*

Lattice Antiisomorphism

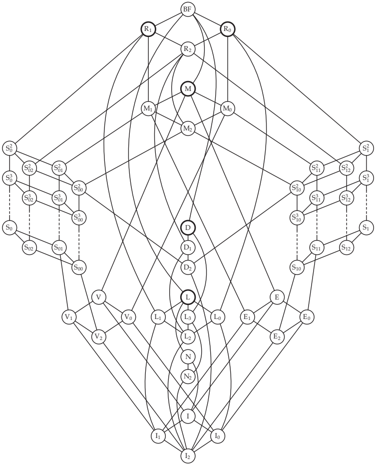
Definition

$R \in \mathbf{R}_k, f \in \mathbf{O}$. f *preserves* R if R is a subuniverse of $(A, f)^k$.
$S \subseteq \mathbf{R}$. $\mathrm{Pol}(S)$, the set of all operations on A that preserve each relation in S (the *polymorphisms* of S).
$F \subseteq \mathbf{O}$. $\mathrm{Inv}(F)$, the set of all relations on A that are preserved by each operation in F (the *invariants* of F).

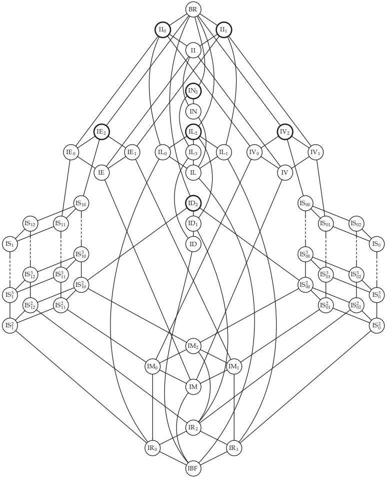
Theorem (Lattice Antiisomorphism)

$\mathrm{Cl}(A)$ *and* $\mathrm{Co}(A)$ *are lattice antiisomorphic via* Pol *(or* Inv*).*

Lattice Antiisomorphism | Boolean Case



Boolean clones (Post lattice).



Boolean coclones.

Relational Presentation of Clones

Notation

$F \subseteq \mathbf{O}$. $[F]$ is the (clone) closure of F, the smallest clone on A containing F. $S \subseteq \mathbf{R}$. $[S]$ is the (coclone) closure of S, the smallest coclone on A containing S.

Corollary

Let $F \subseteq \mathbf{O}$. Then, there exists $S \subseteq \mathbf{R}$, unique up to coclone closure, such that $[F] = \mathrm{Pol}(S)$.

Example (Monotone Boolean Operations)

$F = \{\bot^{\mathbf{2}}, \wedge^{\mathbf{2}}, \vee^{\mathbf{2}}, \top^{\mathbf{2}}\} \subseteq \mathbf{O}_{\{0,1\}}$. $[F] = M = \mathrm{Pol}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$.

Lattice Antiisomorphism

Lemma (Geiger, Bodnarchuk et al. [L06])

$C, C' \in \mathrm{Cl}(A)$, $S, S' \in \mathrm{Co}(A)$. $F \subseteq \mathbf{O}$, $S \subseteq \mathbf{R}$. *Then:*

1. $C \subseteq C' \Rightarrow \mathrm{Inv}(C') \subseteq \mathrm{Inv}(C)$, $S \subseteq S' \Rightarrow \mathrm{Pol}(S') \subseteq \mathrm{Pol}(S)$.
2. $\mathrm{Inv}(F) = \mathrm{Inv}([F]) = [\mathrm{Inv}(F)]$, $\mathrm{Pol}(S) = \mathrm{Pol}([S]) = [\mathrm{Pol}(S)]$.
3. $\mathrm{Pol}(\mathrm{Inv}(F)) = [F]$, $\mathrm{Inv}(\mathrm{Pol}(S)) = [S]$.

Proof of Lattice Antiisomorphism.

By (2), $\mathrm{Inv}\colon \mathrm{Cl}(A) \to \mathrm{Co}(A)$ and $\mathrm{Pol}\colon \mathrm{Co}(A) \to \mathrm{Cl}(A)$. By (1) and (3), Inv and Pol are antitone bijections. □

Proof of Corollary.

Let $S \subseteq \mathbf{R}$ such that $[S] = \mathrm{Inv}(F)$. Then,
$[F] =_{(3)} \mathrm{Pol}(\mathrm{Inv}(F)) = \mathrm{Pol}([S]) =_{(2)} \mathrm{Pol}(S)$. □

Parameterized Boolean Satisfiability

$\mathbf{A} = (\{0,1\}, F)$ algebra (signature σ).

Problem SAT($\mathbf{A}$)

Instance A term $t(x_1, \ldots, x_n)$ on σ.

Question Does there exist $(a_1, \ldots, a_n) \in \{0,1\}^{\{x_1,\ldots,x_n\}}$ such that $t^{\mathbf{A}}(a_1, \ldots, a_n) = 1$?

Theorem (Lewis Dichotomy)

SAT($\mathbf{A}$) *is NP-complete if* $S_1 \subseteq [F]$, *that is, if*

$$(x \wedge \neg y)^2 = \begin{array}{c|cc} b & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 1 & 0 \end{array} \in [F],$$

and in P otherwise.

Parameterized Boolean Satisfiability

Corollary

SAT$(\mathbf{A})$ *is in P iff F is contained in either*

$$R_1 = \mathrm{Pol}\begin{pmatrix} 1 \end{pmatrix} \subseteq \mathbf{O}_{\{0,1\}},$$

$$M = \mathrm{Pol}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \subseteq \mathbf{O}_{\{0,1\}},$$

$$D = \mathrm{Pol}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \subseteq \mathbf{O}_{\{0,1\}}, \text{ or}$$

$$L = \mathrm{Pol}\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \end{pmatrix} \subseteq \mathbf{O}_{\{0,1\}}.$$

Parameterized Boolean Satisfiability

Theorem (Lewis Dichotomy)

$\mathrm{SAT}(\mathbf{A})$ *is NP-complete if* $(x \wedge \neg y)^{\mathbf{2}} \in [F]$*, and in P otherwise.*

Proof.

For hardness, inspection of Post lattice shows $[\{(x \wedge \neg y)^{\mathbf{2}}, \top^{\mathbf{2}}\}] = I_1 \vee S_1 = [\{\wedge^{\mathbf{2}}, \neg^{\mathbf{2}}\}]$*. Let* $\mathbf{A}' = (\{0,1\}, \{(x \wedge \neg y)^{\mathbf{2}}, \top^{\mathbf{2}}\})$*, so* $\mathrm{SAT}(\mathbf{A}')$ *is NP-complete. Reduction* $\mathrm{SAT}(\mathbf{A}') \leq_L \mathrm{SAT}(\mathbf{A})$*: Given* t *on* $x \wedge \neg y, \top$*, return* $z \wedge t[\top/z]$ *with* z *fresh (note* $S_1 \subseteq [F]$ *implies* $\wedge^{\mathbf{2}} \in [F]$*). So,* $\mathrm{SAT}(\mathbf{A})$ *is NP-complete.*

For tractability, inspection of Post lattice yields the following cases:

$F \subseteq R_1$ *(1-reproducing) implies* t *"Yes"instance (*$t^{\mathbf{A}}(1, \ldots, 1) = 1$*).*

$F \subseteq M$ *(monotone) implies* t *"Yes"instance iff* $t^{\mathbf{A}}(1, \ldots, 1) = 1$ *(evaluation, in P).*

$F \subseteq D$ *(selfdual) implies* t *"Yes"instance (*$t^{\mathbf{A}}(0, \ldots, 0) = 1$ *or* $t^{\mathbf{A}}(1, \ldots, 1) = 1$*).*

$F \subseteq L$ *(affine) implies* t *"Yes"instance iff (w.l.o.g.* t *on* $\oplus, \top$ *as* $[\{\oplus^{\mathbf{2}}, \top^{\mathbf{2}}\}] = L$*) either* $\top$ *or some* x_i *have an odd number of occurrences in* t *(counting, in P).* □

Outline

Lewis Dichotomies in Many-Valued Logics

$\mathbf{A} = (A, F)$ algebra (signature σ) such that:

1. $\{0, 1\} \subset A$;
2. $F' = \{f' \mid f'$ restriction of $f \in F$ to $\{0, 1\}\} \subseteq \mathrm{Pol}\begin{pmatrix} 0 & 1 \end{pmatrix}$.

$[F]_n$ is the universe of $\mathbf{F}_{HSP(\mathbf{A})}(n)$, the free n-generated algebra in the variety generated by $\mathbf{A}$ (roughly, the truthtables of the n-variable fragment of a many-valued expansion of a Boolean language).

Problem Give a Lewis dichotomy for "SAT($\mathbf{A}$)".

Lewis Dichotomies in Many-Valued Logics

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Problem Give a Lewis dichotomy for "SAT($\mathbf{A}$)".

Idea Exploit Post lattice.

Kleene Operations

$\mathbf{3} = (\{0,2,1\}, K)$ with $K = \{\wedge^{\mathbf{3}}, \neg^{\mathbf{3}}, \top^{\mathbf{3}}\}$ where: $\top^{\mathbf{3}} = 1$;

$\neg^{\mathbf{3}}(0) = 1, \neg^{\mathbf{3}}(2) = 2, \neg^{\mathbf{3}}(1) = 0$; and,

$\wedge^{\mathbf{3}}$	0	2	1
0	0	0	0
2	0	2	2
1	0	2	1

.

Fact (Kleene Operations)

1. $[K] = \mathrm{Pol}\left(\begin{matrix} 0 & 1 \end{matrix}, \begin{matrix} 0 & 2 & 1 & 2 & 2 \\ 0 & 2 & 1 & 0 & 1 \end{matrix}\right) = \mathrm{Pol}(\mathcal{K})$;
2. $[K]_n$ *universe of* $\mathbf{F}_{HSP(\mathbf{3})}(n)$, *the free* n*-generated Kleene algebra.*

Remark

Propositional semantics: 1, 0 *for "true", "false";* 2 *for "undetermined".*

Satisfiability Problem

σ algebraic signature, $\mathbf{A} = (A, F)$ algebra on σ with:

1. $A = \{0, 2, 1\}$;
2. $F = \{f : A^{\mathrm{ar}(f)} \to A \mid f \in \sigma\} \subseteq [K]$.

Problem KLEENE-SAT($\mathbf{A}$)

Instance A term $t(x_1, \dots, x_n)$ on σ.

Question Does there exist $(a_1, \dots, a_n) \in A^{\{x_1,\dots,x_n\}}$ such that $t^{\mathbf{A}}(a_1, \dots, a_n) = 1$?

Remark

2-KLEENE-SAT($\mathbf{A}$) *is in P: By preservation, there exists* $(a_1, \dots, a_n) \in A^{\{x_1,\dots,x_n\}}$ *such that* $t^{\mathbf{A}}(a_1, \dots, a_n) = 2$ *iff* $t^{\mathbf{A}}(2, \dots, 2) = 2$.

Dichotomy Theorem

Theorem

Kleene-SAT($\mathbf{A}$) *is NP-complete if*

$$\begin{array}{c|ccc} k_1 & 0 & 2 & 1 \\ \hline 0 & 0 & 0 & 0 \\ 2 & 2 & 2 & 2 \\ 1 & 1 & 2 & 0 \end{array} \in [F] \text{ or } \begin{array}{c|ccc} k_2 & 0 & 2 & 1 \\ \hline 0 & 0 & 2 & 0 \\ 2 & 2 & 2 & 2 \\ 1 & 1 & 2 & 0 \end{array} \in [F],$$

and in P otherwise.

Remark

$[k_1]$, $[k_2]$ *incomparable in the lattice of clones on* $\{0, 2, 1\}$.

Dichotomy Theorem

Corollary

KLEENE-SAT(**A**) *tractable iff F is contained in either*

$$K(R_1) = \mathrm{Pol}\left(\begin{matrix} 1 \end{matrix}, \mathcal{K} \right) \subseteq \mathbf{O}_{\{0,2,1\}},$$

$$K(M) = \mathrm{Pol}\left(\begin{matrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{matrix}, \mathcal{K} \right) \subseteq \mathbf{O}_{\{0,2,1\}},$$

$$K(D) = \mathrm{Pol}\left(\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}, \mathcal{K} \right) \subseteq \mathbf{O}_{\{0,2,1\}}, \text{ or}$$

$$K(L) = \mathrm{Pol}\left(\begin{matrix} 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \end{matrix}, \mathcal{K} \right) \subseteq \mathbf{O}_{\{0,2,1\}}.$$

Dichotomy Theorem

Proof of Kleene Dichotomy, Lower Bound.

For the lower bound, suppose $k_1 \in [F]$; the case $k_2 \in [F]$ is similar. $f \in F \subseteq [K]$ implies $f \in \mathrm{Pol}(\{0,1\})$, then $\mathbf{A}' = (\{0,1\}, F')$ with $F' = \{f' | f' \text{ restriction of } f \in F \text{ to } \{0,1\}\}$ is an algebra (display $\mathbf{A}$ and $\mathbf{A}'$ over the same signature). $k_1 \in [F]$ implies $b \in [F']$, so by Lewis dichotomy, $\mathrm{SAT}(\mathbf{A}')$ is NP-complete.

Reduction $\mathrm{SAT}(\mathbf{A}') \leq_L$ KLEENE-SAT$(\mathbf{A})$: Return the input term $t(x_1, \ldots, x_n)$. Let $(a_1, \ldots, a_n) \in \{0,2,1\}^n$ such that $t^{\mathbf{A}}(a_1, \ldots, a_n) = 1$. Pick $(a'_1, \ldots, a'_n) \in \{0,1\}^n$ such that $a'_i = a_i$ if $a_i \in \{0,1\}$. Note $t^{\mathbf{A}}(a_1, \ldots, a_n) = 1$ implies $t^{\mathbf{A}}(a'_1, \ldots, a'_n) = 1$, by preservation of $\mathcal{K}$. But, $t^{\mathbf{A}'}(a'_1, \ldots, a'_n) = t^{\mathbf{A}}(a'_1, \ldots, a'_n)$.

The converse follows as the restriction of $t^{\mathbf{A}}$ to $\{0,1\}$ is equal to $t^{\mathbf{A}'}$. □

Dichotomy Theorem

Proof of Kleene Dichotomy, Upper Bound.

For the upper bound, suppose that neither k_1 nor k_2 are in $[F]$. By direct computation, there are exactly 4 binary operations in $[K]$ whose restriction to $\{0, 1\}$ is the operation b in Lewis dichotomy; in addition to k_1 and k_2,

k_3	0	2	1
0	0	0	0
2	2	2	0
1	1	2	0

and

k_4	0	2	1
0	0	2	0
2	2	2	0
1	1	2	0

.

Neither k_3 nor k_4 are in $[F]$, in fact

$$k_1(x, y) = k_3(x, k_3(x, k_3(x, y))),$$
$$k_2(x, y) = k_4(k_4(x, y), k_4(y, x)).$$

Then, there is no binary operation in $[F]$ whose restriction to $\{0, 1\}$ is b. Thus, if $\mathbf{A}' = (\{0, 1\}, F')$ is as above, $b \notin [F']$, and $\mathrm{SAT}(\mathbf{A}')$ is in P. The trivial reduction KLEENE-SAT$(\mathbf{A}) \leq_L \mathrm{SAT}(\mathbf{A}')$ is correct. □

Gödel Operations

$\mathbf{G} = ([0,1], G)$ with $G = \{\wedge^{\mathbf{G}}, \rightarrow^{\mathbf{G}}, \neg^{\mathbf{G}}, \bot^{\mathbf{G}}\}$ where: $\bot^{\mathbf{G}} = 0$;
$x \wedge^{\mathbf{G}} y = \min\{x, y\}$; $x \rightarrow^{\mathbf{G}} y = \begin{cases} 1 & x \leq y \\ y & \text{otherwise} \end{cases}$; $\neg^{\mathbf{G}} x = x \rightarrow^{\mathbf{G}} 0$.
$m \geq 1$. $\mathbf{G}_m = (\{0, 1/m, 2/m, \ldots, 1\}, G_m)$ subalgebra of $\mathbf{G}$ (easy).

Theorem (Gödel Operations)

1. $[G] = \mathrm{Pol}\left(\{S \mid S \textit{ subuniverse of } \mathbf{G}_m \textit{ or } \mathbf{G}_m^2\}\right)_{m \geq 1} = \mathrm{Pol}(\mathcal{G})$.
2. $[G]_n$ *universe of* $\mathbf{F}_{HSP(\mathbf{G})}(n)$, *the free n-generated Gödel algebra (commutative bounded integral divisible prelinear idempotent residuated lattices).*

Gödel Operations

Proof of Part 1.

If $f \in [G]$, then f is a term operation over $\mathbf{G}$, thus f preserves $\mathcal{G}$.
Suppose $f\colon [0,1]^n \to [0,1] \notin [G]_n \subseteq [G]$. Write $[0,1]_m$ for $\{0, 1/m, \ldots, 1\}$. Let $h\colon [G_{n+1}]_n \to [G]_n$ st $h(g) = g'$ iff, for all $\mathbf{a} \in [0,1]^n_{n+1}$ and $\mathbf{a} \in [0,1]^n$, if $\mathbf{a}$ and $\mathbf{a}'$ "have the same ordered partition", then: $g(\mathbf{a}) = 0$ iff $g'(\mathbf{a}') = 0$, $g(\mathbf{a}) = 1$ iff $g'(\mathbf{a}') = 1$, and, $g(a_1, \ldots, a_n) = a_i$ iff $g'(a'_1, \ldots, a'_n) = a'_i$. h is an isomorphism of $[G]_n$ and $[G_{n+1}]_n$ with op's defined pointwise [AG08]. Note $h(f|_{[0,1]_{n+1}}) = f$, then $f|_{[0,1]_{n+1}} \notin [G_{n+1}]_n$, ow $f \in [G]_n$.

Claim

$f|_{[0,1]_{n+1}} \notin \mathrm{Pol}\left(\{S \mid S \text{ subuniverse of } \mathbf{G}_{n+1} \text{ or } \mathbf{G}^2_{n+1}\}\right) = \mathrm{Pol}(\mathcal{G}_{n+1}) \subseteq \mathrm{Pol}(\mathcal{G})$.

By the claim, $f \notin \mathrm{Pol}(\mathcal{G})$. □

Gödel Operations

Proof of Claim.

We use the combinatorial characterization of $[G_{n+1}]_n$ in [AG08]. If $f(a_1,\ldots,a_n) \notin \{a_1,\ldots,a_n\} \cup \{0,1\}$ for some $(a_1,\ldots,a_n) \in [0,1]^n_{n+1}$, then $f \notin \mathrm{Pol}(\mathcal{G}_{n+1})$. Ow, there exist $\mathbf{a}, \mathbf{b} \in [0,1]^n_{n+1}$ "having the same first i blocks", say $(A_1,\ldots,A_i,\ldots,A_j)$ and $(B_1,\ldots,B_i,\ldots,B_k)$ with $i \leq j,k$. Let v_t be equal to the numerical value of the a_i's in A_t for $1 \leq t \leq j$, and let w_t be equal to the numerical value of the b_i's in B_t for $1 \leq t \leq k$. We have $f(\mathbf{a}) \in A_r$ and $f(\mathbf{b}) \in B_s$ with either $r \neq s \leq i$ or $r \leq i < s$. In both cases, f does not preserve the subuniverse of $\mathbf{G}^2_{n+1}$ given by

$$\{(0,0),(v_1,w_1),\ldots,(v_i,w_i)\} \cup \{v_{i+1},\ldots,v_{j-1},1\} \times \{w_{i+1},\ldots,w_{k-1},1\}.$$

□

Satisfiability Problem

σ algebraic signature, $\mathbf{A} = (A, F)$ algebra on σ with:

1. $A = [0, 1]$;
2. $F = \{f \colon A^{\mathrm{ar}(f)} \to A \mid f \in \sigma\} \subseteq [G]$.

Problem	GÖDEL-SAT($\mathbf{A}$)
Instance	A term $t(x_1, \ldots, x_n)$ on σ.
Question	Does there exist $(a_1, \ldots, a_n) \in A^{\{x_1,\ldots,x_n\}}$ such that $t^{\mathbf{A}}(a_1, \ldots, a_n) = 1$?

Remark

$\epsilon > 0$. ϵ-GÖDEL-SAT($\mathbf{A}$) $\equiv_L$ GÖDEL-SAT($\mathbf{A}$): *By preservation, there exists* $(a_1, \ldots, a_n) \in A^{\{x_1,\ldots,x_n\}}$ *such that* $t^{\mathbf{A}}(a_1, \ldots, a_n) = \epsilon$ *iff there exists* $(a_1, \ldots, a_n) \in A^{\{x_1,\ldots,x_n\}}$ *such that* $t^{\mathbf{A}}(a_1, \ldots, a_n) = 1$.

Dichotomy Theorem

Theorem
GÖDEL-SAT($\mathbf{A}$) *is NP-complete if*

$$(x \wedge \neg y)^{\mathbf{G}} \in [F] \text{ or } (\neg(x \rightarrow y))^{\mathbf{G}} \in [F]$$

and in P otherwise.

Remark
$[(x \wedge \neg y)^{\mathbf{G}}], [(\neg(x \rightarrow y))^{\mathbf{G}}]$ *incomparable in the lattice of clones on* $[0,1]$.

Dichotomy Theorem

Corollary

GÖDEL-SAT($\mathbf{A}$) *tractable iff F is contained in either*

$$G(R_1) = \mathrm{Pol}\left(1 \ , \mathcal{G} \right) \subseteq \mathbf{O}_{[0,1]},$$

$$G(M) = \mathrm{Pol}\left(\begin{matrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{matrix} , \mathcal{G} \right) \subseteq \mathbf{O}_{[0,1]},$$

$$G(D) = \mathrm{Pol}\left(\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} , \mathcal{G} \right) \subseteq \mathbf{O}_{[0,1]}, \text{ or}$$

$$G(L) = \mathrm{Pol}\left(\begin{matrix} 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \end{matrix} , \mathcal{G} \right) \subseteq \mathbf{O}_{[0,1]}.$$

Dichotomy Theorem

$\mathbf{A} = ([0,1]_3, F)$ with $F \subseteq [G_3]$. GÖDEL-SAT($\mathbf{A}$) similarly.

Lemma

GÖDEL-SAT($\mathbf{A}$) *is NP-complete if*

g_1	0	1/3	2/3	1
0	0	0	0	0
1/3	1/3	0	0	0
2/3	2/3	0	0	0
1	1	0	0	0

$\in [F]$ *or*

g_2	0	1/3	2/3	1
0	0	0	0	0
1/3	1	0	0	0
2/3	1	0	0	0
1	1	0	0	0

$\in [F]$,

and in P otherwise.

Remark

$[g_1]$, $[g_2]$ *incomparable in the lattice of clones on* $[0,1]_3$.

Dichotomy Theorem

Proof of Lemma.

Lower Bound, Case $g_1 \in [F]$: Let $\mathbf{A}' = (\{0,1\}, F')$ with $F' = F|_{\{0,1\}}$, correct as $F \subseteq [G]$ implies F preserves the subuniverse $\{0,1\}$. Then, $b \in [F']$, so that $\mathrm{SAT}(\mathbf{A}')$ is NP-complete. Reduction $\mathrm{SAT}(\mathbf{A}') \leq_L$ GÖDEL-SAT$(\mathbf{A})$: return the given term t. If $t^{\mathbf{A}'}(\mathbf{a}) = 1$, then $t^{\mathbf{A}}(\mathbf{a}) = 1$ by the definition of F'. Conversely, let $(a_1, \ldots, a_n) \in [0,1]_3^n$ st $t^{\mathbf{A}}(a_1, \ldots, a_n) = 1$. Let $(a'_1, \ldots, a'_n) \in \{0,1\}^n$ st $a'_i = 0$ if $a_i = 0$ and $a'_i = 1$ ow. As $R = \{(0,0), (1/3,1), (2/3,1), (1,1)\}$ is a subuniverse of $\mathbf{G}_3^2$, $t^{\mathbf{A}}$ preserves R and $t^{\mathbf{A}}(a'_1, \ldots, a'_n) = 1$. Then, $t^{\mathbf{A}'}(a'_1, \ldots, a'_n) = 1$. Case $g_2 \in [F]$: Similar.
Upper Bound: $g_1, g_2 \notin [F]$ imply no operation in $[F]_2$ restricted to $\{0,1\}$ equals b, as g_1 and g_2 are the only such operations in $[G_3]_2 \supseteq [F]_2$. Then, $b \notin [F']$, and $\mathrm{SAT}(\mathbf{A}')$ is in P. As above GÖDEL-SAT$(\mathbf{A}) \leq_L \mathrm{SAT}(\mathbf{A}')$. □

Dichotomy Theorem

Proof of Gödel Dichotomy.

Lower Bound. Case $(x \wedge \neg y)^{\mathbf{G}} \in [F]$: Let $\mathbf{A}' = ([0,1], F')$ with $F' = \{f' \mid f'$ restriction to $[0,1]_3$ of $f \in F\}$. As $g_1 \in [F']$, by the lemma, GÖDEL-SAT$(\mathbf{A}')$ is NP-complete. Reduction GÖDEL-SAT$(\mathbf{A}') \leq_L$ GÖDEL-SAT$(\mathbf{A})$: return the given term t. If $t^{\mathbf{A}'}(\mathbf{a}) = 1$, then $t^{\mathbf{A}}(\mathbf{a}) = 1$ as $t^{\mathbf{A}}|_{[0,1]_3} = t^{\mathbf{A}'}$. Conversely, let $\mathbf{a} = (a_1, \ldots, a_n) \in [0,1]^n$ st $t^{\mathbf{A}}(\mathbf{a}) = 1$. Let $\mathbf{a}' = (a_1', \ldots, a_n') \in \{0,1\}^n \subseteq [0,1]_3^n$ st $a_i' = a_i$ if $a_i = 0$ and $a_i' = 1$ ow. $t^{\mathbf{A}} \in [G]$ implies that $t^{\mathbf{A}}$ preserves the subuniverse $R = \{(0,0), (a,1) \mid 0 < a\}$ of $\mathbf{G}^2$. Also, $(1,a) \notin R$ if $a \neq 1$, then $t^{\mathbf{A}}(\mathbf{a}) = 1$ implies $t^{\mathbf{A}}(\mathbf{a}') = 1$. But $t^{\mathbf{A}}\mid_{\{0,1\}} = t^{\mathbf{A}'}$, then $t^{\mathbf{A}'}(\mathbf{a}') = 1$. Case $(\neg(x \rightarrow y))^{\mathbf{G}} \in [F]$: Similar.
Upper Bound. $(x \wedge \neg y)^{\mathbf{G}}, (\neg(x \rightarrow y))^{\mathbf{G}} \notin [F]$ implies that no operation in $[F]_2$ restricted to $\{0,1\}$ equals b, then $\mathbf{A}' = (\{0,1\}, F')$ with $F' = F|_{\{0,1\}}$ is st SAT$(\mathbf{A}')$ is in P. Reduction GÖDEL-SAT$(\mathbf{A}) \leq_L$ SAT$(\mathbf{A}')$: return the given term t. As above. □

Outline

DeMorgan Operations

$\mathbf{4} = (\{0,2,3,1\}, M)$ with $M = \{\wedge^{\mathbf{4}}, \neg^{\mathbf{4}}, \top^{\mathbf{4}}\}$ where: $\top^{\mathbf{4}} = 1$;

$\neg^{\mathbf{4}}(0) = 1, \neg^{\mathbf{4}}(2) = 2, \neg^{\mathbf{4}}(3) = 3, \neg^{\mathbf{4}}(1) = 0;$

$\wedge^{\mathbf{3}}$	0	2	3	1
0	0	0	0	0
2	0	2	0	2
3	0	0	3	3
1	0	2	3	1

Fact (DeMorgan Operations)

1. $[M] = \mathrm{Pol}\left(\begin{matrix} 0 & 2 & 3 & 1 \\ 0 & 3 & 2 & 1 \end{matrix}, \begin{matrix} 0 & 2 & 3 & 1 & 2 & 2 & 2 & 0 & 1 \\ 0 & 2 & 3 & 1 & 0 & 1 & 3 & 3 & 3 \end{matrix}\right)$.
2. $[M]_n$ *universe of* $\mathbf{F}_{HSP(\mathbf{4})}(n)$, *the free n-generated DeMorgan algebra.*

Remark

1, 0 *for "true", "false"*; 2, 3 *for "undetermined", "overdetermined".*

Satisfiability Problem

σ algebraic signature, $\mathbf{A} = (A, F)$ algebra on σ with:

1. $A = \{0, 2, 3, 1\}$;
2. $F = \{f \colon A^{\mathrm{ar}(f)} \to A \mid f \in \sigma\} \subseteq [M]$.

Problem DEMORGAN-SAT($\mathbf{A}$)

Instance A term $t(x_1, \dots, x_n)$ on σ.

Question Does there exist $(a_1, \dots, a_n) \in A^{\{x_1, \dots, x_n\}}$ such that $t^{\mathbf{A}}(a_1, \dots, a_n) = 1$?

Remark

Complexity of 2-DEMORGAN-SAT($\mathbf{A}$), 3-DEMORGAN-SAT($\mathbf{A}$)*?*

Hard and Easy Cases

Proposition

DEMORGAN-SAT($\mathbf{A}$) *is NP-complete if*

$$\begin{array}{c|cccc} d_1 & 0 & 2 & 3 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 3 \\ 1 & 1 & 2 & 3 & 0 \end{array} \in [F] \text{ or } \begin{array}{c|cccc} d_2 & 0 & 2 & 3 & 1 \\ \hline 0 & 0 & 2 & 3 & 0 \\ 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 3 \\ 1 & 1 & 2 & 3 & 0 \end{array} \in [F],$$

and in P if $[F|_{\{0,1\}}] \subseteq R_1, D$.

Remark

Complexity of DEMORGAN-SAT($\mathbf{A}$) *if* $I_0 \subseteq [F]$*?*

m-Valued Łukasiewicz Operations

$\mathbf{L} = ([0,1], L)$ with $L = \{\odot^{\mathbf{L}}, \rightarrow^{\mathbf{L}}, \perp^{\mathbf{L}}\}$ where: $\perp^{\mathbf{L}} = 0$;
$x \odot^{\mathbf{L}} y = \max\{0, x+y-1\}$; $x \rightarrow^{\mathbf{L}} y = \min\{1, y+1-x\}$.

$m \geq 1$. $\mathbf{L}_m = (\{0, 1/m, 2/m, \ldots, 1\}, L_m)$ subalgebra of $\mathbf{L}$ (easy).

Theorem (m-Valued Łukasiewicz Operations)

1. $[L_m] = \mathrm{Pol}(\{R \mid R \textit{ subuniverse of } \mathbf{L}_m\})$
   $= \mathrm{Pol}(\{dk/m \mid 0 \leq k \leq m/d\})_{1 \leq d | m}$.
2. $[L_m]_n$ *universe of* $\mathbf{F}_{HSP(\mathbf{L}_m)}(n)$, *the free n-generated algebra in the variety generated by* $\mathbf{L}_m$ *(m-valued Łukasiewicz algebras).*

Satisfiability Problem

$m \geq 1$, σ algebraic signature, $\mathbf{A} = (A, F)$ algebra on σ with:

1. $A \subseteq [0,1]$;
2. $F = \{f \colon A^{\mathrm{ar}(f)} \to A \mid f \in \sigma\} \subseteq [L_m]$.

Problem ŁUKASIEWICZ-SAT($\mathbf{A}$)

Instance A term $t(x_1, \ldots, x_n)$ on σ.

Question Does there exist $(a_1, \ldots, a_n) \in A^{\{x_1,\ldots,x_n\}}$ such that $t^{\mathbf{A}}(a_1, \ldots, a_n) = 1$?

Remark

$0 < l < m$. *Complexity of* $\frac{l}{m}$-ŁUKASIEWICZ-SAT($\mathbf{A}$)?

Hard and Easy Cases

Proposition

Łukasiewicz-SAT$(\mathbf{A})$ *is NP-complete if*

$$\begin{array}{c|cccc} l & 0 & 1/m & \cdots & 1 \\ \hline 0 & 0 & 0 & \cdots & 0 \\ 1/m & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 0 & \cdots & 0 \end{array} \in [F],$$

and in P if $[F|_{\{0,1\}}] \subseteq R_1, D$.

Remark

Complexity of Łukasiewicz-SAT$(\mathbf{A})$ *if* $I_0 \subseteq [F]$*?*

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Thank you!