Finite Projective Kleene Algebras

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Outline

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deMorgan Algebras, or Lattices with Involution [K58]

$\mathbf{A} = (A, \wedge, \vee, ', 0, 1)$ of type $(2, 2, 1, 0, 0)$.

$\mathbf{A}$ is a *deMorgan* algebra ($\mathbf{A} \in \mathcal{M}$) if:

1. $(A, \wedge, \vee, 0, 1)$ is a bounded distributive lattice (in $\mathcal{BDL}$);
2. $\mathbf{A} \models x = x''$ and $\mathbf{A} \models (x \wedge y)' = x' \vee y'$ ($'$ called *involution*).

$\mathbf{A}$ is a *Kleene* algebra ($\mathbf{A} \in \mathcal{K}$) if:

1. $\mathbf{A}$ is a deMorgan algebra;
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$\mathbf{A}$ is a *Boolean* algebra ($\mathbf{A} \in \mathcal{B}$) if:

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Remark

(Additive) lattice-ordered groups ($x' = -x$, [K58]) and MV-algebras ($x' = \neg x$) are Kleene algebras.

Injective and Projective deMorgan Algebras

Fact (Balbes and Horn [BH70], Sikorski [S51])

1. $\mathbf{A} \in \mathcal{B}$ injective iff complete.
2. $\mathbf{A} \in \mathcal{B}$ projective iff countable.

Fact (Cignoli [C75])

1. $\mathbf{A} \in \mathcal{M}$ injective iff retract of $\mathbf{M}^{\kappa}$ $(0 < \kappa)$.
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Question

Projective Kleene and deMorgan algebras?

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Subdirectly Irreducible deMorgan Algebras

Define $\mathbf{B}, \mathbf{K}, \mathbf{M} \in \mathcal{M}$ (call $x \in \mathbf{A} \in \mathcal{M}$ a *fixpoint* if $x' = x$):

1. $\mathbf{B} = (\{0, 1\}, \wedge, \vee, ', 0, 1) \in \mathcal{M}$ with no fixpoints;
2. $\mathbf{K} = (\{0, 2, 1\}, \wedge, \vee, ', 0, 1) \in \mathcal{M}$ with one fixpoint, 2;
3. $\mathbf{M} = (\{0, 2, 3, 1\}, \wedge, \vee, ', 0, 1) \in \mathcal{M}$ with two fixpoints, $2, 3$.

Theorem (Kalman, [K58])
$\mathbf{A} \in \mathcal{M}$ *(nontrivial) subdirectly irreducible iff* $\mathbf{A}$ *is* $\mathbf{B}$*,* $\mathbf{K}$*, or* $\mathbf{M}$*.*

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Proof (Sketch).

$(\Leftarrow)$ $\mathbf{B}$, $\mathbf{K}$, $\mathbf{M}$ *simple* ($\mathcal{M}$ *semisimple).*

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($\Rightarrow$) $\mathbf{A} \in \mathcal{M}$ *(nontrivial) subdirectly irreducible.*
x *and* x' *comparable for all* $x \in A$.
Let $x \in A$ *such that wlog* $x > x'$ ($\mathbf{A}$ *trivial if* $x = x'$ *for all* $x \in A$). *Then,* $x = 1$.
For all $y \in A$, *if* $y \neq 0, 1$, *then* $y = y'$ (*y fixpoint*).
$\mathbf{A}$ *has either* 0, 1, *or* 2 *fixpoints.*
$\mathbf{A}$ *is* $\mathbf{B}$, $\mathbf{K}$, *or* $\mathbf{M}$. □

Subdirectly Irreducible deMorgan Algebras

Corollary

(Nontrivial) deMorgan varieties form a 3-element chain:

$$\begin{array}{ccccc} SP(\mathbf{B}) & \subset & SP(\mathbf{K}) & \subset & SP(\mathbf{M}) \\ \| & & \| & & \| \\ \mathcal{B} & \subset & \mathcal{K} & \subset & \mathcal{M} \end{array}$$

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For $\subseteq$, $\mathbf{B} \in S(\mathbf{K})$ *and* $\mathbf{K} \in S(\mathbf{M})$.
For $\neq$, $\mathbf{K} \not\models x \wedge x' = 0$ $(x = 2)$, *and* $\mathbf{M} \not\models x \wedge x' \leq y \vee y'$ $(x = 2, y = 3)$. □

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Remark

deMorgan varieties are locally finite.

Free (Finitely Generated) Kleene Algebras

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The free n-generated Kleene algebra $\mathbf{F}_{\mathcal{K}}(n)$ is the subalgebra of $\mathbf{K}^{\mathbf{K}^n}$ generated by the projections.

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Definition

An operation $f: A^n \to A$ *preserves* a relation $R \subseteq A^k$
if R is a subalgebra of $(A, f)^k$.

Example

f preserves $\theta \subseteq A^2$ iff θ congruence on (A, f).

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Theorem

The universe of $\mathbf{F}_{\mathcal{K}}(n)$ is the set of all n-ary operations on $\{0, 2, 1\}$ preserving:

$$\left\{ \begin{matrix} 0 & 1 \end{matrix}, \begin{matrix} 0 & 2 & 1 & 2 & 2 \\ 0 & 2 & 1 & 0 & 1 \end{matrix} \right\} = \{ \begin{matrix} 0 & 1 \end{matrix}, \mathcal{R} \}.$$

(Finite) Projective Algebras

Definition (Retract)

$\mathcal{V}$ variety. $\mathbf{B} \in \mathcal{V}$ *retract* of $\mathbf{A} \in \mathcal{V}$ if,
there exist homomorphisms $f\colon \mathbf{A} \to \mathbf{B}$ and $g\colon \mathbf{B} \to \mathbf{A}$
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$\mathbf{B}$ is projective if every homomorphism of $\mathbf{B}$ to a quotient "lifts to the numerator".

Definition (Projective)

$\mathcal{V}$ variety. $\mathbf{B} \in \mathcal{V}$ *projective* if, for every $\mathbf{A}, \mathbf{C} \in \mathcal{V}$,
every surjective homomorphism $f \colon \mathbf{A} \to \mathbf{C}$,
and every homomorphism $h \colon \mathbf{B} \to \mathbf{C}$,
there exists a homomorphism $g \colon \mathbf{B} \to \mathbf{A}$ such that $f \circ g = h$.

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$\mathcal{V}$ variety. $\mathbf{B} \in \mathcal{V}$ projective iff
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Proof (Sketch).

($\Rightarrow$) If $\mathbf{B}$ is projective,
then $\mathbf{B}$ is a retract of every algebra that homomorphically maps onto it,
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($\Leftarrow$) *If $\mathbf{A}$ is a retract of $\mathbf{F}_V(\kappa)$,*
since $\mathbf{F}_V(\kappa)$ is projective (claim, any free algebra is projective),
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Corollary

$\mathcal{V}$ locally finite variety. $\mathbf{B} \in \mathcal{V}_{\text{fin}}$ projective iff
$\mathbf{B}$ retract of $\mathbf{F}_{\mathcal{V}}(n)$ for $n < \omega$.

Outline

Motivation

Background

Contribution

Open

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Notation | $n = 2$

$\mathbf{F}_{\mathcal{K}}(2) \ni$

f	0	2	1
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2	2	2	2
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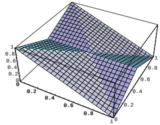

(Puzzling) Notation | $n = 2$

$$\left\{ \begin{matrix} a & 2_a & a \\ 2 & 2 & 2_a \\ b & 2 & a \end{matrix} \;\middle|\; 2_a \in \{2, a\} \right\}_{a \neq b \in \{0,1\}}$$

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Quotients

Proposition

θ is a congruence on $\mathbf{F}_{\mathcal{K}}(n)$ iff, there exists $X \subseteq \{0, 2, 1\}^n$ such that $f \equiv_\theta g$ iff $f(x) = g(x)$ for all $x \in X$ (write θ_X).

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Example ($n = 1$)

The congruence on $\mathbf{F}_{\mathcal{K}}(1)$ corresponding to $X = \{0, 2, 1\}$ is $\theta_X = \{\{(0, 0, 0)\}, \{(a, 2, b) \mid a, b \in \{0, 1\}\}, \{(1, 1, 1)\}\}$.

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Proof (Sketch).

($\Leftarrow$) For every $x \subseteq \{0, 2, 1\}^n$, the relation θ on $\mathbf{F}_{\mathcal{K}}(n)$ such that, $(f, g) \in \theta$ iff $f(x) = g(x)$, is a maximal congruence on $\mathbf{F}_{\mathcal{K}}(n)$, say θ_x; and, $\theta_X = \bigwedge_{x \in X} \theta_x$.

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($\Rightarrow$) Assume that θ is a congruence on $\mathbf{F}_{\mathcal{K}}(n)$ but for every $x \subseteq \{0,2,1\}^n$ there exist $f \equiv_\theta g$ such that $f(x) \neq g(x)$, then preservation fails (contradiction). $\square$

Quotients | *Example* $n = 2$

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Characterize $X \subseteq \{0, 2, 1\}^n$ such that $\mathbf{F}_K(n)/\theta_X \in S(\mathbf{F}_K(n))$.

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A *retraction* of a poset $\mathbf{P}$ onto $X \subseteq P$ is a map of P such that $r(P) = X$, $r|_X = \mathrm{id}_X$, and $r(x) \leq r(y)$ if $x \leq y$.

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Proposition

Let $X \subseteq \{0,2,1\}^n$. Then, $\mathbf{F}_K(n)/\theta_X \in S(\mathbf{F}_K(n))$ iff there is a retraction r of $\mathcal{R}^n$ onto X such that $r|_{\{0,1\}^n} \subseteq \{0,1\}^n$.

Projective

Characterize $X \subseteq \{0,2,1\}^n$ such that $\mathbf{F}_K(n)/\theta_X \in S(\mathbf{F}_K(n))$.

$\mathcal{R}^n = (\{0,2,1\}^n, \leq)$, n-th power of the poset $\mathcal{R}$, a meet semilattice. $\mathcal{X} = (X, \leq|_X)$ for $X \subseteq \{0,2,1\}^n$.

A *retraction* of a poset $\mathbf{P}$ onto $X \subseteq P$ is a map of P such that $r(P) = X$, $r|_X = \mathrm{id}_X$, and $r(x) \leq r(y)$ if $x \leq y$.

Proposition

Let $X \subseteq \{0,2,1\}^n$. Then, $\mathbf{F}_K(n)/\theta_X \in S(\mathbf{F}_K(n))$ iff there is a retraction r of $\mathcal{R}^n$ onto X such that $r|_{\{0,1\}^n} \subseteq \{0,1\}^n$.

Proof (Sketch).

For each block B of θ_X, pick the $f_B \in B$ such that, for all $y \in \{0,2,1\}^n \setminus X$,

$$f_B(y) = f_B(r(y)).$$

Check that $\{f_B \mid B \text{ block of } \theta_X\}$ is a subuniverse of $\mathbf{F}_K(n)$. □

Projective | Example $n = 2$

$$\{0,2,1\}^2 \supseteq X = \left\{ \begin{array}{ccc} (0,0) & (2,0) & (1,0) \\ (0,2) & (2,2) & (1,2) \\ (0,1) & (2,1) & (1,1) \end{array} \right\}$$

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Subuniverse of $\mathbf{F}_K(2)\ldots$ found!

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Main Result

Theorem

$\mathcal{R}^n$ retracts onto $X \subseteq \{0,2,1\}^n$ "respecting" $\{0,1\}^n$ iff:

(R1) $\bigwedge X \in X$.

(R2) *For all $x \in X \setminus \{0,1\}^n$, there is $y \in X \cap \{0,1\}^n$ st $x \leq y$.*

(R3)[1] *For all $Y \subseteq X$, if Y has an upper bound in $\mathcal{R}^n$, then Y has an upper bound in X.*

[1] Called *well-embeddability* in [BB89].

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Proof (Sketch).

($\Rightarrow$) *Counterexamples.* For (R1), $X = \{0,1\} \subseteq \{0,2,1\}$.
For (R2), $X = \{2\} \subseteq \{0,2,1\}$.
For (R3), $X = \{(2,2,2),(2,2,0),(2,0,2),(1,1,0),(1,0,1)\} \subseteq \{0,2,1\}^3$.

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For (R3), $X = \{(2,2,2),(2,2,0),(2,0,2),(1,1,0),(1,0,1)\} \subseteq \{0,2,1\}^3$.
($\Leftarrow$) *For $x \in \mathcal{R}^n$, let $\{0,2,1\}^n \supseteq Y = (x] \cap X$. Let $U = \{x, \dots\} \neq \emptyset$ be the set of upper bounds of Y in $\mathcal{R}^n$. By (R3), Y has an upper bound in X, say $b = \bigwedge_{\mathcal{X}}(U \cap X)$, noticing that $\mathcal{X}$ is a finite meet semilattice (hence complete). The desired retraction sends x to b.* □

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Outline

Motivation

Background

Contribution

Open

Open

Generalize:

1. retracts of $\mathbf{F}_{\mathcal{K}}(\kappa)$ for κ infinite cardinal,
   ie, projective Kleene algebras;
2. retracts of $\mathbf{F}_{\mathcal{M}}(n)$ for $n < \omega$,
   ie, finite projective deMorgan algebras.

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Thank you!