Model Checking Existential Logic on Partially Ordered Sets

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Outline

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Model Checking

We study a restriction of the *model checking* problem:

Problem	$\mathrm{MC}(\mathcal{S}, \mathcal{L})$
Instance	A finite structure $\mathbf{A} \in \mathcal{S}$ and a logical sentence $\phi \in \mathcal{L}$.
Question	$\mathbf{A} \models \phi$?

where:

- $\mathcal{S}$ is a class of *partial orders* (*posets*), ie,
  reflexive antisymmetric transitive digraphs
  (or, reflexo transitive closure of DAGs);
- $\mathcal{L} = \mathcal{FO}(\exists, \wedge, \vee, \neg)$ is *existential logic*, ie,
  prenex $\mathcal{FO}$-sentences with existential prefix and unrestricted matrix.

Parameterized Complexity

For any class $\mathcal{X}$ of finite structures, $\mathrm{MC}(\mathcal{X}, \mathcal{FO})$ is decidable in time

$$O(n^k)$$

where n is the size of the instance and k is the size of the $\mathcal{FO}$-sentence.

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We are interested in *fixed-parameter tractable* (*FPT*) cases of the problem, ie, decidable in time

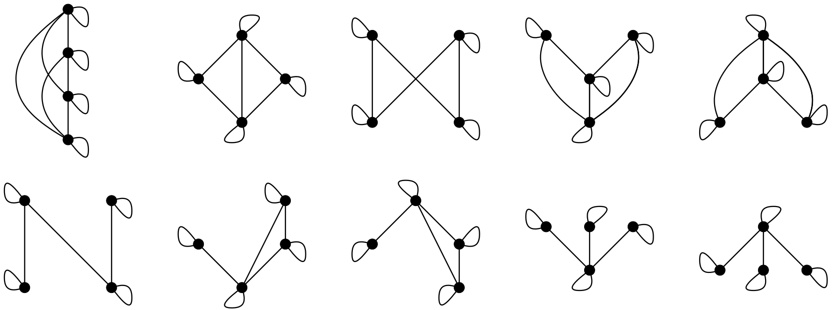
$$f(k) \cdot \mathrm{poly}(n)$$

for some fixed computable function f.

Outline

Posets

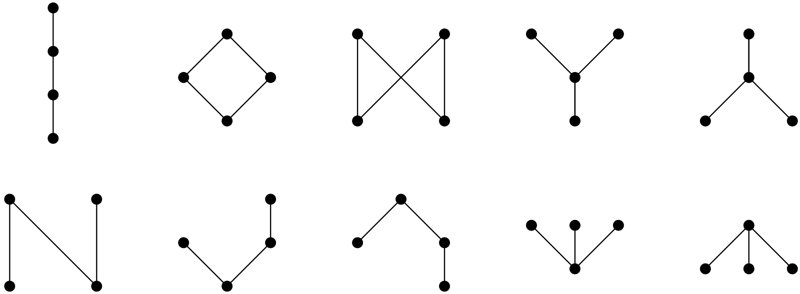
Reflexive antisymmetric transitive digraphs, ie *posets*. . .



4-element connected posets. Edges are directed upwards.

Cover Relations

. . . and their *cover relations* (reflexo transitive reductions).



Cover relations of 4-element connected posets. Edges are directed upwards.

Poset Properties

We model check $\mathcal{FO}$-properties of posets.

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(Not of their cover relations.)

Hardness on Digraphs

How hard is model checking $\mathcal{FO}$-sentences on digraphs?

	classical complexity	parameterized complexity
$\mathrm{MC}(\mathcal{H}, \mathcal{FO})$	PSPACE-complete	AW[$*$]-complete

where:

- $\mathcal{H}$ is the class of all digraphs;

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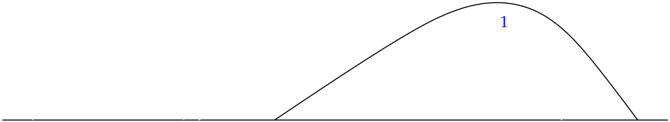
- $\mathcal{E}''$ is a nontrivial unbounded size class of digraphs. Bounded degree?

Digraphs versus Posets

Which digraph properties help in parameterized model checking?

Let $\mathcal{S}$ be a class of digraphs (unbounded size).

1 $\mathcal{S}$ "nowhere dense" $\Rightarrow$ $\mathrm{MC}(\mathcal{S}, \mathcal{FO})$ tractable*



*Grohe, Kreutzer, Siebertz (2014). Example: Bounded degree digraphs

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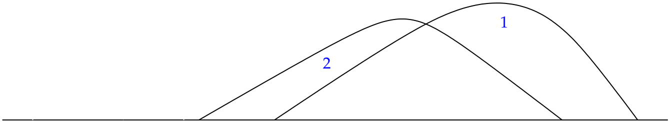

†Courcelle, Makowsky, Rotics (2000). Example: Acyclic tournaments

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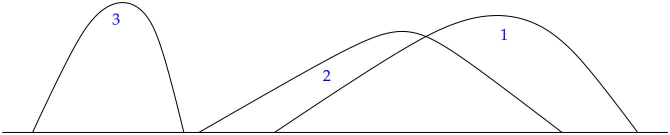
3. $\mathcal{S}$ "somewhere dense", closed under substructures $\Rightarrow$ MC$(\mathcal{S}, \mathcal{FO})$ hard[‡]



[‡] Dvořák, Král, Thomas (2010); Kreutzer (2011). Example: DAGs

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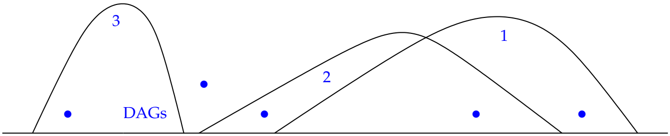

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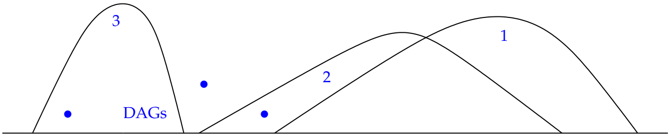
Posets?

$\neg 1$ "somewhere dense"



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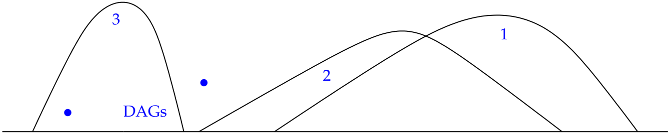
Posets?

$\neg$1 "somewhere dense"

$\neg$2 unbounded "directed cliquewidth"



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$\neg 1$ "somewhere dense"

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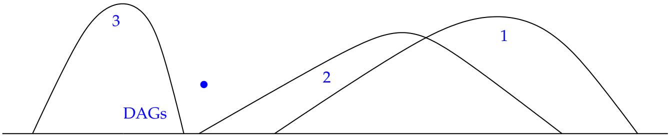
$\neg 3$ not closed under substructures



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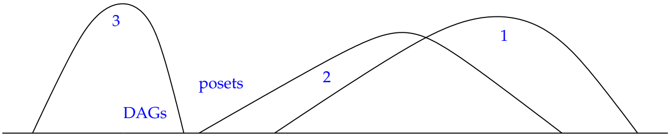
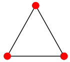


Hardness on Posets

Model checking $\mathcal{FO}$-sentences on posets is as hard as on graphs:



$\models \forall x \exists y Exy$

$\models$

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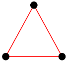

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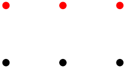

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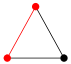
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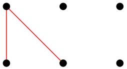
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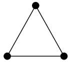
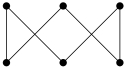
 $\models \forall x \exists y Exy$

 $\models$

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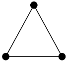
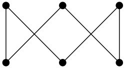
 $\models \forall x \exists y Exy$

 $\models (\forall x$ "minimal ")$(\exists y$ "minimal")

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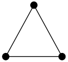

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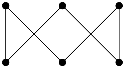

$\models (\forall x \text{ “minimal ”})(\exists y \text{ “minimal”})(x \neq y \wedge \exists z(x < z \wedge y < z))$

Hardness on Posets

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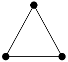
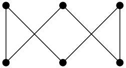
 $\models \forall x \exists y Exy$

$\Updownarrow$

 $\models (\forall x \text{ “minimal ”})(\exists y \text{ “minimal”})(x \neq y \land \exists z(x < z \land y < z))$

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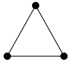 $\models \forall x \exists y Exy$

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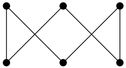 $\models (\forall x \text{ "minimal "})(\exists y \text{ "minimal"})(x \neq y \land \exists z(x < z \land y < z))$

So, as for graphs, the model checking problem

- is unlikely in PTIME on any nontrivial class of posets,
- but is in FPT on certain nontrivial (unbounded size) classes of posets.

Poset Invariants

Which poset properties help in parameterized model checking?

Let $\mathbf{P} = (P, \leq^{\mathbf{P}})$ be a poset.

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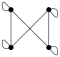
- $\text{size}(\mathbf{P}) = |P|$. Eg size() = 4.

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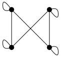
- $\mathrm{width}(\mathbf{P}) = \max\{|A| : A \text{ antichain in } \mathbf{P}\}$. Eg width() = 2.

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- $\text{degree}(\mathbf{P}) = \text{degree}(\leq^{\mathbf{P}})$.

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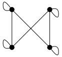
- $\text{degree}(\mathbf{P}) = \text{degree}(\leq^{\mathbf{P}})$. Eg degree() $= 3$.

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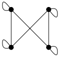
- $\text{cover-degree}(\mathbf{P}) = \mathrm{degree}(\mathrm{cover}(\leq^{\mathbf{P}}))$. Eg cover-degree() = 2.

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- $\text{cover-degree}(\mathbf{P}) = \text{degree}(\text{cover}(\leq^{\mathbf{P}}))$.
- $\text{depth}(\mathbf{P}) = \max\{|C| \colon C \text{ chain in } \mathbf{P}\}$.

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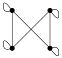
- $\text{depth}(\mathbf{P}) = \max\{|C| : C \text{ chain in } \mathbf{P}\}$. Eg depth() $= 2$.

Known and Easy Facts

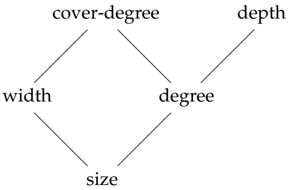


Figure: Relations between poset invariants.

Known and Easy Facts

$\mathcal{S}$ ranges over classes of posets.

- $(\forall \mathcal{S})(\mathcal{S}$ bounded degree $\Rightarrow$ MC$(\mathcal{S}, \mathcal{FO})$ is FPT$)$. From Seese (1996).

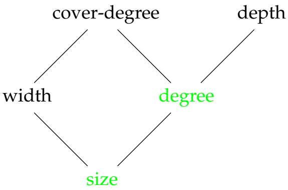


Figure: Parameterized model checking on posets, known and easy facts.

Known and Easy Facts

$\mathcal{S}$ ranges over classes of posets.

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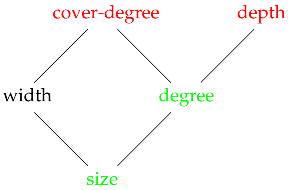

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Bounded Width Posets have Unbounded Cliquewidth

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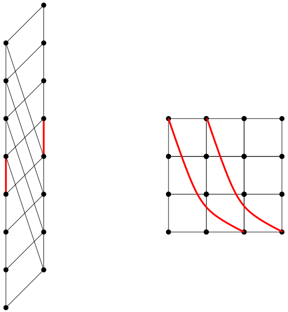
Posets of width 2 have unbounded directed cliquewidth.

Idea: There exists a class $\mathcal{G}$ of width 2 posets (*folded grids*) such that undirected(cover($\mathcal{G}$)) $= \mathcal{G}'$ have unbounded treewidth (plus theory. . .).



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"Hard enough" fragments are interesting by theirselves
(and maybe help understanding the general case).

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Outline

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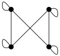
- MC({  }, $\mathcal{FO}(\exists, \wedge, \neg)$) is NP-hard. Uses Pratt and Tiuryn (1996).

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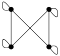
- MC($\{$  $\}$, $\mathcal{FO}(\exists, \wedge, \neg)$) is NP-hard. Uses Pratt and Tiuryn (1996).

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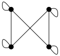
- MC($\{$  $\}, \mathcal{FO}(\exists, \wedge, \neg)$) is NP-hard. Uses Pratt and Tiuryn (1996).

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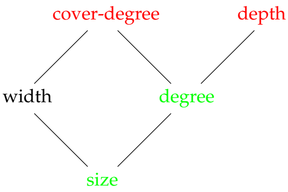

Embedding is FPT on Bounded Width Posets

How hard is the embedding problem on bounded width posets?

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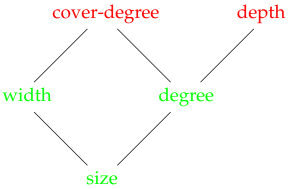

Figure: Parameterized complexity of embedding wrt poset invariants.

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Theorem [B, Ganian, Szeider '14]. Embedding is FPT on bounded width posets.

Idea. For every poset **P**, every "coordinatization" of **P**,
and every "coloring" of **P**, let the *compilation* of **P** be the structure

$$\mathbf{P}^* = \text{compile}(\mathbf{P}, \text{a "coordinatization" of } \mathbf{P}, \text{a "coloring" of } \mathbf{P})$$

such that:

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2. Let $\mathbf{Q}$ and $\mathbf{P}$ be posets. The following are equivalent:
   – $\mathbf{Q}$ embeds into $\mathbf{P}$
   – ($\forall$ compilations $\mathbf{P}^*$ of $\mathbf{P}$) ($\exists$ compilation $\mathbf{Q}^*$ of $\mathbf{Q}$) ($\mathbf{Q}^* \in \text{HOM}(\mathbf{P}^*)$)

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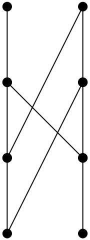
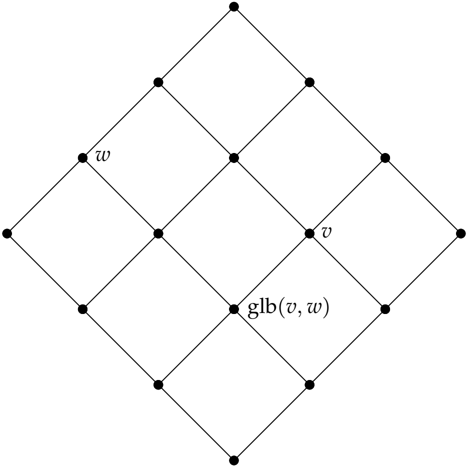


P



$\mathbf{P}^* \models$ "glb(v, w) semilattice polymorphism"

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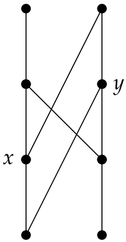

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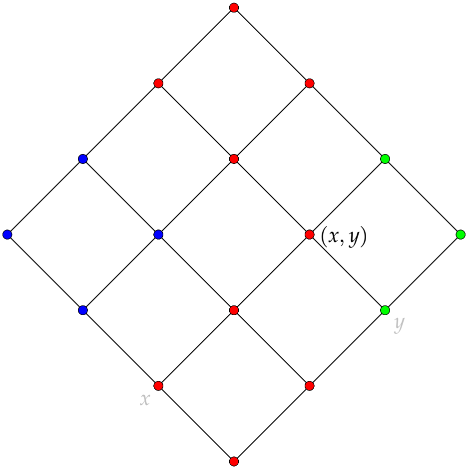

P*

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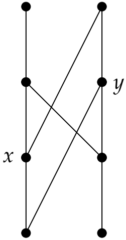
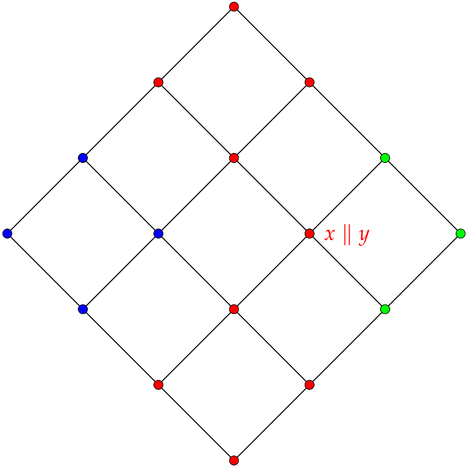


$\mathbf{P} \models x \parallel y$



$\mathbf{P}^* \models \textit{Incomparable}(x, y)$

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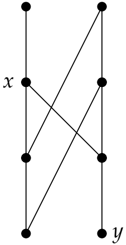
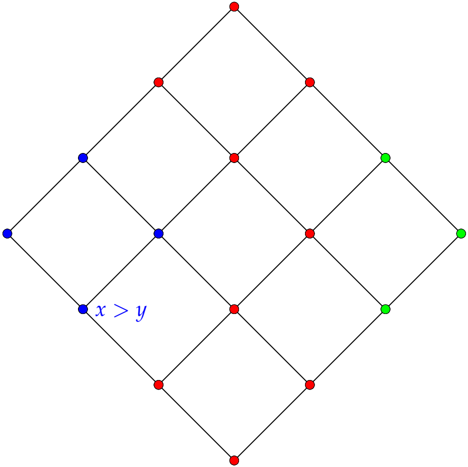


$\mathbf{P} \models x > y$



$\mathbf{P}^* \models Above(x, y)$

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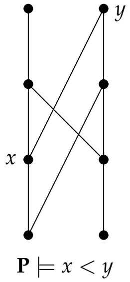

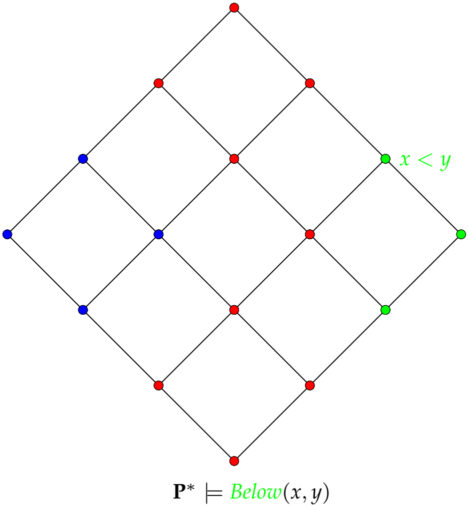

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Idea (Cont'd). Given posets $\mathbf{Q}$ and $\mathbf{P}$.

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The algorithm runs in $2^{O(\|\mathbf{Q}\| \log \|\mathbf{Q}\|)} n^{O(\text{width}(\mathbf{P}))}$ time.

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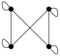
- $\text{EMB}(\mathcal{S})$ in PTIME $\not\Rightarrow$ $\text{MC}(\mathcal{S}, \mathcal{FO}(\exists, \wedge, \neg))$ in PTIME. $\mathcal{S} = \{$  $\}$.

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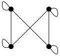
- EMB($\mathcal{S}$) in PTIME $\not\Leftarrow$ MC($\mathcal{S}, \mathcal{FO}(\exists, \wedge, \neg)$) in PTIME. $\mathcal{S} = \{$  $\}$.

Two involved reductions complete the classical complexity classification:

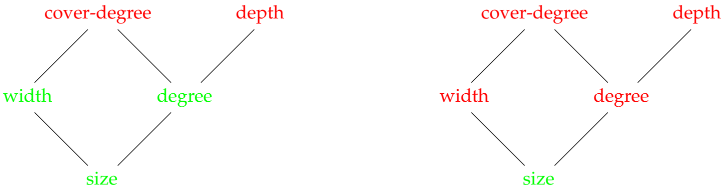


Figure: Parameterized vs. classical complexity of embedding wrt poset invariants.

Isomorphism of Bounded Width Posets in Polytime

Proposition [B, Ganian, Szeider '14].
The isomorphism problem is in PTIME on bounded width posets.

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The algorithm runs in $n^{O(w)}$ time.

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More Fragments, More Posets

This paper classifies all $\mathcal{FO}$-fragments of the form $\mathcal{FO}(Q, \mathcal{C})$, where $Q \in \{\forall, \exists\}$ and $\mathcal{C} \subseteq \{\wedge, \vee, \neg\}$, wrt natural poset invariants.

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This paper classifies all $\mathcal{FO}$-fragments of the form $\mathcal{FO}(Q, \mathcal{C})$, where $Q \in \{\forall, \exists\}$ and $\mathcal{C} \subseteq \{\wedge, \vee, \neg\}$, wrt natural poset invariants.

More fragments:

- A companion paper classifies $\mathcal{FO}(\forall, \exists, \wedge)$ wrt same poset invariants.
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- Bounded dimension posets (above width and degree)?
  We know that $\mathcal{FO}$ is hard. Embedding?
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Thank you for your attention!

Bounded Width Posets have Unbounded Cliquewidth

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Posets of width 2 have unbounded directed cliquewidth [BGS'14].

Idea (Cont'd): cover($\mathcal{G}$) has bounded degree.

If $\mathcal{S}$ is a class of digraphs of bounded degree,
then undirected($\mathcal{S}$) has bounded treewidth
iff $\mathcal{S}$ has bounded directed cliquewidth (Courcelle).
$\Longrightarrow$ cover($\mathcal{G}$) has unbounded directed cliquewidth.

If a class of DAGs has unbounded directed cliquewidth,
then their reflexo transitive closures have unbounded directed cliquewidth (Courcelle).
$\Longrightarrow$ $\mathcal{G}$ has unbounded directed cliquewidth.