A Decision Algorithm for Hájek’s Basic Logic

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Contribution

The tautology problem of Basic Logic is exponential-time decidable through a semantic algorithm [BHMV02, MPT03].

We describe a decision algorithm based on a different formalization of the problem.

The algorithm runs in $2^{O(n)}$ worst-case time.
Outline

Introduction 6 slides
  A Complete Semantics for BL
  BL Decidability

A Decision Algorithm for BL 11 slides
  The Semantic Calculus RWBL
  The Algorithm CHECKBL
  Correctness and Complexity

Conclusion 2 slides
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Outline

- Introduction
  - A Complete Semantics for BL
  - BL Decidability

  A Decision Algorithm for BL
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A Complete Semantics for BL

$L = (\bot, \top, \otimes, \to)$. $p_0, p_1, \ldots$ variables. $A, B, C \in L$.

**Definition** [H98]. The BL Hilbert system has the axioms:

(A1) $(A \to B) \to ((B \to C) \to (A \to C))$

(A2) $(A \otimes B) \to A$

(A3) $(A \otimes B) \to (B \otimes A)$

(A4) $(A \otimes (A \to B)) \to (B \otimes (B \to A))$

(A5a) $(A \to (B \to C)) \to ((A \otimes B) \to C)$

(A5b) $((A \otimes B) \to C) \to (A \to (B \to C))$

(A6) $((A \to B) \to C) \to (((B \to A) \to C) \to C)$

(A7) $\bot \to A$

and the *Modus Ponens* inference rule.

**Definition.** $\text{BL-TAUT} = \{ \langle A \rangle : \text{BL} \vdash_{HBL} A \} \subseteq \{0, 1\}^*$. 

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A Complete Semantics for BL

For all $x \in \mathbb{R}$, $\lfloor x \rfloor$ is the integer part of $x$. $\lfloor \infty \rfloor = \infty$.

**Definition [MPT03]**. $(\omega)[0, 1] = ([0, \infty], \ast, \Rightarrow_\ast, 0, \infty)$, where:

$$x \ast y = \begin{cases} 
  x & \text{if } \lfloor x \rfloor < \lfloor y \rfloor \\
  y & \text{if } \lfloor x \rfloor > \lfloor y \rfloor \\
  x + y - \lfloor x \rfloor - 1 & \text{if } \lfloor x \rfloor = \lfloor y \rfloor < \infty \text{ and } 1 \leq x - \lfloor x \rfloor + y - \lfloor y \rfloor \\
  \lfloor x \rfloor & \text{if } \lfloor x \rfloor = \lfloor y \rfloor < \infty \text{ and } 1 > x - \lfloor x \rfloor + y - \lfloor y \rfloor \\
  \infty & \text{if } x = y = \infty 
\end{cases}$$

$$x \Rightarrow_\ast y = \begin{cases} 
  y & \text{if } \lfloor y \rfloor < \lfloor x \rfloor \\
  \lfloor x \rfloor + 1 - x + y & \text{if } \lfloor x \rfloor = \lfloor y \rfloor \text{ and } y < x \\
  \infty & \text{if } x \leq y 
\end{cases}$$
A Complete Semantics for BL

**Definition [MPT03]**. A valuation of $L$ into $(\omega)[0, 1]$ is a map $v$ such that:

- $v(\bot) = 0$, $v(\top) = \infty$ and $v(p_i) \in [0, \infty]$, $i \in \mathbb{N}$;
- $v(A \odot B) = v(A) \ast v(B)$ and $v(A \rightarrow B) = v(A) \Rightarrow v(B)$.

**Theorem [MPT03]**. $\text{BL} \vdash_{HBL} A$ iff $(\forall v)v(A) = \infty$, i.e., $(\omega)[0, 1]$ is a complete semantics for $\text{BL}$.

Write $A^v \equiv v(A)$, $A^i \equiv \lfloor v(A) \rfloor$, $A^d \equiv A^v - A^i$. $\top^v \equiv \top^i$. $\text{size}(A)$ is the circuit complexity of $A$. 

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Theorem [~MPT03]. BL-TAUT ∈ EXP.

Input: \( A \in L, \text{size}(A) > 0. \)

Question: \((\forall v)A^v = \top^v?\)

Answer: Divide-and-conquer approach:

- **Divide**: Choose a *pivot* formula in the question and reduce the question to *simpler* subquestions, applying the definition by cases of \( v \) to the pivot.

- **Conquer**: Answer the subquestions:
  - recursively, if they are *reducible*;
  - easily, if they are *irreducible*.

- **Combine**: Answer “Yes” iff the answer to all the subquestions is “Yes”. 
BL Decidability

Example (trivial). $size(A) = 0, A = p_i$:

Q: $(\forall v)p^v_i = \top^v$?

A: “No”.

Example (hard). $size(A) > 0, A = (B \rightarrow C)$:

Q: $(\forall v)(B \rightarrow C)^v = \top^v$?

A: “Yes” iff the answer to all the subquestions:

$Q_1$: $(\forall v)(C^v = \top^v \iff C^i < B^i)$?

$Q_2$: $(\forall v)(1 - B^d + C^v = \top^v \iff (B^i = C^i \land C^d < B^d))$?

$Q_3$: $(\forall v)(\top^v = \top^v \iff B^v \leq C^v)$?

is “Yes”.
BL Decidability

Example (hard). $\text{size}(A) > 0$, $A = (B \odot C)$:

Q: $(\forall v) \left( (B \odot C)^v = T^v \right)$?

A: “Yes” iff the answer to all the subquestions:

- $Q_1$: $(\forall v) (B^v = T^v \iff B^i < C^i)$?
- $Q_2$: $(\forall v) (C^v = T^v \iff C^i < B^i)$?
- $Q_3$: $(\forall v) (B^d + C^v - 1 = T^v \iff (B^i = C^i < T^v \land 1 \leq B^d + C^d))$?
- $Q_4$: $(\forall v) (B^i = T^v \iff (B^i = C^i < T^v \land B^d + C^d < 1))$?
- $Q_5$: $(\forall v) \left( T^v = T^v \iff (B^v = T^v \land C^v = T^v) \right)$?

is “Yes”.

Issue: Generalize to clause matrices.
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The Semantic Calculus RWBL

Goal: Formalize efficiently the algorithm sketched above.

Idea (vague): Simplify the subquestions as much as possible.

Issues:

- **Subgoal 1**: Formalize the problem.
- **Subgoal 2**: Formalize the divide step and the recursive case of the conquer step.
- **Subgoal 3**: Formalize the easy case of the conquer step.
Subgoal 1: Find a set of *predicates* to formalize the questions.

Informal questions are clauses, *e.g.*:

\[ 1 - B^d + C^v = \top^v \lor B^i \neq C^i \lor B^d \leq C^d. \]

Informal literals are relations over *pairs* of reals, *e.g.*:

\[ B^d \leq C^d, \]

and *sums of tuples* of reals, *e.g.*:

\[ 1 - B^d + C^v = \top^v. \]
The Semantic Calculus RWBL

The relations chosen are ($z \in \mathbb{Z}$):

- $B \ll_v C$ iff $B^i < C^i$
- $B \prec_v C$ iff $B^i = C^i \land B^d < C^d$
- $B \preceq_v C$ iff $B^i = C^i \land B^d \leq C^d$
- $A_1, \ldots, A_n \prec_{v,z} B_1, \ldots, B_m$ iff $A^i_1 = \cdots = B^i_m < \infty$ and
  \[
  \sum_{i=1}^{n} (A^d_i - 1) < z + \sum_{i=1}^{m} (B^d_i - 1)
  \]
- $A_1, \ldots, A_n \preceq_{v,z} B_1, \ldots, B_m$ iff $A^i_1 = \cdots = B^i_m < \infty$ and
  \[
  \sum_{i=1}^{n} (A^d_i - 1) \leq z + \sum_{i=1}^{m} (B^d_i - 1)
  \]
The Semantic Calculus RWBL

∅ contributes 0.

Example.

- $A^v = \top^v$ becomes $\top \preceq_v A$.
- $\top^v = \top^v \lor C^v < B^v$ becomes $\top \preceq_v \top \lor C \preceq_v B \lor C \ll_v B$.
- $B^d + C^d < 1$ becomes $B, C \preceq_{v, -1} \emptyset$.
- $1 \leq B^d + C^d$ becomes $\emptyset \preceq_{v, 1} B, C$.

A question is reducible if it has at least one formula of complexity $> 0$, and irreducible otherwise.

Example. $\top \preceq_v p_i$ is irreducible, $\top \preceq_v p_i \odot p_j$ is reducible.
The Semantic Calculus RWBL

Subgoal 2. Find a set of rules to simplify recursively a formalized question (minimizing the recursions).

Example. By the interpretation of $\odot$:

- if $B^i = C^i$, then $(B \odot C)^i = B^i = C^i$;
- if $B^i = C^i < \infty$ and $1 \leq B^d + C^d$, then
  $$(B \odot C)^d - 1 = (B^d - 1) + (C^d - 1).$$

Let, e.g., $B^i = C^i < \infty$ and $1 \leq B^d + C^d$.

- $B \odot C \ll_v A$ if and only if $B \ll_v A$
- $B \odot C \lessdot_v A$ if and only if $B, C \ll_v, 0 A$
- $\Gamma, B \odot C \lessdot_{z, v} \Delta$ if and only if $\Gamma, B, C \ll_{z, v} \Delta$

where $\ll \in \{\lessdot, \ll\}$. 
The Semantic Calculus RWBL

**Idea (definite):** Generate subquestions with no occurrences of the *pivot*.

The *ReWriting Basic Logic calculus* has two *rewriting rules*:

\[
(B \odot C) \xrightarrow{\odot,1} Q \xrightarrow{\odot,2} Q \xrightarrow{\odot,3} Q \xrightarrow{\odot,4} Q \xrightarrow{\odot,5} Q
\]

\[
(B \rightarrow C) \xrightarrow{\rightarrow,1} Q \xrightarrow{\rightarrow,2} Q \xrightarrow{\rightarrow,3} Q
\]

where \( Q \) is the clause matrix of a reducible question, \((B \odot C)\) is the pivot and each \( Q_{\odot,j} \) is a clause matrix, \( \odot \in \{ \odot, \rightarrow \} \).

**Remark:** The divide step and recursive case of the conquer step are settled.
The Semantic Calculus RWBL

The rules meet logical and complexity requirements.

Claim 1. The rewriting rules are sound and invertible.

Proof (sketch). The rewriting rules satisfy:

\[ (\forall v)(Q_{o,1} \land \cdots \land Q_{o,k_o}) \Rightarrow (\forall v)Q, \text{ and} \]
\[ (\forall v)Q \Rightarrow (\forall v)(Q_{o,1} \land \cdots \land Q_{o,k_o}) \]

where \( o \in \{\odot, \rightarrow\}, k_{\odot} = 5, k_{\rightarrow} = 3. \]

Claim 2. The rewriting rules eliminate the pivot.

Proof (sketch). The pivot can be eliminated exploiting the consequences of the interpretation of \( \odot \) and \( \rightarrow \) while deriving the subquestions of a question.
Subgoal 3. Find an irreducible questions checker.

Claim 3. $\text{CHECKAX}(\langle Q \rangle) = 1$ iff $Q$ is irreducible and the answer to the question $(\forall v)Q$ is “Yes”.

*Proof (sketch).* If $Q$ is reducible, output 0. Otherwise, the negation $\neg Q$ is a conjunction of atomic literals and, by the interpretation of the language, there exists a linear program $P$ such that $P$ is feasible iff $(\exists v)\neg Q$. So, output 1 iff $P$ is unfeasible. □

*Remark:* The easy case of the conquer step is settled.
The Algorithm CHECKBL

Input: $\langle A \rangle \in \{0, 1\}^*$.  
Output: 1 iff $\text{BL} \vdash_{HBL} A$.

CHECKBL($\langle A \rangle$)

$\begin{align*}
&\quad A = A_1 \succ_c \cdots \succ_c A_n, \text{size}(A_i) > 0; \\
&\quad \text{initialize a labelled tree } T_A \text{ with root } A^v = \top^v; \\
&\quad \text{for } i = 1, \ldots, n, \text{ extend the leaves of } T_A \text{ applying the rewriting rule with pivot } A_i, \text{ until all the leaves are irreducible or the loop terminates}; \\
&\quad \text{after } m \leq n \text{ steps, } T_A \text{ has leaves } \{Q_1, \ldots, Q_k\}; \\
&\quad \text{output 1 iff the checker outputs 1 for all the } Q_j \text{'s.}
\end{align*}$
Correctness and Complexity

Theorem [BM]. $\langle A \rangle \in \text{BL-TAUT}$ iff $\text{CHECKBL}(\langle A \rangle) = 1$.

Proof (sketch). By Claim 1, the rewriting procedure

$$\{Q\} = T_0 \xrightarrow{A_1} T_1 \xrightarrow{A_2} \ldots \xrightarrow{A_{m-1}} T_m = \{Q_1, \ldots, Q_k\},$$

satisfies $(\forall v)Q$ iff $(\forall v)Q_i$ for all $i = 1, \ldots, k$.

By Claim 2, $Q_1, \ldots, Q_k$ are irreducible, since the pivot $A_i$ of the $i$th rewriting does not occur in the external nodes of $T_{i+1}$.

By Claim 3, the output is 1 iff $(\forall v)Q$. $\Box$
Correctness and Complexity

Theorem [BM]. BL-TAUT $\in$ DTIME($2^{O(n)}$).

Proof (sketch). In the worst case, $T_A$ has $5^{n+1} - 1$ nodes. Each node can be encoded in $O(n^2)$ space. So, $T_A$ can be encoded in $2^{O(n)}$ space.

The construction is feasible in time polynomial in the size of $T_A$, and each leaf can be checked in time polynomial in $n$ and the size of the corresponding linear program [Y91]. $\square$
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RWBL Pros/Cons

**Pro:** the size of the proof trees is improved:

\[
\begin{align*}
\text{height} & \leq n^3 \quad n \\
\text{width} & \leq 2^{O(n^3)} \quad 2^{O(n)}
\end{align*}
\]

**Pro:** suitable for *automatic proof search*.

**Con:** *slower* than Boolean logic propositional proof systems.

**Con:** *strongly coupled* with linear programming.
References

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