Hájek's Basic Logic: Decision and Representation

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December 19, 2007
Logic Seminar
University of Barcelona (Spain)

Outline

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1. Motivation
   - Vague Notions
   - Basic Logic
2. Decision Problems
   - Derivability and Validity
   - Complexity and Algorithms
3. Functional Representation
   - Free Algebras and Normal Forms
   - Open Problems

Sorite's Paradox

$X_i \rightleftharpoons$ "a collection of i grains is a heap", $N \rightleftharpoons 1000000$

Tentative axiomatization of the notion of heap ($i = 0, \ldots, N-1$):

(H1) X_N

(H2) $\neg X_0$

(H3.i) $X_{i+1} \rightarrow X_i$

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The theory is inconsistent:

1 X_N

2 X_{N-1}

... ...

$N+1$ X_0

$N+2$ $\neg X_0$

Bivalence *versus* Vagueness

We can either reject vagueness . . .

$$0 = X_0 = \cdots = X_{500000} < X_{500001} = ... = X_N = 1$$

“500001 grains form a heap, whether 500000 do not”

Bivalence *versus* Vagueness

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. . . or abjure bivalence:

$$0 = X_0 < X_1 < ... < X_{N-1} < X_N = 1$$

"i grains form a heap" is less true than "j grains form a heap", if $i < j$

Hájek's Paradigm | Fuzzy Logic

Fuzzy logics are propositional logics over $\top, \bot, \odot, \rightarrow$ st:

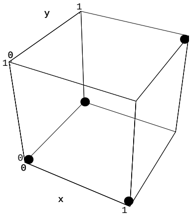
- variables $X, Y, \ldots$ are interpreted over $[0, 1]$;
- $\top$ and $\bot$ are interpreted over 1 and 0;
- $\odot$ and $\rightarrow$ are interpreted over binary functions on $[0, 1]$;
- $\neg X \rightleftharpoons X \rightarrow \bot$.

Fuzzy conjunction and implication *must* maintain:

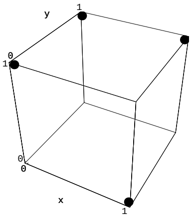
- the behavior of Boolean counterparts over $\{0, 1\}^2$;
- *intuitive* properties of Boolean counterparts over $[0, 1]^2$;
- the validity of *fuzzy modus ponens*.

Hájek's Paradigm | Boolean Logic

Intuitive properties of Boolean conjunction and implication:



Boolean conjunction is commutative, associative, weakly increasing in both arguments, and has 1 as unit.



Boolean implication, x implies y, is 1 iff $x \leq y$, weakly decreasing in x, weakly increasing in y.

Hájek's Paradigm | *t*-Norms and Residua

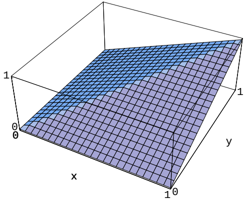
Definition (Continuous *t*-Norm, Residuum)

A continuous *t-norm* $\odot_*$ is a continuous binary function on $[0,1]$ that is associative, commutative, monotone ($x \leq y$ implies $x \odot_* z \leq y \odot_* z$) and has 1 as unit ($x \odot_* 1 = x$). Given a continuous *t*-norm $\odot_*$, its *residuum* is the binary function $\rightarrow_*$ on $[0,1]$ defined by $x \rightarrow_* y = max\{z : x \odot_* z \leq y\}$.

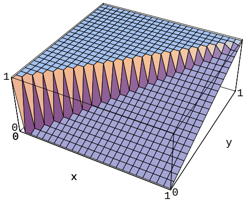
t-norms and their residua provide suitable interpretations for fuzzy conjunction and implication.

Hájek's Paradigm | Gödel Logic

$\odot_G$ and $\rightarrow_G$ over $[0,1]^2$:



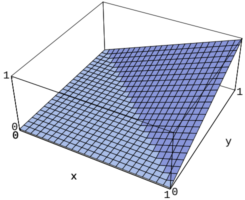
$$x \odot_G y = \min(x, y)$$



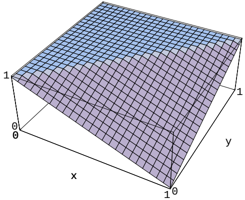
$$x \rightarrow_G y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}$$

Hájek's Paradigm | Łukasiewicz Logic

$\odot_L$ and $\rightarrow_L$ over $[0,1]^2$:



$$x \odot_L y = \max(0, x + y - 1)$$



$$x \rightarrow_L y = \min(1, -x + y + 1)$$

Basic Logic | Logical Calculus

$\vdash_{BL} \phi$ iff ϕ is derivable in the following Hilbert calculus:

(A1) $(\phi \to \chi) \to ((\chi \to \psi) \to (\phi \to \psi))$

(A2) $(\phi \odot \chi) \to \phi$

(A3) $(\phi \odot \chi) \to (\chi \odot \phi)$

(A4) $(\phi \odot (\phi \to \chi)) \to (\chi \odot (\chi \to \phi))$

(A5) $((\phi \odot \chi) \to \psi) \leftrightarrow (\phi \to (\chi \to \psi))$

(A6) $((\phi \to \chi) \to \psi) \to (((\chi \to \phi) \to \psi) \to \psi)$

(A7) $\bot \to \phi$

(R1) $\phi, \phi \to \chi \vdash \chi$

Basic Logic | Semantic Completeness

BL is the logic of all continuous *t*-norms and their residua [Cignoli et al., 2000]:

(*i*) $\vdash_{BL} \chi$ iff,
for every *t*-norm $\odot_*$ and every assignment *v*,
χ evaluates to 1 with respect to $\odot_*$ and *v*.

(*ii*) $\phi_1, \ldots, \phi_n \vdash_{BL} \chi$ iff,
for every *t*-norm $\odot_*$ and every assignment *v*,
if $\phi_1, \ldots, \phi_n$ evaluate to 1 with respect to $\odot_*$ and *v*,
then χ evaluates to 1 with respect to $\odot_*$ and *v*.

Outline

Derivability and Validity

Let $\phi_1, \ldots, \phi_n, \chi$ be formulas over $X_1, \ldots, X_n$.

$$\text{BL-CONS}_n = \{\langle(\{\phi_1, \ldots, \phi_m\}, \{\chi\})\rangle : \phi_1, \ldots, \phi_m \vdash_{BL} \chi\}$$

$$\text{BL-TAUT}_n = \{\langle\chi\rangle : (\emptyset, \{\chi\}) \in \text{BL-CONS}_n\} \subseteq \text{BL-CONS}_n$$

Derivability and Validity

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There are infinitely many *t*-norms and infinitely many assignments of *n* propositional variables over $[0, 1]$.

Derivability and Validity

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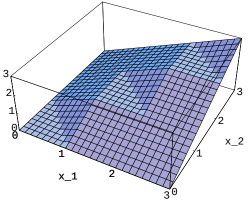
There are infinitely many *t*-norms and infinitely many assignments of *n* propositional variables over $[0, 1]$.

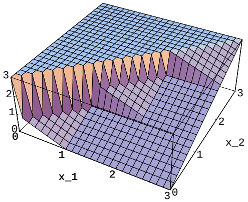
Question: Is BL-TAUT_n decidable? And BL-CONS_n?

Generic t-Norms | 2-Variate Fragment

$3[0,1]_{MV} = ([0,3], \odot_2, \rightarrow_2, \bot)$:



$$x_1 \odot_2 x_2 = \begin{cases} \max(x_1 + x_2 - 1, 0) & \text{if } 0 \leq x_1, x_2 < 1 \\ \max(x_1 + x_2 - 2, 1) & \text{if } 1 \leq x_1, x_2 < 2 \\ \max(x_1 + x_2 - 3, 2) & \text{if } 2 \leq x_1, x_2 \leq 3 \\ \min(x_1, x_2) & \text{if } \lfloor x_1 \rfloor \neq \lfloor x_2 \rfloor \end{cases}$$



$$x_1 \rightarrow_2 x_2 = \begin{cases} 3 & \text{if } x_1 \leq x_2 \\ x_2 - x_1 + 1 & \text{if } 0 \leq x_1, x_2 < 1 \\ x_2 - x_1 + 2 & \text{if } 1 \leq x_1, x_2 < 2 \\ x_2 - x_1 + 3 & \text{if } 2 \leq x_1, x_2 \leq 3 \\ x_2 & \text{if } \lfloor x_2 \rfloor < \lfloor x_1 \rfloor \end{cases}$$

Generic t-Norms | n-Variate Fragment

$(n+1)[0,1]_{MV} = ([0,n+1], \odot_n, \to_n, \bot)$:

$$x \odot_n y = \begin{cases} \max(x+y-(i+1), i) & \text{if } \lfloor x \rfloor = \lfloor y \rfloor = i \\ \min(x,y) & \text{if } \lfloor x \rfloor \neq \lfloor y \rfloor \end{cases}$$

$$x \to_n y = \begin{cases} n+1 & \text{if } x \leq y \\ y+(i+1)-x & \text{if } \lfloor x \rfloor = \lfloor y \rfloor = i \\ y & \text{if } \lfloor y \rfloor < \lfloor x \rfloor \end{cases}$$

Let $\bot_0 \leftrightharpoons \bot$, $\bot_1 \leftrightharpoons 1, \ldots, \bot_n \leftrightharpoons n$, $\bot_{n+1} \leftrightharpoons \top = n+1$.

Generic t-Norms | Decidability and Complexity

Theorem (∼ Aglianò and Montagna, 2003)

$\odot_n$ is generic for the n-variate fragment of BL, that is:

(i) $\vdash_{BL} \chi(X_1, \ldots, X_n)$ *iff, for every assignment* v, $v(\chi) = n + 1$ *wrt* $\odot_n$.

(ii) $\phi_1(X_1, \ldots, X_n), \ldots, \phi_m(X_1, \ldots, X_n) \vdash_{BL} \chi(X_1, \ldots, X_n)$ *iff, for every assignment* v*, if* $v(\phi_1) = \cdots = v(\phi_n) = n + 1$ *wrt* $\odot_n$*, then* $v(\chi) = n + 1$ *wrt* $\odot_n$.

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Corollary (Baaz et al., 2002; $\sim$ Aguzzoli and Gerla, 2002)

BL-CONS$_n$ $\in$ coNP.

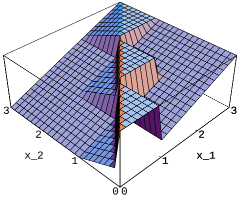
"No" instances of BL-CONS$_n$ have small witnesses wrt $\odot_n$.

Generic t-Norms | Decidability and Complexity

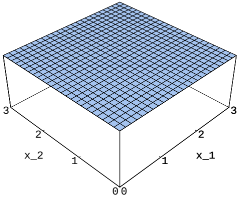
Example: $((X_1 \to X_2) \to X_2) \to X_1 \rightleftharpoons \psi \in \text{BL-TAUT}_2$?

Generic t-Norms | Decidability and Complexity

Example: $((X_1 \to X_2) \to X_2) \to X_1 \rightleftharpoons \psi \in$ BL-TAUT$_2$? No:



$= \psi^{BL_2} \neq$



Sample witnesses of $\psi \notin$ BL-TAUT$_2$:

(i) $v(X_1) = v(X_2) = 4/3$;

(ii) any v st $0 \leq v(X_2) < 1 < v(X_1) < n+1$;

(iii) any v st $2 \leq v(X_1), v(X_2) \leq 3$ and $v(X_1) < v(X_2)$;

(iv) $\bot \leq X_2 = X_1 \to X_2 < \bot_1 < X_1 = \psi < \top = (X_1 \to X_2) \to X_2$.

Generic t-Norms | Decidability and Complexity

Definition (Subformulae Order)

Let $\chi(X_1, \ldots, X_n)$ be a formula with l connectives. A *subformulae order* for χ is a partition of the subformulae of χ, $\bot_0, \ldots, \bot_{n+1}$ and $\top$ into $\leq n+2$ blocks. For $j = 0, \ldots, n+1$, the block B_j is linearly ordered with least element $\bot_j$, and holds a linear program of $O(l)$ constraints over $x_1, \ldots, x_n$. The order is *consistent* if and only if there exists an assignment v of the variables in $[0, n+1]$ that satisfies the linear orders and the linear programs.

Fact

$\chi < \top$ holds in a consistent order iff, for some v, $v(\phi) < v(\top)$.

Generic t-Norms | Decidability and Complexity

Question: How many witnesses do we have to check, in the worst case, to conclude that a given instance χ of size l is not in BL-TAUT$_n$? What about BL-CONS$_n$?

Generic t-Norms | Decidability and Complexity

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We know that testing $2^{3l} \leq l!$ witnesses suffices wrt BL-TAUT$_n$ [Bova and Montagna, 2007]. Wrt BL-CONS$_n$, the bound $l!$ still resists [Baaz et al., 2002].

Outline

Algebraic Logic

Definition (BL-Algebras)

A *BL-algebra* is an algebra $(A, \vee, \wedge, \odot, \rightarrow, \top, \bot)$ of type $(2, 2, 2, 2, 0, 0)$ such that:

(*i*) $(A, \odot, \top)$ is a commutative monoid;

(*ii*) $(A, \vee, \wedge, \top, \bot)$ is a bounded lattice;

(*iii*) $x \odot y \leq z$ if and only if $y \leq x \rightarrow z$ (residuation);

(*iv*) $(x \rightarrow y) \vee (y \rightarrow x)$ (prelinearity).

BL-algebras form an algebraic variety.

Algebraic Logic

The variety of *BL*-algebras forms the algebraic semantics of *BL*.

Thus, the free *n*-generated *BL*-algebra, BL_n, *encodes* the *n*-variate fragment of *BL*, in the precise sense that BL_n is isomorphic to the Lindenbaum-Tarski algebra of the *n*-variate fragment of *BL*.

Algebraic Logic

The variety of BL-algebras forms the algebraic semantics of BL.

Thus, the free n-generated BL-algebra, BL_n, *encodes* the n-variate fragment of BL, in the precise sense that BL_n is isomorphic to the Lindenbaum-Tarski algebra of the n-variate fragment of BL.

Question: Is there an explicit description of BL_n?

Functional Definition

Fact (Aglianò and Montagna, 2002)

The free n-generated BL-algebra, BL_n, is the subalgebra of

$$((n+1)[0,1]_{MV})^{((n+1)[0,1]_{MV})^n},$$

generated by the projections, with pointwise defined operations.

The explicit description of BL_n amounts to the *characterization* of the class F of functions $f : [0, n+1]^n \to [0, n+1]$ st:

- (i) f is either a projection $x_1, \ldots, x_n$ or the constant 0;
- (ii) f has the form $g_1 \circ_n g_2$, where $g_1, g_2 \in F$, $\circ_n \in \{\odot_n, \to_n\}$, and $(g_1 \circ_n g_2)(\cdot) = g_1(\cdot) \circ_n g_2(\cdot)$.

BL_1 | Functional Characterization

The explicit description of BL_1 amounts to the characterization of the functions $f : [0,2] \to [0,2]$ that are definable as arbitrary compositions of the projection x and the constant 0 via the operations $\odot_1$ and $\to_1$:

$$x \odot_1 y = \begin{cases} \max(x+y-1, 0) & \text{if } 0 \leq x, y < 1 \\ \max(x+y-2, 1) & \text{if } 1 \leq x, y \leq 2 \\ \min(x, y) & \text{if } \lfloor x \rfloor \neq \lfloor y \rfloor \end{cases}$$

$$x \to_1 y = \begin{cases} 2 & \text{if } x \leq y \\ y+1-x & \text{if } 0 \leq x, y < 1 \\ y+2-x & \text{if } 1 \leq x, y \leq 2 \\ y & \text{if } \lfloor y \rfloor < \lfloor x \rfloor \end{cases}$$

BL_1 | McNaughton Functions

Definition (McNaughton Function)

A continuous n-variate function over $[0, 1]$ is a *McNaughton function* iff there are linear polynomials $p_1, \ldots, p_k$ with integer coefficients such that, for every $x \in [0, 1]^n$, there is $j \in [k]$ such that $f(x) = p_j(x)$.

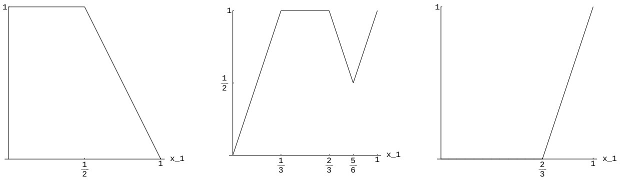

Figure: 1-variate McNaughton functions $f, g_1, g_2 : [0, 1] \to [0, 1]$.

BL_1 | Lifting

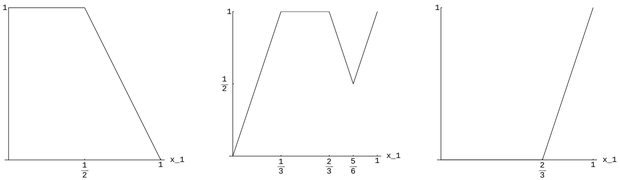

Figure: $f, g_1, g_2 : [0,1] \to [0,1]$.

BL_1 | Lifting

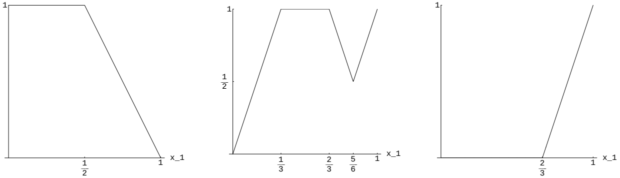

Figure: $f, g_1, g_2 : [0, 1] \to [0, 1]$.

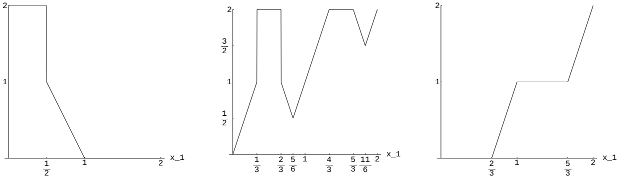

Figure: $lift_1(f), lift_2(g_1), lift_2(g_2) : [0, 2] \to [0, 2]$.

BL_1 | Masking

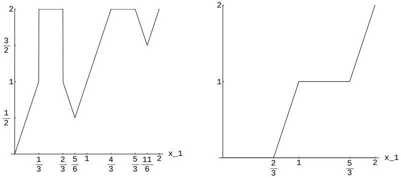

Figure: $lift_2(g_1), lift_2(g_2) : [0,2] \to [0,2]$.

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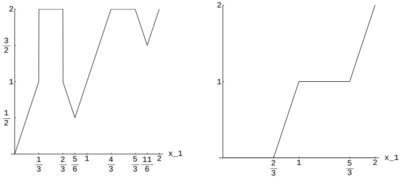


Figure: $lift_2(g_1), lift_2(g_2) : [0,2] \to [0,2]$.

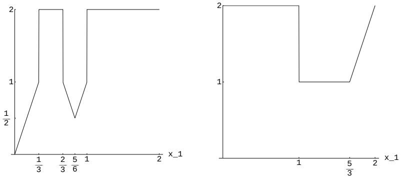
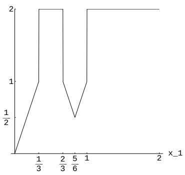


Figure: $mask_1(lift_2(g_1)), mask_2(lift_2(g_2)) : [0,2] \to [0,2]$.

BL_1 | $\wedge$'ing



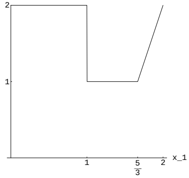


Figure: $mask_1(lift_2(g_1)), mask_2(lift_2(g_2)) : [0, 2] \rightarrow [0, 2]$.

BL_1 | $\wedge$'ing

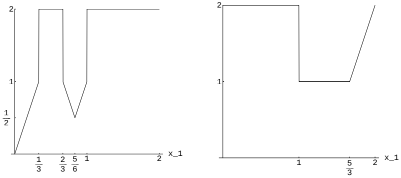


Figure: $mask_1(lift_2(g_1)), mask_2(lift_2(g_2)) : [0,2] \to [0,2]$.

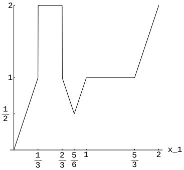
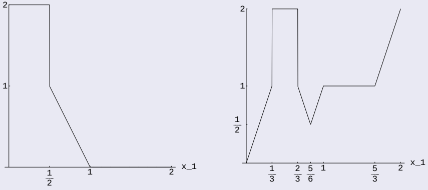


Figure: $mask_1(lift_2(g_1)) \wedge mask_2(lift_2(g_2)) : [0,2] \to [0,2]$.

BL_1 | Explicit Description

Theorem ($\sim$ Montagna, 2000)

Let f, g_1, g_2 be McNaughton functions, st $f(1) = 0$, $g_1(1) = g_2(1) = 1$. The free 1-generated BL-algebra, BL_1, is the algebra of 1-variate functions over $[0, 2]$ of the form $lift_1(f)$ or $mask_1(lift_2(g_1)) \wedge mask_2(lift_2(g_2))$:



with pointwise defined operations $\odot_1$ and $\rightarrow_1$.

Open Problems

(*i*) Give the functional characterization of BL_n for $2 \leq n < \omega$.

(*ii*) Compute deductive interpolants in *BL*.

(*iii*) Provide a combinatorial characterization of *finite* *n*-generated free *BL*-algebras.

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