Functional Representation of Basic Logic

Simone Bova
bova@unisi.it
www.mat.unisi.it/~bova

Department of Mathematics and Computer Science
University of Siena (Italy)

April 24-26, 2008
Non-Classical Logics: from Foundations to Applications
Centro di Ricerca Matematica E. de Giorgi, Pisa (Italy)

Outline

Outline

Language

X set of *variables*
$\emptyset \neq I \subseteq \{1, \ldots, n\}$, $X_I = \{x_i \mid i \in I\}$

T *(propositional) language* over X and $\{\odot, \rightarrow, \bot\}$
T^+ fragment of T over $\{\odot, \rightarrow\}$
T_I fragment of T over X_I
T_I^+ fragment of T over X_I and $\{\odot, \rightarrow\}$

$\top = x_1 \rightarrow x_1$
$\neg t = t \rightarrow \bot$
$t_1 \wedge t_2 = t_1 \odot (t_1 \rightarrow t_2)$
$t_1 \vee t_2 = ((t_1 \rightarrow t_2) \rightarrow t_2) \wedge ((t_2 \rightarrow t_1) \rightarrow t_1)$
$t_1 \leftrightarrow t_2 = (t_1 \rightarrow t_2) \odot (t_2 \rightarrow t_1)$

Basic Logic and Łukasiewicz Logic

Basic logic, $\vdash_{BL}$, is defined by the MP rule and the axiom schemata:

(A1) $(A \to B) \to ((B \to C) \to (A \to C))$

(A2) $(A \odot B) \to A$

(A3) $(A \odot B) \to (B \odot A)$

(A4) $(A \odot (A \to B)) \to (B \odot (B \to A))$

(A5) $((A \to (B \to C)) \leftrightarrow ((A \odot B) \to C))$

(A6) $((A \to B) \to C) \to (((B \to A) \to C) \to C)$

(A7) $\bot \to A$

Łukasiewicz logic, $\vdash_{L}$, extends Basic logic by adding:

(A8) $\neg\neg A \to A$

Semantics

A *semantics* for T is an algebra $\mathbf{A} = (A, \odot, \rightarrow, \bot)$ of type $(2, 2, 0)$.

Fact (Truthfunctionality)

Let $\mathbf{A}$ *be a semantics for* T *and let* $t \in T_n$. *Then,*
t *uniquely determines an n-ary function* $t^{\mathbf{A}}$ *over* A*, by putting, for every* $\mathbf{a} = (a_1, \ldots, a_n) \in A^n$*:*

(*i*) *if* $t = x_i$*, then* $t^{\mathbf{A}}(\mathbf{a}) = a_i$*;*

(*ii*) *if* $t = \bot$*, then* $t^{\mathbf{A}}(\mathbf{a}) = \bot^{\mathbf{A}}$*;*

(*iii*) *if* $t = r \circ s$*, then* $(r \circ s)^{\mathbf{A}}(\mathbf{a}) = r^{\mathbf{A}}(\mathbf{a}) \circ^{\mathbf{A}} s^{\mathbf{A}}(\mathbf{a})$*,*

where $\circ^{\mathbf{A}}$ *realizes* $\circ$ *in* $\mathbf{A}$ *and* $\circ \in \{\bot, \odot, \rightarrow\}$*.*

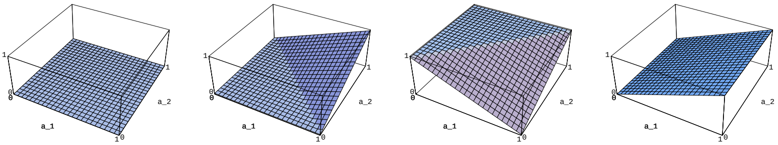
We say that $t^{\mathbf{A}}$ is the function *computed* by t over $\mathbf{A}$.

Semantics | Łukasiewicz Logic

Definition (Łukasiewicz Semantics)

$[0,1] = ([0,1], \odot, \to, \bot)$ given by $\bot^{[0,1]} = 0$ and:

$$a_1 \odot^{[0,1]} a_2 = \max(0, a_1 + a_2 - 1)$$
$$a_1 \to^{[0,1]} a_2 = \min(1, a_2 + 1 - a_1)$$



(a) $t = \bot$. (b) $t = x_1 \odot x_2$. (c) $t = x_1 \to x_2$. (d) $t = x_1$.

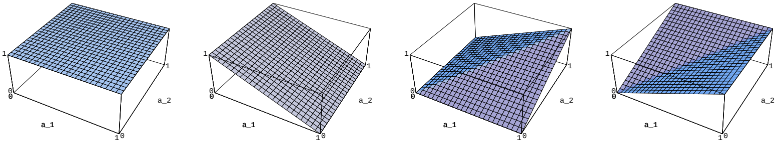
Figure: $t^{[0,1]} : [0,1]^2 \to [0,1]$ for sample $t \in T_2$.

Semantics | Łukasiewicz Logic

Fact (Abbreviations)

$\top^{[0,1]} = 1$, $\neg^{[0,1]} a = 1 - a$, *and:*

$$a_1 \wedge^{[0,1]} a_2 = \min(a_1, a_2)$$
$$a_1 \vee^{[0,1]} a_2 = \max(a_1, a_2)$$



(a) $t = \top$. (b) $t = \neg x_1$. (c) $t = x_1 \wedge x_2$. (d) $t = x_1 \vee x_2$.

Figure: $t^{[0,1]} : [0,1]^2 \to [0,1]$ for sample $t \in T_2$.

Completeness | Łukasiewicz Logic

Let $t \in T_n$. Then, t is a *Łukasiewicz tautology*, $[0,1] \models t$, iff $t^{[0,1]}(\mathbf{a}) = 1$ for every $\mathbf{a} \in [0,1]^n$.

Theorem (Chang)

Let $t \in T$. Then,

$$[0,1] \models t \text{ iff } \vdash_L t.$$

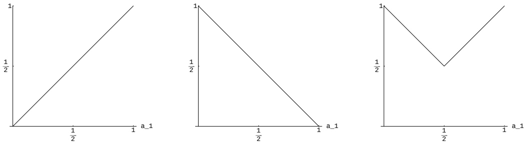
Fuzziness | Łukasiewicz Logic

Fact (Fuzziness)

$\nvdash_{L} t \vee \neg t$. *Thus,* $\nvdash_{BL} t \vee \neg t$.

Proof.

$t \in T_n$ for some $n \geq 1$. Check (in NP) that $(t \vee \neg t)^{[0,1]} \neq \top^{[0,1]}$. Note that $\nvdash_{L} t \vee \neg t$ implies $\nvdash_{BL} t \vee \neg t$ for every t. □



(a) x_1. (b) $\neg x_1$. (c) $x_1 \vee \neg x_1$.

Figure: $t^{[0,1]} : [0,1] \to [0,1]$ for sample $t \in T_1$.

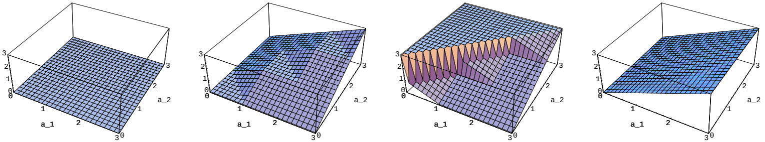
Semantics | Basic Logic

Definition (Basic Semantics)

$[0, n+1] = ([0, n+1], \odot, \to, \bot)$ given by $\bot^{[0,n+1]} = 0$ and:

$$a_1 \odot^{[0,n+1]} a_2 = \begin{cases} \min(a_1, a_2) & \text{if } \lfloor a_1 \rfloor \neq \lfloor a_2 \rfloor \\ \max(\lfloor a_1 \rfloor, a_1 + a_2 - \lfloor a_1 \rfloor - 1) & \text{otherwise} \end{cases}$$

$$a_1 \to^{[0,n+1]} a_2 = \begin{cases} a_2 & \text{if } \lfloor a_2 \rfloor < \lfloor a_1 \rfloor \\ a_2 + \lfloor a_1 \rfloor + 1 - a_1 & \text{if } \lfloor a_1 \rfloor = \lfloor a_2 \rfloor \text{ and } a_2 < a_1 \\ n+1 & \text{otherwise} \end{cases}$$



(a) $t = \bot$. (b) $t = x_1 \odot x_2$. (c) $t = x_1 \to x_2$. (d) $t = x_1$.

Semantics | Basic Logic

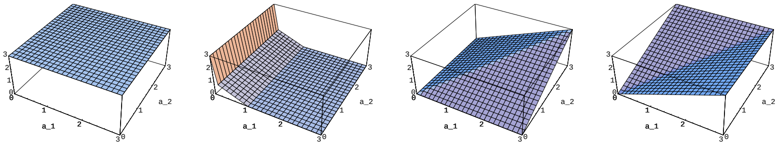
Fact (Abbreviations)

$\top^{[0,n+1]} = n+1$ *and:*

$$\neg^{[0,n+1]} a = \begin{cases} n+1 & \text{if } a = 0 \\ 1-a & \text{if } 0 < a < 1 \\ 0 & \text{if } 1 \leq a \end{cases}$$

$$a_1 \wedge^{[0,n+1]} a_2 = \min(a_1, a_2)$$

$$a_1 \vee^{[0,n+1]} a_2 = \max(a_1, a_2)$$



(a) $t = \top$. (b) $t = \neg x_1$. (c) $t = x_1 \wedge x_2$. (d) $t = x_1 \vee x_2$.

Completeness | Basic Logic

Let $t \in T_n$. Then, t is a *Basic tautology*, $[0, n+1] \models t$, iff $t^{[0,n+1]}(\mathbf{a}) = 1$ for every $\mathbf{a} \in [0, n+1]^n$.

Theorem (Aglianó and Montagna)

Let $t \in T_n$. Then,

$$[0, n+1] \models t \text{ iff } \vdash_{BL} t.$$

Outline

Functional Representation | Problem Statement

Fix $n \geq 1$ and $\mathbf{A} \in \{[0,1], [0, n+1]\}$.

Let $t \in T_n$. We know that $t^{\mathbf{A}}$ is an n-ary operation over A, but not every n-ary operation over A is computable by means of some $t \in T_n$.

A natural problem is then to characterize *explicitly* the set

$$F_{\mathbf{A},n} = \{f : A^n \to A \mid f = t^{\mathbf{A}} \text{ for some } t \in T_n\} \subseteq A^{A^n}.$$

Functional Representation | Solution Schema

Let $n \geq 1$ and $\mathbf{A} \in \{[0,1], [0, n+1]\}$ be given.

Step 1: Guess $F_{\mathbf{A},n} \subseteq A^{A^n}$, and provide an *effective* encoding $\langle \cdot \rangle \in \{0,1\}^*$ of functions in $F_{\mathbf{A},n}$.

Step 2: Check that $t^{\mathbf{A}} \in F_{\mathbf{A},n}$ for every $t \in T_n$, i.e: by induction on t, show that $t^{\mathbf{A}} = f$ for some $f \in F_{\mathbf{A},n}$.

Step 3: Check that for every $f \in F_{\mathbf{A},n}$ there is $t \in T_n$ s.t. $t^{\mathbf{A}} = f$, i.e: describe a *terminating* and *correct* algorithm that receives $\langle f \rangle$ and returns $t \in T_n$ such that $t^{\mathbf{A}} = f$.

Functional Representation | Łukasiewicz Logic

Definition (McNaughton Function)

A continuous function $f : [0,1]^n \to [0,1]$ is an *n-ary McNaughton function* iff there are linear polynomials with integer coefficients $p_1, \ldots, p_u : \mathbb{R}^n \to \mathbb{R}$ s.t. for every $\mathbf{a} \in [0,1]^n$, there is $j \in \{1, \ldots, u\}$ s.t. $f(\mathbf{a}) = p_j(\mathbf{a})$.

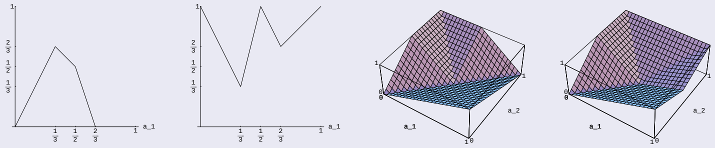


Figure: Unary and binary McNaughton functions samples.

Functional Representation | Łukasiewicz Logic

Definition (Unimodular Triangulation)

A *unimodular triangulation* U of $[0,1]^n$ is a *finite* set of n-dimensional *unimodular* simplexes with rational vertices, such that the union of all simplexes in U coincides with $[0,1]^n$ and any two simplexes intersect in a common face.
Let f be an n-ary McNaughton function with linear components $p_1, \ldots, p_u$, and let U be a unimodular triangulation of $[0,1]^n$.
We say that U *linearizes* f if for every simplex $S \in U$, there is $j \in \{1, \ldots, u\}$ such that $f \upharpoonright S = p_j$.

Theorem (Mundici)

Let f be an n-ary McNaughton function. Then, there is a unimodular triangulation of $[0,1]^n$ linearizing f.

Functional Representation | Łukasiewicz Logic

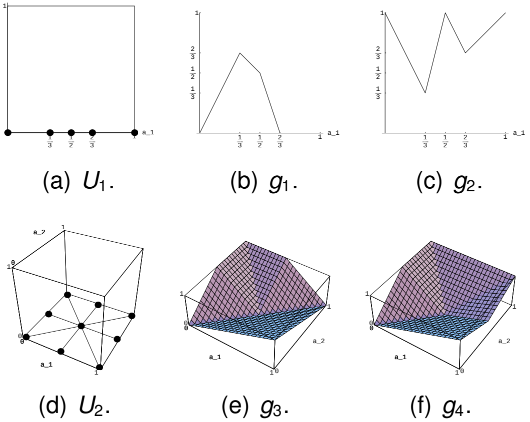


Figure: (a)-(c) U_1 linearizes g_1 and g_2. (d)-(f) U_2 linearizes g_3 and g_4.

Functional Representation | Łukasiewicz Logic

Goal: Characterize the set:

$$F_{[0,1],n} = \{f : [0,1]^n \to [0,1] \mid f = t^{[0,1]} \text{ for some } t \in T_n\} \subseteq [0,1]^{[0,1]^n}.$$

Step 1: Guess

$$F_{[0,1],n} = \{f \mid f \text{ } n\text{-ary McNaughton function}\} \subseteq [0,1]^{[0,1]^n}.$$

Let $f \in F_{[0,1],n}$ with polynomials $p_1, \ldots, p_u$, let $U = \{S_1, \ldots, S_m\}$ be a unimodular triangulation linearizing f, and for $i = 1, \ldots, m$ let $q_i \in \{p_1, \ldots, p_u\}$ be such that $f \restriction S_i = q_i$. Encode f via:

$$\langle f \rangle = \{(S_i, q_i) \mid i = 1, \ldots, m\}.$$

Step 2: Nontrivial, since $F_{[0,1],n} \subset [0,1]^{[0,1]^n}$ (by induction on t).

Step 3: For every $f \in F_{[0,1],n}$, there is $t \in T_n$ such that $t^{[0,1]} = f$ (Mundici).

Functional Representation | Łukasiewicz Logic

Corollary

Let g be an n-ary McNaughton function s.t. $g(\mathbf{1}) = 1$.
Then, there exists $t \in T_n^+$ s.t. $t^{[0,1]} = g$.

Proof.

Take $s \in T_n$ s.t. $s^{[0,1]} = g$, and derive $t \in T_n^+$ s.t. $t^{[0,1]} = s^{[0,1]}$ by substitutions:

(*i*) $r \odot \bot \ \Leftarrow \ \bot$, $\bot \odot r \ \Leftarrow \ \bot$, $\bot \to r \ \Leftarrow \ r \to r$, *and* $r \to \bot \ \Leftarrow \ \neg r$;

(*ii*) $(r \odot \neg s) \ \Leftarrow \ \neg(r \to s)$ *and* $(\neg r \odot s) \ \Leftarrow \ \neg(s \to r)$;

(*iii*) $(r \to \neg s) \ \Leftarrow \ \neg(r \odot s)$ *and* $(\neg r \to s) \ \Leftarrow \ (r \to (r \odot s)) \to s$;

(*iv*) $\neg\neg r \ \Leftarrow \ r$.

□

If $g(\mathbf{1}) = 1$ and $t^{[0,1]} = g$, we *always* assume w.l.o.g. $t \in T_n^+$.

Basic Logic | 1-Variate Fragment | Goal

Memo: T_1 interpreted over $[0,2]$, $t \in T_1$ computes $t^{[0,2]} \in [0,2]^{[0,2]}$.

Goal: Characterize the set of unary *basic* functions:

$$F_{[0,2],1} = \{f : [0,2] \to [0,2] \mid f = t^{[0,2]} \text{ for some } t \in T_1\} \subseteq [0,2]^{[0,2]}.$$

Idea: Provide a *blockwise* description of $f \in F_{[0,2],1}$ by means of McNaughton functions. Exploit the construction of terms computing unary McNaughton functions.

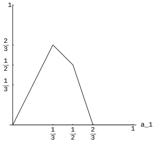
Basic Logic | 1-Variate Fragment | Step 1

Step 1: Let (g_1, g_2) be unary McNaughton functions, $g_2(\mathbf{1}) = 1$.

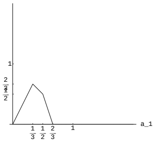
Case 1: If $g_1(\mathbf{1}) = 0$, then f is specified by:

$$\mathbf{a} \in [0,1) \Rightarrow f(\mathbf{a}) = \begin{cases} g_1(\mathbf{a}) & \text{if } g_1(\mathbf{a}) < 1 \\ 2 & \text{otherwise} \end{cases}$$

$$\mathbf{a} \in [1,2] \Rightarrow f(\mathbf{a}) = 0$$



(a) g_1.



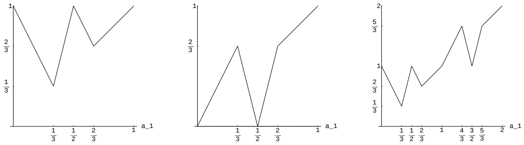
(b) f.

Basic Logic | 1-Variate Fragment | Step 1

Case 2: If $g_1(\mathbf{1}) = 1$, then f is specified by:

$$\mathbf{a} \in [0,1) \Rightarrow f(\mathbf{a}) = \begin{cases} g_1(\mathbf{a}) & \text{if } g_1(\mathbf{a}) < 1 \\ 2 & \text{otherwise} \end{cases}$$

$$\mathbf{a} \in [1,2] \Rightarrow f(\mathbf{a}) = g_2(\mathbf{a} - \mathbf{1}) + 1$$



(a) g_1. (b) g_2. (c) f.

Basic Logic | 1-Variate Fragment | Step 1 (Finished)

Guess:

$$F_{[0,2],1} = \{f \mid f \text{ specified by some } (g_1, g_2)\} \subseteq [0,2]^{[0,2]},$$

and let $\langle f \rangle = (\langle g_1 \rangle, \langle g_2 \rangle)$ be the encoding of $f \in F_{[0,2],1}$.

Basic Logic | 1-Variate Fragment | Step 2 (Finished)

Step 2: Nontrivial, since $F_{[0,2],1} \subset [0,2]^{[0,2]}$.

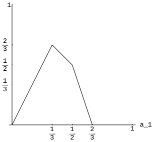
We claim that $t^{[0,2]} \in F_{[0,2],1}$ for every $t \in T_1$ (by induction on t, provide a pair (g_1, g_2) describing $t^{[0,2]}$ in terms of Step 1).

Basic Logic | 1-Variate Fragment | Step 3

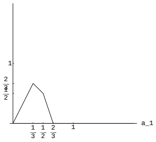
Fact

Let g be a unary McNaughton function and let $t \in T_1$ be such that $t^{[0,1]} = g$. If $g(\mathbf{1}) = 0$, then:

$$\langle t^{[0,2]} \rangle = (\langle g \rangle, \langle 0 \rangle).$$



(a) $g = t^{[0,1]}$.



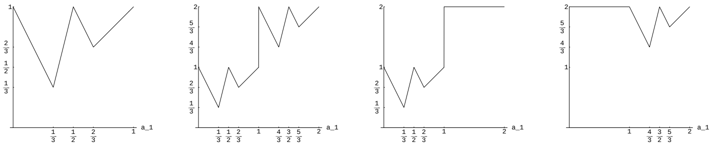
(b) $t^{[0,2]}$.

Basic Logic | 1-Variate Fragment | Step 3

Fact (Cont'd)

Otherwise, if $g(\mathbf{1}) = 1$ *(w.l.o.g.* $t \in T_1^+$*), then:*

$$\langle t^{[0,2]} \rangle = (\langle g \rangle, \langle g \rangle),$$
$$\langle (\neg\neg t)^{[0,2]} \rangle = (\langle g \rangle, \langle 1 \rangle),$$
$$\langle (\neg\neg t \to t)^{[0,2]} \rangle = (\langle 1 \rangle, \langle g \rangle).$$



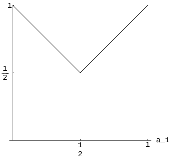
(a) $g = t^{[0,1]}$. (b) $t^{[0,2]}$. (c) $(\neg\neg t)^{[0,2]}$. (d) $(\neg\neg t \to t)^{[0,2]}$.

Basic Logic | 1-Variate Fragment | Step 3

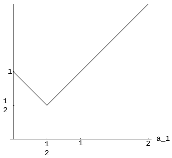
Assuming $t \in T_1^+$ is necessary, e.g. if $g = \max(x_1, 1 - x_1)$,

$$(x_1 \vee \neg x_1)^{[0,1]} = r^{[0,1]} = g = s^{[0,1]} = (x_1 \to (x_1 \odot x_1))^{[0,1]},$$

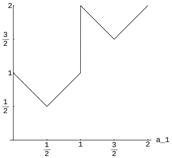
but $(\langle g \rangle, \langle x_1 \rangle) = \langle r^{[0,2]} \rangle \neq \langle s^{[0,2]} \rangle = (\langle g \rangle, \langle g \rangle)$,



(a) $r^{[0,1]} = s^{[0,1]}$.



(b) $r^{[0,2]}$.



(c) $s^{[0,2]}$.

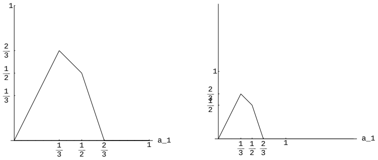
Basic Logic | 1-Variate Fragment | Step 3

Step 3: Let $f \in F_{[0,2],1}$ be given by (g_1, g_2),
and let $t_1, t_2 \in T_1$ be s.t. $g_1 = t_1^{[0,1]}$ and $g_2 = t_2^{[0,1]}$.

Case 1: If $g_1(\mathbf{1}) = 0$, we put:

$$t = t_1,$$

and we claim that $t^{[0,2]} = f$. Indeed,



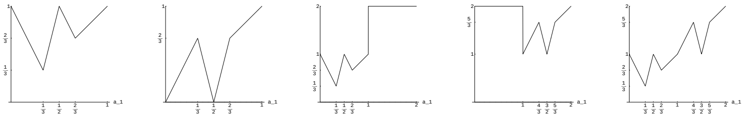
(a) $t^{[0,1]} = g_1$. (b) $t^{[0,2]} = f$.

Basic Logic | 1-Variate Fragment | Step 3 (Finished)

Case 2: If $g_1(\mathbf{1}) = 1$, we put:

$$t = (\neg\neg t_1) \wedge (\neg\neg t_2 \to t_2),$$

and we claim that $t^{[0,2]} = f$. Indeed,



(a) $t_1^{[0,1]} = g_1$. (b) $t_2^{[0,1]} = g_2$. (c) (d) (e) $t^{[0,2]} = f$.

Figure: $t_1, t_2 \in T_1$. (c) $(\neg\neg t_1)^{[0,2]}$ (d) $(\neg\neg t_2 \to t_2)^{[0,2]}$.

Basic Logic | 1-Variate Fragment | Summary

Goal: Characterize the set of unary basic functions:

$$F_{[0,2],1} = \{f : [0,2] \to [0,2] \mid f = t^{[0,2]} \text{ for some } t \in T_1\}.$$

Step 1: Guess

$$F_{[0,2],1} = \{f : [0,2] \to [0,2] \mid f \text{ specified by } (g_1, g_2)\}.$$

Step 2: Every term $t \in T_1$ computes a function $t^{[0,2]} \in F_{[0,2],1}$.

Step 3: Every function $f \in F_{[0,2],1}$ is computed by a term $t \in T_1$.

Basic Logic | 2-Variate Fragment | Goal

Memo: T_2 interpreted over $[0,3]$, $t \in T_2$ computes $t^{[0,3]} \in [0,3]^{[0,3]^2}$.

Goal: Characterize the set of binary basic functions:

$$F_{[0,3],2} = \{f : [0,3]^2 \to [0,3] \mid f = t^{[0,3]} \text{ for some } t \in T_2\} \subseteq [0,3]^{[0,3]^2}.$$

Idea: Provide a *blockwise* description of $f \in F_{[0,3],2}$ by means of binary McNaughton functions and unary basic functions. Exploit the construction of terms computing binary McNaughton functions and unary basic functions.

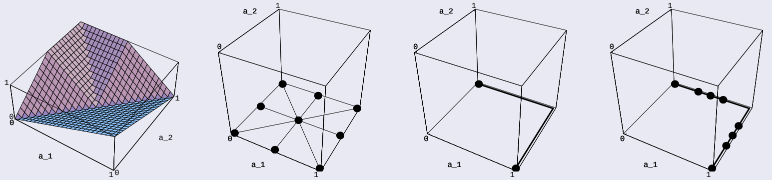
Basic Logic | 2-Variate Fragment | Step 1

Definition (Interface)

Let g be a binary McNaughton function linearized by U (over vertices V). Let

$$I_g = I_1 \cup I_2 = \{\mathbf{a} \mid a_1 = 1\} \cup \{\mathbf{a} \mid a_2 = 1\} \subseteq [0,1]^2.$$

An *interface* of g is an arbitrary fixed *finite* set R_g of rational points and open line segments with rational endpoints s.t. R_g forms a partition of I_g, and $(V \cap I_g) \subseteq R_g$ (notation, $R_{g,1} = R_g \cap I_1$ and $R_{g,2} = R_g \cap I_2$).



(a) g. (b) U and V. (c) I_g. (d) R_g.

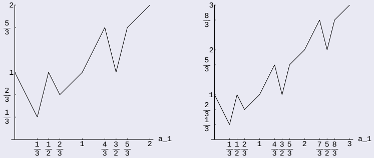
Basic Logic | 2-Variate Fragment | Step 1

Definition (Supplement)

Let g be a binary McNaughton function and let R_g be an interface of g. Then, the *supplement* of g is a set K_g containing a pair (R, t_R) for every $R \in R_g$ s.t. $g \upharpoonright R = 1$, where $t_R \in T_{\{i\}}$ if $R \subseteq I_i$ for $i = 1, 2$ and $t_R^{[0,2]}(\mathbf{1}) = 1$.

Fact

Let $t \in T_1$ be s.t. $t^{[0,2]}(\mathbf{1}) = 1$. Then,



(a) $t^{[0,2]}$.

(b) $t^{[0,3]} \upharpoonright \{\mathbf{a} \mid a_2 = 0\}$.

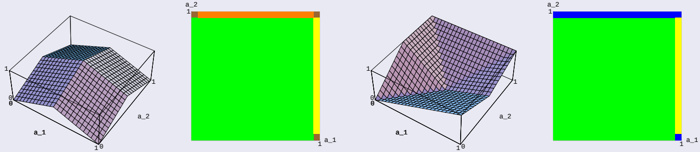
Basic Logic | 2-Variate Fragment | Step 1

Notation

Let h be a function in $F_{[0,1],2}$ *or in* $F_{[0,3],2}$. *Then* $\mathbf{a} \in dom(h)$ *has color:*

- 🟩 *if* $h(\mathbf{a}) = p_{\mathbf{a}}(\mathbf{a})$
- 🟧 *if* $h(\mathbf{a}) = p_{\mathbf{a}}((a_1, \cdot))$
- 🟨 *if* $h(\mathbf{a}) = p_{\mathbf{a}}((\cdot, a_2))$
- 🟫 *if* $h(\mathbf{a}) = 0$
- 🟦 *if* $h(\mathbf{a}) = 1$

where $p_{\mathbf{a}}$ *is a linear polynomial with integer coefficients.*



(a) h_1. (b) h_1 colors. (c) h_2. (d) h_2 colors.

Basic Logic | 2-Variate Fragment | Step 1

Step 1: Let (g_1, g_2) be binary McNaughton functions s.t. $g_2(\mathbf{1}) = 1$, and let (K_1, K_2) their supplements.

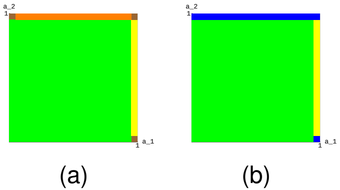


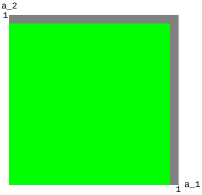
Figure: (a) g_1 candidate. (b) g_1 or g_2 candidate.

We describe f *blockwise*. We use the information encoded by g_1 and g_2, and their supplements K_1 and K_2, to *cover* the whole of $[0, 3]^2$.

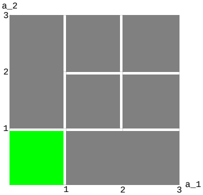
Basic Logic | 2-Variate Fragment | Step 1

Block 1: $g_1 \upharpoonright [0,1)^2$ *covers* the region $[0,1)^2$, in the following sense.

$$\mathbf{a} \in [0,1)^2 \Rightarrow f(\mathbf{a}) = \begin{cases} g_1(\mathbf{a}) & \text{if } g_1(\mathbf{a}) < 1 \\ 3 & \text{if } g_1(\mathbf{a}) = 1 \end{cases}$$



(a) $g_1 \upharpoonright [0,1)^2$.



(b) $f \upharpoonright [0,1)^2$.

Basic Logic | 2-Variate Fragment | Step 1

Block 2: $g_1 \upharpoonright I_{g_1,2} = \{(a_1, 1) \mid 0 \leq a_1 < 1\}$ *covers* $[0,1) \times [1,3]$,

$$\mathbf{a} \in [0,1) \times [1,3] \Rightarrow f(\mathbf{a}) = \begin{cases} p_R((a_1)) & \text{if } g_1((a_1,1)) = p_R((a_1)) < 1 \\ t_R^{[0,3]}(\mathbf{a}) & \text{if } (a_1,1) \subseteq R \in R_{g_1,2} \end{cases}$$

and $g_1 \upharpoonright I_{g_1,1} = \{(1, a_2) \mid 0 \leq a_2 < 1\}$ *covers* $[1,3] \times [0,1)$,

$$\mathbf{a} \in [1,3] \times [0,1) \Rightarrow f(\mathbf{a}) = \begin{cases} p_R((a_2)) & \text{if } g_1((1,a_2)) = p_R((a_2)) < 1 \\ t_R^{[0,3]}(\mathbf{a}) & \text{if } (1,a_2) \subseteq R \in R_{g_1,1} \end{cases}$$

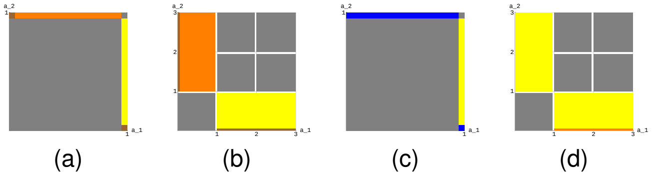


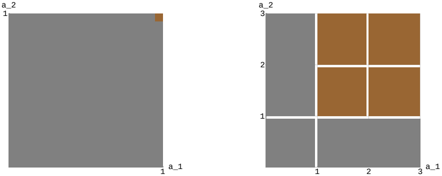
Figure: (b)-(d) $f \upharpoonright ([0,1) \times [1,3] \cup [1,3] \times [0,1))$, for $g_1 \upharpoonright I_{g_1,1} \cup I_{g_1,2}$ as in (a)-(c).

Basic Logic | 2-Variate Fragment | Step 1

Block 3: $g_1 \upharpoonright \{\mathbf{1}\}$ *covers* the region $[1,3]^2$, in the following sense.

Case 1: If $g_1(\mathbf{1}) = 0$, then

$$\mathbf{a} \in [1,3]^2 \Rightarrow f(\mathbf{a}) = 0. \quad (1)$$



(a) $g_1 \upharpoonright \{\mathbf{1}\}$. (b) $f \upharpoonright [1,3]^2$.

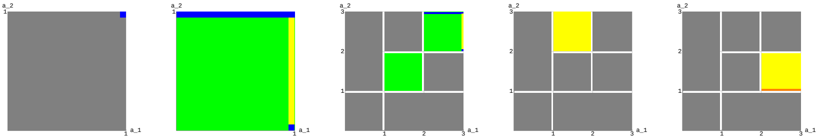
Basic Logic | 2-Variate Fragment | Step 1

Case 2: If $g_1(\mathbf{1}) = 1$, then g_1 *delegates* to g_2 the coverage of $[1,3]^2$:

$$\mathbf{a} \in [1,2)^2 \cup [2,3]^2 \Rightarrow f(\mathbf{a}) = \begin{cases} g_2(\mathbf{a}-\mathbf{1})+1 & \text{if } \mathbf{a} \in [1,2)^2 \text{ and } g_2(\mathbf{a}-\mathbf{1}) < 1 \\ g_2(\mathbf{a}-\mathbf{2})+2 & \text{if } \mathbf{a} \in [2,3]^2 \text{ and } g_2(\mathbf{a}-\mathbf{2}) < 1 \\ 3 & \text{otherwise} \end{cases} \tag{2}$$

$$\mathbf{a} \in [1,2) \times [2,3] \Rightarrow f(\mathbf{a}) = \begin{cases} p_R((a_1-1))+1 & \text{if } g_2((a_1-1,1)) = p_R((a_1-1)) < 1 \\ t_R^{[0,3]}(\mathbf{a}) & \text{if } (a_1-1,1) \subseteq R \in R_{g_2,2} \end{cases} \tag{3}$$

$$\mathbf{a} \in [2,3] \times [1,2) \Rightarrow f(\mathbf{a}) = \begin{cases} p_R((a_2-1))+1 & \text{if } g_2((1,a_2-1)) = p_R((a_2-1)) < 1 \\ t_R^{[0,3]}(\mathbf{a}) & \text{if } (1,a_2-1) \subseteq R \in R_{g_2,1} \end{cases} \tag{4}$$

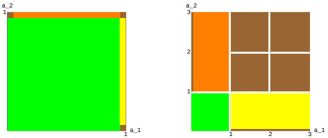


(a) $g_1 \restriction \{\mathbf{1}\}$. (b) $g_2 \restriction [0,1]^2$. (c) Eq. (2). (d) Eq. (3). (e) Eq. (4).

Basic Logic | 2-Variate Fragment | Step 1

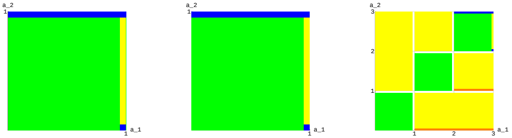
Summarizing the previous blockwise description:

Case 1: If $g_1(\mathbf{1}) = 0$, then f is colored as follows:



(a) g_1. (b) f.

Case 2: If $g_1(\mathbf{1}) = 1$, then f is colored as follows:



(a) g_1. (b) g_2. (c) f.

Basic Logic | 2-Variate Fragment | Step 1 (Finished)

Step 1: Guess:

$$F_{[0,3],2} = \{f \mid f \text{ given by some } (g_1, g_2), (K_1, K_2)\} \subseteq [0,3]^{[0,3]^2},$$

and let $\langle f \rangle = (\langle g_1 \rangle, \langle g_2 \rangle, \langle K_1 \rangle, \langle K_2 \rangle)$ be the encoding of $f \in F_{[0,3],2}$.

Basic Logic | 2-Variate Fragment | Step 2 (Finished)

Step 2: Nontrivial, since $F_{[0,3],2} \subset [0,3]^{[0,3]^2}$.

By induction on $t \in T_2$, it is possible to check that $t^{[0,3]} \in F_{[0,3],2}$ by providing pairs (g_1, g_2) and (K_1, K_2) that describe $t^{[0,3]}$ in terms of Step 1.

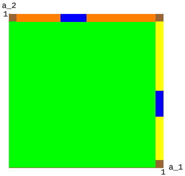
Basic Logic | 2-Variate Fragment | Step 3

Fact

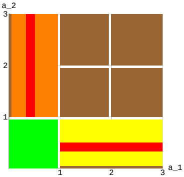
Let $f \in F_{[0,3],2}$ be given by pairs (g_1, g_2) and (K_1, K_2) with $g_1(\mathbf{1}) = 0$. Let $t_1 \in T_2$ be s.t. $t_1^{[0,1]} = g_1$. Then,

$$t = t_1 \tag{5}$$

satisfies $t^{[0,3]} = f$, excluding points covered by R's in R_{g_1} s.t. $g_1 \restriction R = 1$.



(a) $g_1 = t_1^{[0,1]}$.



(b) $f \neq t^{[0,3]}$.

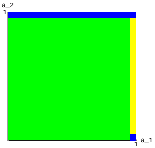
Basic Logic | 2-Variate Fragment | Step 3

Fact (Cont'd)

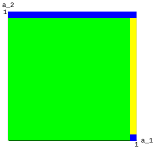
Let $f \in F_{[0,3],2}$ *be given by pairs* (g_1, g_2) *and* (K_1, K_2) *with* $g_1(\mathbf{1}) = 1$. *Let* $t_1, t_2 \in T_2^+$ *be s.t.* $t_1^{[0,1]} = g_1$ *and* $t_2^{[0,1]} = g_2$. *Then,*

$$t = ((\neg\neg t_1) \wedge (\neg\neg t_2 \rightarrow t_2)) \tag{6}$$

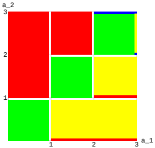
satisfies $t^{[0,3]} = f$, *excluding points covered by R's in* R_{g_i} *s.t.* $g_i \restriction R = 1$, $i \in \{1, 2\}$.



(a) $g_1 = t_1^{[0,1]}$.



(b) $g_2 = t_2^{[0,1]}$.



(c) $f \neq t^{[0,3]}$.

Basic Logic | 2-Variate Fragment | Step 3

Theorem (Masking)

Let $f \in F_{[0,3],2}$ be given by pairs (g_1, g_2) and (K_1, K_2), and let t be as in (5)-(6). Then, there exist $r, s \in T_2$ s.t. for every $\mathbf{a} \in [0,3]^2$:

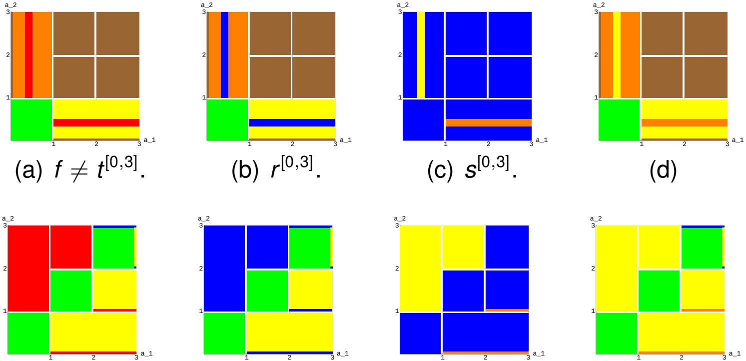
$$r^{[0,3]}(\mathbf{a}) = \begin{cases} t^{[0,3]}(\mathbf{a}) & \text{if } t^{[0,3]}(\mathbf{a}) = f(\mathbf{a}) \\ \top^{[0,3]} & \text{otherwise} \end{cases} \tag{7}$$

$$s^{[0,3]}(\mathbf{a}) = \begin{cases} f(\mathbf{a}) & \text{if } t^{[0,3]}(\mathbf{a}) \neq f(\mathbf{a}) \\ \top^{[0,3]} & \text{otherwise} \end{cases} \tag{8}$$

Then, $r \wedge s$ is s.t.

$$(r \wedge s)^{[0,3]} = f.$$

Basic Logic | 2-Variate Fragment | Step 3



(a) $f \neq t^{[0,3]}$. (b) $r^{[0,3]}$. (c) $s^{[0,3]}$. (d)

(e) $f \neq t^{[0,3]}$. (f) $r^{[0,3]}$. (g) $s^{[0,3]}$. (h)

Figure: The masking theorem. (d) and (h) show $(r \wedge s)^{[0,3]} = f$.

Basic Logic | 2-Variate Fragment | Step 3

The key skill to implement the masking theorem is the following.

Lemma (Gadget)

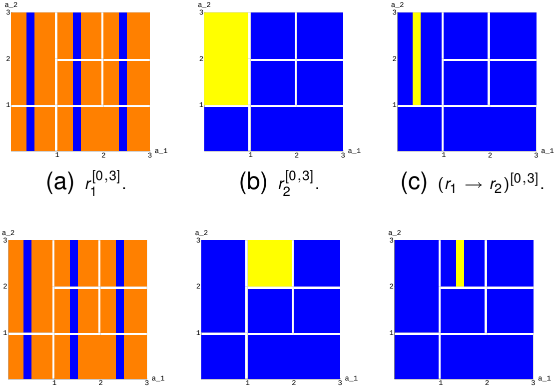
Let $f \in F_{[0,3],2}$ be given by pairs (g_1, g_2) and (K_1, K_2).
Let $R \in R_{g_1}$ s.t. $g_1 \upharpoonright R = 1$ and suppose w.l.o.g. that $R \subseteq I_2$. Then, there exists $r \in T_2$ s.t. for every $\mathbf{a} \in [0,3]^2$:

$$r^{[0,3]}(\mathbf{a}) = \begin{cases} x_2^{[0,3]}(\mathbf{a}) & \textit{if } (a_1, 1) \subseteq R \\ \top^{[0,3]} & \textit{otherwise} \end{cases} \tag{9}$$

Let $R \in R_{g_2}$ s.t. $g_2 \upharpoonright R = 1$ and suppose w.l.o.g. that $R \subseteq I_2$. Then, there exists $s \in T_2$ s.t. for every $\mathbf{a} \in [0,3]^2$:

$$s^{[0,3]}(\mathbf{a}) = \begin{cases} x_2^{[0,3]}(\mathbf{a}) & \textit{if } (a_1 - 1, 1) \subseteq R \\ \top^{[0,3]} & \textit{otherwise} \end{cases} \tag{10}$$

Basic Logic | 2-Variate Fragment | Step 3 (Finished)



(a) $r_1^{[0,3]}$. (b) $r_2^{[0,3]}$. (c) $(r_1 \to r_2)^{[0,3]}$.

(d) $s_1^{[0,3]}$. (e) $s_2^{[0,3]}$. (f) $(s_1 \to s_2)^{[0,3]}$.

Figure: The gadget lemma.

Basic Logic | 2-Variate Fragment | Summary

Goal: Characterize the set of binary basic functions:

$$F_{[0,3],2} = \{f : [0,3]^2 \rightarrow [0,3] \mid f = t^{[0,3]} \text{ for some } t \in T_2\}.$$

Step 1: Guess

$$F_{[0,3],2} = \{f : [0,3]^2 \rightarrow [0,3] \mid f \text{ specified by } (g_1, g_2), (K_1, K_2)\}.$$

Step 2: Every term $t \in T_2$ computes a function $t^{[0,3]} \in F_{[0,3],2}$.

Step 3: Every function $f \in F_{[0,3],2}$ is computed by a term $t \in T_2$.

Outline

1. Basic Logic
2. Functional Representation
3. Conclusion
   - Universal Algebra
   - References

Universal Algebra | *BL*-Algebras

Definition (*BL*-Algebras)

A *(commutative bounded) GBL-algebra* is an algebra $(A, \vee, \wedge, \odot, \rightarrow, \top, \bot)$ of type $(2, 2, 2, 2, 0, 0)$ s.t.:

(*i*) $(A, \odot, \top)$ is a commutative monoid;

(*ii*) $(A, \vee, \wedge, \top, \bot)$ is a bounded lattice;

(*iii*) *residuation* holds, i.e. $x \odot y \leq z$ iff $y \leq x \rightarrow z$;

(*iv*) *divisibility* holds, i.e. $x \wedge y = x \odot (x \rightarrow y)$.

A *BL-algebra* is a *prelinear GBL*-algebra i.e., $(x \rightarrow y) \vee (y \rightarrow x) = \top$ holds. An *MV-algebra* is an *involutive BL*-algebra i.e., $\neg\neg x = x$ holds ($\neg x = x \rightarrow \bot$).

Universal Algebra | Free *BL*-Algebra

Theorem (Generic *BL*-Algebra)

$[0, n+1]$ *generates the variety of n-generated BL-algebras.*

Corollary (Free *BL*-Algebra)

The free n-generated BLalgebra is isomorphic to the algebra having domain $F_{[0,n+1],n}$ *and pointwise defined operations* $\odot^{[n+1]}$ *and* $\rightarrow^{[n+1]}$.

References

P. Aglianò and F. Montagna.
Varieties of BL-Algebras I: General Properties.
Journal of Pure and Applied Algebra, 181:105–129, 2003.

S. Aguzzoli and B. Gerla.
Normal Forms for the One-Variable Fragment of Hájek's Basic Logic.
Proceedings of ISMVL'05, 284–289, 2005.

S. Aguzzoli and S. Bova.
The Free *n*-Generated *BL*-Algebra.
Submitted.

F. Montagna
The Free BL-Algebra on One Generator.
Neural Network World, 5:837–844, 2000.

D. Mundici.
A Constructive Proof of McNaughton's Theorem in Infinite-Valued Logics.
The Journal of Symbolic Logic, 59:596–602, 1994.