Width Minimization for Existential Positive Queries

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Outline

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Expressibility

Notation:

- $\mathrm{FO} = \{\phi \mid \phi$ relational first-order sentence$\}$, $\mathrm{L} \subseteq \mathrm{FO}$, $k \in \mathbb{N}$;
- ϕ *uses at most* k *variables* if $|\{x \mid x$ variable occurring in $\phi\}| \leq k$;
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the problem of *minimizing variable usage in first-order logic*:

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L^k-EXPRESS is restriction of L-EXPRESS to instances in $\mathrm{L} \times \{k\}$.

Expressibility | Example

$$\gamma = \exists x_1 \ldots \exists x_9 \Big(\bigwedge_{i=2,4,6,8} E_{5i} x_5 x_i \wedge \bigwedge_{i=1,3} E_{2i} x_2 x_i \wedge \bigwedge_{i=1,7} E_{4i} x_4 x_i \wedge \bigwedge_{i=3,9} E_{6i} x_6 x_i \wedge \bigwedge_{i=7,9} E_{8i} x_8 x_i \Big)$$

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$\gamma \in \mathrm{FO}^4$-EXPRESS because $\gamma \equiv \gamma'$ and $\gamma' \in \mathrm{FO}^4$.

Model Checking

Variable usage is important in the *algorithmic* and *complexity* study of the model checking problem:

Problem	MODELCHECKING(L)
Instance	A finite structure **A** and $\phi \in$ L.
Question	$\mathbf{A} \models \phi$?

A pertinent example of model checking is *(Boolean) query evaluation*, evaluating a *(Boolean) query* ϕ over a relational *database* **A**.

Model Checking | Algorithmics

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Minimizing the variables used in ϕ, also minimizes the exponent in the runtime of the natural query evaluation algorithm.

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This approach yields a relaxation of polynomial-time tractability, called *fixed-parameter tractability*, capable of exploiting this asymmetry of the database setting.

With respect to basic and fundamental classes of queries in database theory, such as conjunctive queries and existential positive queries, "expressibility characterizes fixed-parameter tractability".

Model Checking | Complexity

EP $= \mathrm{FO}(\exists, \vee, \wedge)$ is the class of *existential positive* sentences (semantically equivalent to *union of conjunctive queries*).

---

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Model Checking | Complexity

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Theorem (Chen)

Let L $\subseteq$ EP *be a class of sentences. The following are equivalent:* *

- MODELCHECKING(L) *is fixed-parameter tractable.*
- *There exists* $k \geq 1$ *st* L $\subseteq$ EP^k-EXPRESS.

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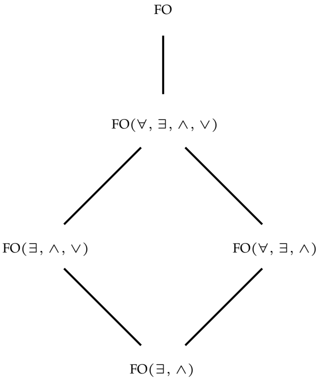
Outline

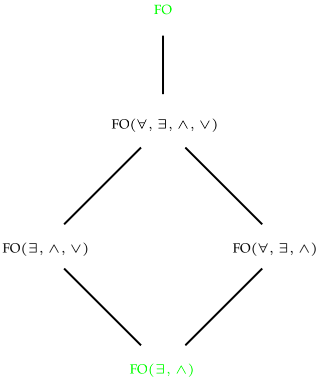
Expressibility Classification | Syntactic Fragments



S	FO(S)-EXPRESS
$\forall, \exists, \wedge, \vee, \neg$	
$\forall, \exists, \wedge, \vee$	
$\forall, \exists, \wedge$	
$\exists, \wedge, \vee$	
$\exists, \wedge$	

FO(S) denotes FO-sentences with logical vocabulary $S \subseteq \{\forall, \exists, \vee, \wedge, \neg\}$.

Expressibility Classification | Previous Work



S	FO(S)-EXPRESS
∀, ∃, ∧, ∨, ¬	undecidable, $k \geq 2$ [Folklore]
∀, ∃, ∧, ∨	
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∃, ∧, ∨	
∃, ∧	NP-complete, $k \geq 2$ [Dalmau et al.]

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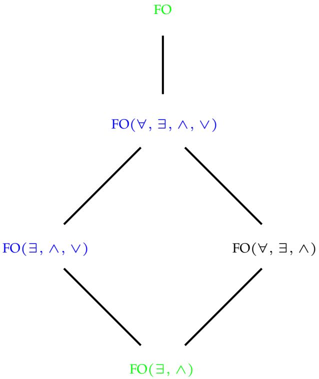
Expressibility Classification | Our Work



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∀, ∃, ∧, ∨, ¬	undecidable, $k \geq 2$ [Folklore]
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Primitive Positive Logic

PP $= \mathrm{FO}(\exists, \wedge)$ is primitive positive logic (*conjunctive queries*).

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A combinatorial characterization of k-variable expressibility for PP-logic.

Theorem (Dalmau, Kolaitis, and Vardi)

Let $\phi \in \mathrm{PP}_\sigma$. The following are equivalent:

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- $\phi \in \mathrm{PP}^k$-EXPRESS
- $\mathrm{tw}(\mathrm{core}(\mathbf{C}[\phi])) < k$, *where:*
  - $\mathbf{C}[\phi]$ *is the* canonical structure *of* ϕ*;*
  - $\mathrm{core}(\mathbf{A})$ *is the* core *of the structure* $\mathbf{A}$*;*
  - $\mathrm{tw}(\mathbf{A})$ *is the* treewidth *of the structure* $\mathbf{A}$*.*

Canonical Structure

Conjunctive queries naturally correspond to relational structures.

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Example (Canonical Structure of a Query)

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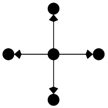

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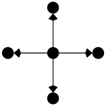
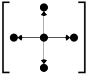
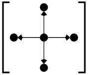


Example (Canonical Query of a Structure)

F [] $= E x_3 x_1 \wedge E x_3 x_2 \wedge E x_3 x_4 \wedge E x_3 x_5$

Q [] $= \exists x_1 \exists x_2 \exists x_3 \exists x_4 \exists x_5 (E x_3 x_1 \wedge E x_3 x_2 \wedge E x_3 x_4 \wedge E x_3 x_5)$

Cores

Let **A** and **B** be σ-structures. A *homomorphism* from **A** to **B** is a mapping $h\colon A \to B$ such that for all $R \in \sigma$ and all $(a_1, \ldots, a_{\mathrm{ar}(R)}) \in A^{\mathrm{ar}(R)}$, if $(a_1, \ldots, a_{\mathrm{ar}(R)}) \in R^{\mathbf{A}}$, then $(h(a_1), \ldots, h(a_{\mathrm{ar}(R)})) \in R^{\mathbf{B}}$.

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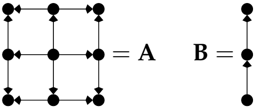

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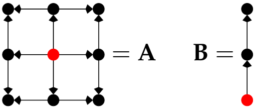

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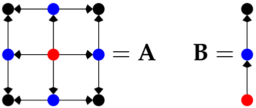

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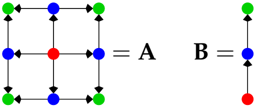

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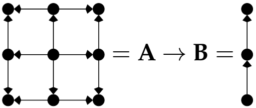

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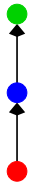
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$\mathbf{B}$ is a *core of* $\mathbf{A}$ if (i) $\mathbf{B}$ is a core, (ii) $\mathbf{B}$ is a substructure of $\mathbf{A}$, (iii) $\mathbf{A} \leftrightarrow \mathbf{B}$.

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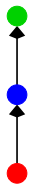

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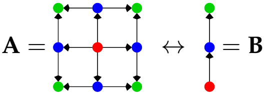

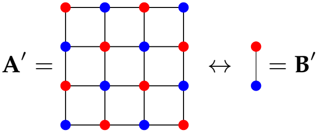

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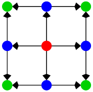
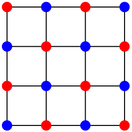
Every finite structure $\mathbf{A}$ has a unique core up to isomorphism, $\mathrm{core}(\mathbf{A})$.

Example

$\mathbf{A} =$  $\leftrightarrow$  $= \mathbf{B} = \mathrm{core}(\mathbf{A})$

$\mathbf{A}' =$  $\leftrightarrow$  $= \mathbf{B}' = \mathrm{core}(\mathbf{A}')$

Treewidth

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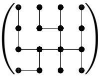
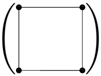
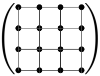
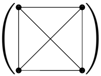
Low treewidth indicates high similarity with trees.

Example (Treewidth)

tw  = 1 tw  = 2 tw  = 4 tw  = 3

tw(a tree) = 1 tw(a cycle) = 2 tw(a $k \times k$-grid) = k tw(a k-clique) = $k - 1$

Example

$$\exists x_1 \dots \exists x_9 \left(\bigwedge_{i=2,4,6,8} E x_5 x_i \wedge \bigwedge_{i=1,3} E x_2 x_i \wedge \bigwedge_{i=1,7} E x_4 x_i \wedge \bigwedge_{i=3,9} E x_6 x_i \wedge \bigwedge_{i=7,9} E x_8 x_i \right) \in \mathrm{PP}^2\text{-}\textsc{Express}$$

Example

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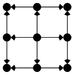
tw (core ()) < 2

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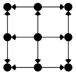
$\mathrm{tw}\Big(\mathrm{core}\Big($  $\Big)\Big) < 2$

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$1 = \mathrm{tw}\Big($  $\Big) < 2$

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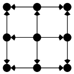
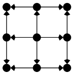
tw(core()) $< 2 \leq$ tw() $= 3$

$$\Updownarrow$$

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Implicants

Let $\phi \in$ EP.

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where ϕ'_i's are PP-formulas, $\{\tau_j \mid j \in J\} \subseteq \text{PP}$, $\tau_j \not\models \tau_{j'}$ $(j, j' \in J, j \neq j')$.

Characterization

A combinatorial characterization of expressibility in EP.

Theorem (B, Chen)

Let $\phi \in \mathrm{EP}_\sigma$*. Then,* $\phi \in \mathrm{EP}^k_\sigma$-EXPRESS *if and only if* $\mathrm{tw}(\mathrm{core}(\mathbf{C}[\tau])) < k$*, for all implicants* τ *in an irredundant disjunctive form of* ϕ*.*

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Proof (Sketch).

Combine the combinatorial characterization of k-expressibility in PP-logic and the following combinatorial characterization of equivalence in EP-logic: If $\phi' = \bigvee_{i \in [m]} \phi'_i$ and $\phi'' = \bigvee_{j \in [n]} \phi''_j$ are irredundant disjunctive forms of $\phi \in \mathrm{EP}$, then there exists a bijection $\pi \colon [m] \to [n]$ such that, for all $i \in [m]$, $\mathbf{C}[\phi'_i] \leftrightarrow \mathbf{C}[\phi''_{\pi(i)}]$. □

Classification

$\Pi_2^p = \{S \mid S \leq_{\mathrm{m}}^{\mathrm{poly}} \Pi_2\text{-3CNF-SAT}\}.$

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Theorem (B, Chen)

- EP-Express *is in* Π_2^p.
- EP_σ^k-Express *is* Π_2^p*-hard:*
  - *if* $k \geq 3$ *and* $\sigma \supseteq \{U_i \mid i \in \mathbb{N}\} \cup \{E\}$;
  - *if* $k \geq 6$ *and* $\sigma \supseteq \{E\}$.

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Proof (Sketch).

The upper bound follows from the characterization of expressibility in EP ("*for every* implicant *there exists* an entailed implicant of small treewidth"). A reduction from a Π_2^p-complete quantified version of the graph k-colorability problem gives the lower bound for all $k \geq 3$ (extra work required if $\sigma = \{E\}$). □

Reduction

$\mathbf{K}_k = ([k], E^{\mathbf{K}_k})$, where $E^{\mathbf{K}_k} = [k]^2 \setminus \{(i,i) \mid i \in [k]\}$.

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Reduction from the following Π_2^p-hard problem ($k \geq 3$):

Problem Π_2-k-COLORABILITY

Instance $\psi = \forall y_1 \ldots \forall y_m \exists x_1 \ldots \exists x_n F[\mathbf{G}]$,
where $\mathbf{G} = (\{y_1, \ldots, y_m, x_1, \ldots, x_n\}, E^{\mathbf{G}})$ is a (simple) graph.

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The following are equivalent:

- $\mathbf{K}_k \models \psi$
- Each $f\colon \{y_1, \ldots, y_m\} \to [k]$ extends to a homomorphism $\mathbf{G} \to \mathbf{K}_k$ (ie, a k-coloring of $\mathbf{G}$).

Reduction

$\psi = \forall y_1 \ldots \forall y_m \exists x_1 \ldots \exists x_n F[\mathbf{G}]$ instance of Π_2-k-COLORABILITY.

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Reduction maps ψ to

$$\chi = \exists 1 \ldots \exists k \exists y_1 \ldots \exists y_m \exists x_1 \ldots \exists x_n \mathrm{matrix}(\chi) \in \mathrm{EP}_{\{E, U_1, \ldots, U_k, U_{y_1}, \ldots, U_{y_m}\}}$$

such that

$$\mathbf{K}_k \models \psi \iff \chi \in \mathrm{EP}^k\text{-EXPRESS}$$

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where

- $\mathrm{matrix}(\chi) = F[\mathbf{G} \cup \mathbf{K}_k^k] \wedge \bigwedge_{i \in [m]} \bigvee_{j \in [k]} F[\mathbf{L}_{y_i \mapsto j} \cup \mathbf{M}_{y_i \mapsto j}]$;
- $\mathbf{K}_k^k = \mathbf{C}[F[\mathbf{K}_k] \wedge U_1 1 \wedge \cdots \wedge U_k k]$;
- $\mathbf{L}_{y_i \mapsto j} = \mathbf{C}[U_{y_i} j]$;
- $\mathbf{M}_{y_i \mapsto j} = \mathbf{C}[\bigwedge_{c \in [k], c \neq j} (E y_i c \wedge E c y_i)]$.

Distributivity

Using distributivity, χ encodes k^m maps $f : \{y_1, \ldots, y_m\} \to [k]$ in $O(mk)$ space:

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Example ($k = 3$, $m = 2$)

$$\bigwedge_{i \in [2]} \bigvee_{j \in [3]} y_i \mapsto j = (y_1 \mapsto 1 \vee y_1 \mapsto 2 \vee y_1 \mapsto 3) \wedge (y_2 \mapsto 1 \vee y_2 \mapsto 2 \vee y_2 \mapsto 3)$$

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Irredundant Form

$$\begin{aligned} \mathrm{matrix}(\chi) &= F[\mathbf{G} \cup \mathbf{K}_k^k] \wedge \bigwedge_{i \in [m]} \bigvee_{j \in [k]} F[\mathbf{L}_{y_i \mapsto j} \cup \mathbf{M}_{y_i \mapsto j}] \\ &\equiv \bigvee_{f:\ \{y_1, \ldots, y_m\} \to [k]} F[\mathbf{G} \cup \mathbf{K}_k^k \cup \bigcup_{i \in [m]} (\mathbf{L}_{y_i \mapsto f(y_i)} \cup \mathbf{M}_{y_i \mapsto f(y_i)})] \end{aligned}$$

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By the characterization, suffices to show the following:

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Item 1: The disjunctive form $\bigvee_{f\colon \{y_1,\ldots,y_m\} \to [k]} Q[\mathbf{H}_f]$ of χ is irredundant.

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By the characterization, suffices to show the following:

Item 1: The disjunctive form $\bigvee_{f\colon \{y_1,\ldots,y_m\} \to [k]} Q[\mathbf{H}_f]$ of χ is irredundant.

Item 2: The following are equivalent:

- Each $f\colon \{y_1, \ldots, y_m\} \to [k]$ extends to a hom $\mathbf{G} \to \mathbf{K}_k$.
- $\mathrm{tw}(\mathrm{core}(\mathbf{H}_f)) < k$ for all $f\colon \{y_1, \ldots, y_m\} \to [k]$.

Outline

Undecidability of Positive Logic

PFO $= \mathrm{FO}(\forall, \exists, \wedge, \vee)$ is *positive* logic.

Theorem (B, Chen)
PFO^k_σ-EXPRESS *is undecidable* ($k \geq 3$, $\sigma \supseteq \{U_i \mid i \in \mathbb{N}\} \cup \{E_1, E_2, E_3\}$).

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Proof (Sketch).
Reduction from the decision problem for Kahr sentences (undecidable). □

Sketch of the Proof

Problem Kahr-SAT

Instance $\phi \in \mathrm{FO}_{\{E_1, U_i | i \in \mathbb{N}\}}$ in prefix form with prefix $\forall x \exists y \forall z$.

Question Is there a structure $\mathbf{A}$ such that $\mathbf{A} \models \phi$?

Sketch of the Proof

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$$\leq_m^{\log}$$

Problem $\mathrm{PFO}^3_{\{E_1, E_2, U_i \mid i \in \mathbb{N}\}}$-ENTAILMENT

Instance $(\phi, \psi) \in \mathrm{PFO}^3_{\{E_1, E_2, U_i \mid i \in \mathbb{N}\}}$.

Question $\phi \models \psi$?

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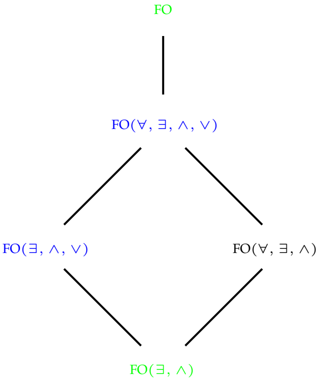
Problem $\mathrm{PFO}^3_{\{E_1, E_2, E_3, U_i \mid i \in \mathbb{N}\}}$-EXPRESS

Instance $\phi \in \mathrm{PFO}^3_{\{E_1, E_2, E_3, U_i \mid i \in \mathbb{N}\}}$.

Question Is ϕ logically equivalent to some $\psi \in \mathrm{PFO}^3_{\{E_1, E_2, E_3, U_i \mid i \in \mathbb{N}\}}$?

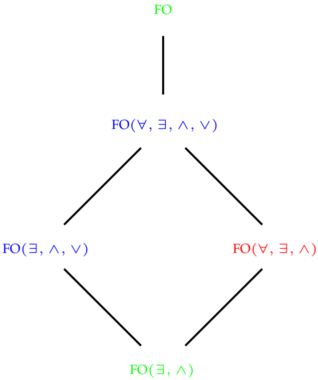
Outline

Results



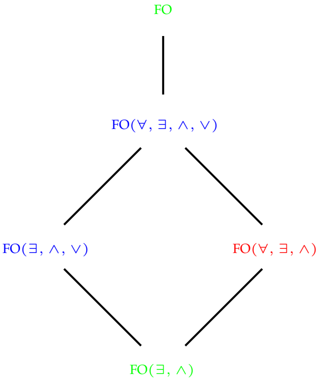
S	FO(S)-EXPRESS
∀, ∃, ∧, ∨, ¬	undecidable, $k \geq 2$ [Folklore]
∀, ∃, ∧, ∨	undecidable, $k \geq 3$ [B, Chen]
∀, ∃, ∧	
∃, ∧, ∨	Π_2^p-complete, $k \geq 3$ [B, Chen]
∃, ∧	NP-complete, $k \geq 2$ [Dalmau et al.]

Open



S	FO(S)-EXPRESS
∀, ∃, ∧, ∨, ¬	undecidable, $k \geq 2$ [Folklore]
∀, ∃, ∧, ∨	undecidable, $k \geq 3$ [B, Chen]
∀, ∃, ∧	open
∃, ∧, ∨	Π_2^p-complete, $k \geq 3$ [B, Chen]
∃, ∧	NP-complete, $k \geq 2$ [Dalmau et al.]

Open



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∃, ∧	NP-complete, $k \geq 2$ [Dalmau et al.]

Thank you for your attention!

Main Reduction | *Idea Item* 1

The disjunctive form $\bigvee_{f\colon \{y_1,\ldots,y_m\}\to[k]} Q[\mathbf{H}_f]$ of χ is irredundant.

Example ($k = 3$, $m = 2$)

$f(y_1) = f'(y_1) = f'(y_2) = 2, f(y_2) = 1.$

$\mathbf{H}_f$ $\mathbf{H}_{f'}$

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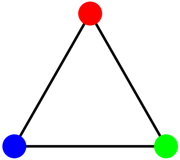
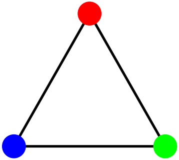
$f(y_1) = f'(y_1) = f'(y_2) = 2, f(y_2) = 1.$

$U_1^{\mathbf{H}_f} = U_1^{\mathbf{H}_{f'}} = \{{\color{red}\bullet}\}, U_2^{\mathbf{H}_f} = U_2^{\mathbf{H}_{f'}} = \{{\color{blue}\bullet}\}, U_3^{\mathbf{H}_f} = U_3^{\mathbf{H}_{f'}} = \{{\color{green}\bullet}\}.$



$\subseteq \mathbf{H}_f$



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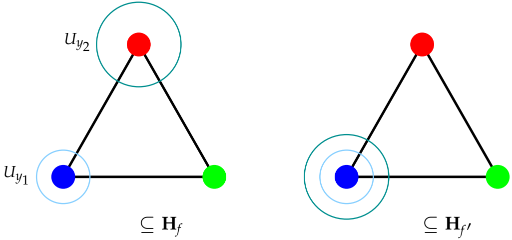

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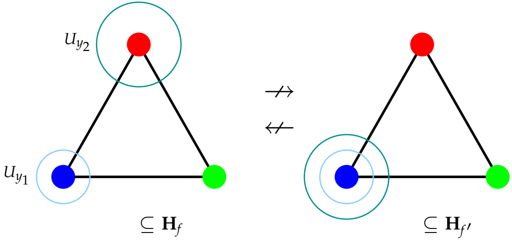
$\mathbf{H}_f \nrightarrow \mathbf{H}_{f'}$ and $\mathbf{H}_{f'} \nrightarrow \mathbf{H}_f$, ie, $Q[\mathbf{H}_f] \not\models Q[\mathbf{H}_{f'}]$ and $Q[\mathbf{H}_{f'}] \not\models Q[\mathbf{H}_f]$.



Main Reduction | Sketch Item 2

f extends to k-coloring of $\mathbf{G} \iff \mathrm{tw}(\mathrm{core}(\mathbf{H}_f)) < k$.

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Then $\mathrm{core}(\mathbf{H}_f) \to \mathbf{K}_k$, taking the $\{E\}$-reduct on the left (Picture 1).

Main Reduction | *Picture* 1

$\mathrm{tw}(\mathrm{core}(\mathbf{H}_f)) < k \Longrightarrow \mathrm{core}(\mathbf{H}_f) \rightarrow \mathbf{K}_k$, ie, $\mathrm{core}(\mathbf{H}_f)$ is k-colorable.

Main Reduction | *Picture* 1

$\mathrm{tw}(\mathrm{core}(\mathbf{H}_f)) < k \Longrightarrow \mathrm{core}(\mathbf{H}_f) \to \mathbf{K}_k$, ie, $\mathrm{core}(\mathbf{H}_f)$ is k-colorable.

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Is  3-colorable?

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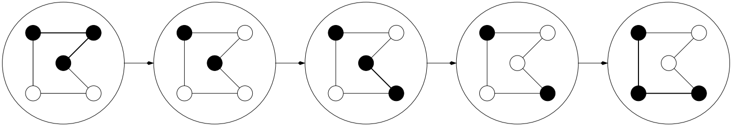
$\Downarrow$



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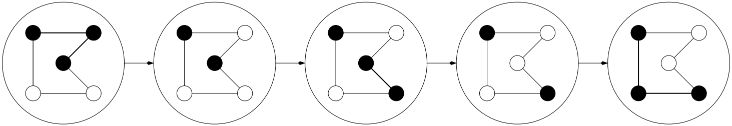

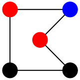

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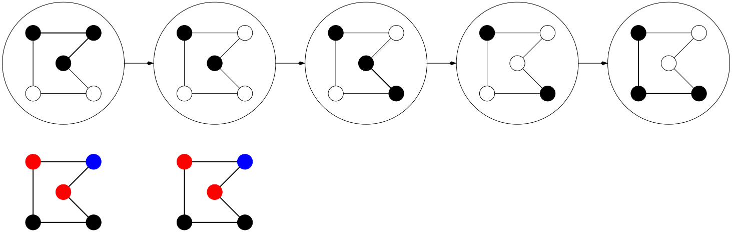

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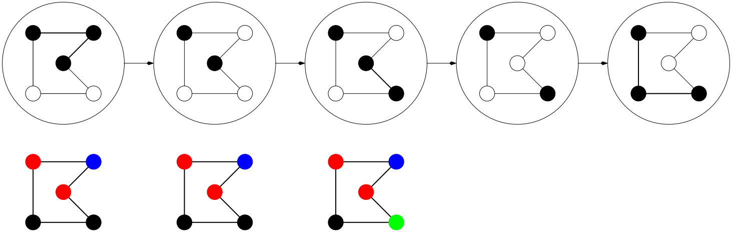

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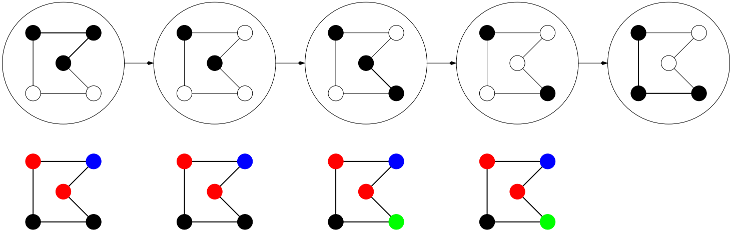

Is  3-colorable?

Main Reduction | *Picture* 1

$\mathrm{tw}(\mathrm{core}(\mathbf{H}_f)) < k \implies \mathrm{core}(\mathbf{H}_f) \to \mathbf{K}_k$, ie, $\mathrm{core}(\mathbf{H}_f)$ is k-colorable.

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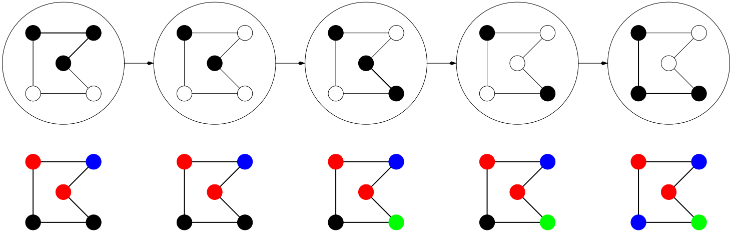
$\Downarrow$



$\Downarrow$

 is 3-colorable

Main Reduction | *Sketch Item* 2

f extends to k-coloring of $\mathbf{G} \iff \mathrm{tw}(\mathrm{core}(\mathbf{H}_f)) < k$.

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implies $\mathbf{H}_f \to \mathbf{K}_k^k$ via homomorphism acting as h on G and identically on $[k]$.
Thus, $\mathbf{H}_f \leftrightarrow \mathbf{K}_k^k$. Clearly $\mathbf{K}_k^k$ is a core.
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Thus $\mathrm{tw}(\mathrm{core}(\mathbf{H}_f)) = \mathrm{tw}(\mathbf{K}_k^k) = k - 1 < k$.

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Then $\mathrm{core}(\mathbf{H}_f) \rightarrow \mathbf{K}_k$, taking the $\{E\}$-reduct on the left (Picture 1).
We have $\mathbf{H}_f \rightarrow \mathrm{core}(\mathbf{H}_f)$, taking $\{E\}$-reducts.

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f extends to k-coloring of $\mathbf{G} \iff \mathrm{tw}(\mathrm{core}(\mathbf{H}_f)) < k$.

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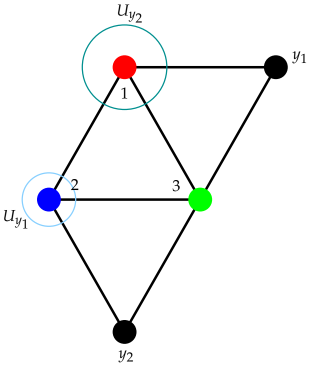
we have that h extends f (Picture 2).

Main Reduction | Picture 2

$\mathbf{H}_f \to \mathbf{K}_k$ via $h \Longrightarrow h$ extends f.

Example ($k = 3$, $m = 2$)

$f(y_1) = 2$ and $f(y_2) = 1$. Thus $U_{y_1}^{\mathbf{H}_f} = \{2\}$, $U_{y_2}^{\mathbf{H}_f} = \{1\}$,
$E^{\mathbf{H}_f} \supseteq \{(y_1, 1), (y_1, 3), (1, y_1), (3, y_1), (y_2, 2), (y_2, 3), (2, y_2), (3, y_2)\}$.



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$\mathbf{H}_f \to \mathbf{K}_k$ via $h \implies h$ extends f.

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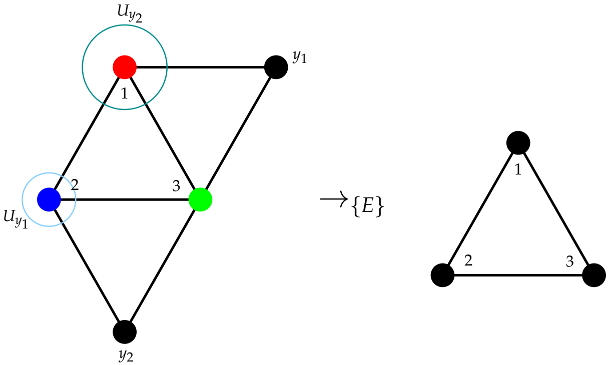
Assume $\mathbf{H}_f \to \mathbf{K}_k$ via h.



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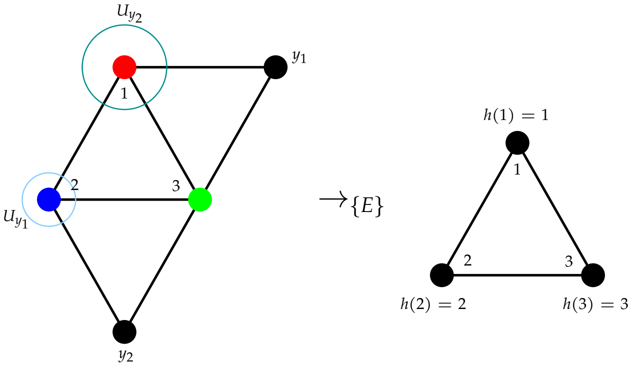

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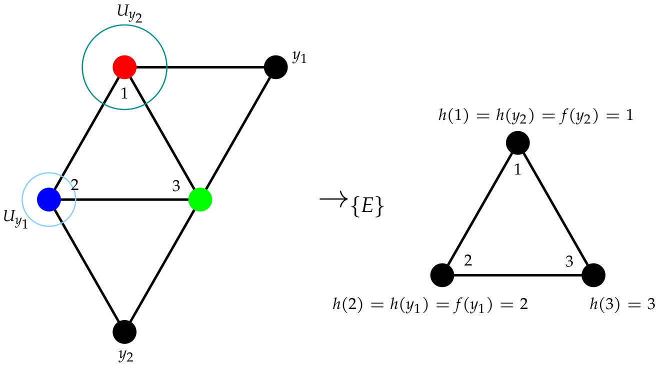
Then, $h(y_j) = f(y_j)$ for all $j \in [2]$, ie, h extends f.



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Hence $\mathbf{H}_f \to \mathbf{K}_k$, taking the $\{E\}$-reduct on the left, say via map h.
Since $\mathbf{K}_k, h \models \bigcup_{i \in [m]} (\mathbf{L}_{y_i \mapsto f(y_i)} \cup \mathbf{M}_{y_i \mapsto f(y_i)})$,
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we have that h extends f (Picture 2).
Moreover, $\mathbf{K}_k, h \models F[\mathbf{G}]$.