Width Minimization for Existential Positive Queries

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Technische Universität Wien

joint work with Hubie Chen

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Outline

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Research Motivation

Previous Work

Our Result

Other Results and Open Problems

Minimization (or Expressibility) Problem

Notation

- FO *denotes relational first-order sentences;*
- FO^k *denotes* FO-*sentences using at most k variable symbols;*
- $\mathrm{width}(\phi)$ *is the the maximum number of free variables over subformulas of* ϕ.

Minimizing number of variable symbols in FO-sentences (decision version):

Problem FO-Express

Instance $\phi \in \mathrm{FO}$ and $k \in \mathbb{N}$.

Question Is there $\psi \in \mathrm{FO}^k$ such that ϕ is logically equivalent to ψ?

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Theorem (Folklore)
FO-EXPRESS *is undecidable.*

Minimization | Model Checking

However, minimization and expressibility are important wrt *algorithmic* and *complexity* aspects of the ModelChecking problem:

Given a finite structure $\mathbf{A}$ *and a* FO-*sentence* ϕ, *decide whether* $\mathbf{A} \models \phi$.

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The MODELCHECKING problem is PSPACE-complete. But, analyzing the time complexity of the natural recursive evaluation of ϕ in $\mathbf{A}$ . . .

Proposition (Vardi)

Let $(\mathbf{A}, \phi)$ *be an instance of* MODELCHECKING.
The question, $\mathbf{A} \models \phi$?, *is decidable in time* $O(|A|^{\mathrm{width}(\phi)})$.

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Minimization in Model Checking: Minimizing the number of variables in ϕ also minimizes the exponent in the runtime of the natural model checking algorithm.

Expressibility | Complexity Criterion

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Expressibility as Complexity Criterion: With respect to basic and fundamental classes of queries in database theory, "expressibility characterizes tractability" in a precise sense.

Expressibility | Complexity Criterion

Let PP $\subseteq$ FO be the class of *primitive positive* ($\exists$, $\wedge$) sentences (aka *conjunctive queries* in database theory).

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Theorem (Dalmau, Kolaitis, and Vardi; Grohe)

Let L $\subseteq$ PP *be a class of sentences. The following are equivalent:* *

- MODELCHECKING *restricted to* L *is fixed-parameter tractable.*
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Theorem (B, Chen)

EP^k-EXPRESS *is* Π_2^p*-complete* ($k \geq 3$).

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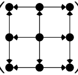
Outline

Primitive Positive Logic | Minimization

Example (Grid)

$$\gamma = \exists x_1 \ldots \exists x_9 \Big(\quad \Big)$$



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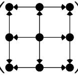

$$= \exists x_1 \ldots \exists x_9 \Big(\bigwedge_{i=2,4,6,8} E x_5 x_i \wedge \bigwedge_{i=1,3} E x_2 x_i \wedge \bigwedge_{i=1,7} E x_4 x_i \wedge \bigwedge_{i=3,9} E x_6 x_i \wedge \bigwedge_{i=7,9} E x_8 x_i \Big)$$

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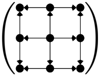
$$\gamma = \exists x_1 \ldots \exists x_9 \Big($$  $$\Big)$$

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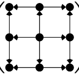

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$$\equiv \exists x_1 \exists x_2 \exists x_3 \exists x_4 (E x_4 x_1 \wedge E x_2 x_1 \wedge E x_2 x_3$$
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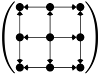
$\gamma = \exists x_1 \dots \exists x_9 \Big($  $\Big)$

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$$\begin{aligned}
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\end{aligned}$$

$\gamma \in \mathrm{PP}^4$-EXPRESS, but $\gamma \in \mathrm{PP}^2$-EXPRESS (claim), ie, the result is *not* optimal.

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Example (Grid, Cont'd)

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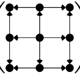
Semantic Minimization (Hard): Observe in "modest complexity" that

$$\gamma = \exists x_1 \ldots \exists x_9 \left(\right. $$  $$ \left. \right) \equiv \exists x_1 \exists x_2 \exists x_3 \left(\right. $$  $$ \left. \right). $$

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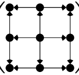
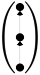
$\gamma = \exists x_1 \ldots \exists x_9$ () $\equiv \exists x_1 \exists x_2 \exists x_3$ ().

Syntactic Minimization (Easy): Minimize the variables in $\exists x_1 \exists x_2 \exists x_3$ ()

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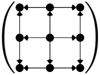
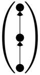
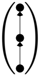
$\gamma = \exists x_1 \ldots \exists x_9$  $\equiv \exists x_1 \exists x_2 \exists x_3$ .

Syntactic Minimization (Easy): Minimize the variables in $\exists x_1 \exists x_2 \exists x_3$ 

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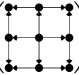
$$\gamma = \exists x_1 \ldots \exists x_9 \Bigg($$  $$\Bigg) \underset{\text{“NP away”}}{\equiv} \exists x_1 \exists x_2 \exists x_3 \Bigg($$  $$\Bigg).$$

Syntactic Minimization (Easy): Minimize the variables in $\exists x_1 \exists x_2 \exists x_3 \Bigg($  $\Bigg)$

by optimal polytime syntactic rewriting.

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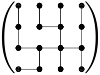
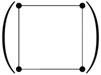
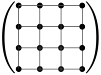
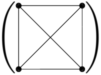
Low treewidth indicates high similarity with trees.

Example (Treewidth)

tw  $= 1$

tw  $= 2$

tw  $= 4$

tw  $= 3$

$\text{tw}(\text{a tree}) = 1$ $\text{tw}(\text{a cycle}) = 2$ $\text{tw}(\text{a } k\text{-grid}) = k$ $\text{tw}(\text{a } k\text{-clique}) = k - 1$

Primitive Positive Logic | Homomorphisms

Let **A** and **B** be σ-structures. A *homomorphism* from **A** to **B** is a mapping $h\colon A \to B$ such that for all $R \in \sigma$ and all $(a_1, \ldots, a_{\mathrm{ar}(R)}) \in A^{\mathrm{ar}(R)}$, if $(a_1, \ldots, a_{\mathrm{ar}(R)}) \in R^{\mathbf{A}}$, then $(h(a_1), \ldots, h(a_{\mathrm{ar}(R)})) \in R^{\mathbf{B}}$.

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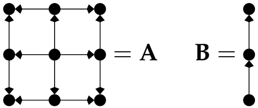

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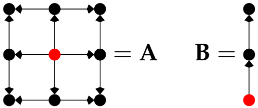

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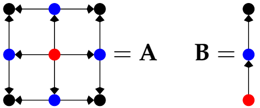

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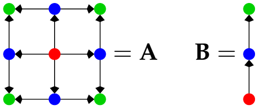

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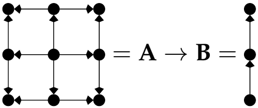

Primitive Positive Logic | Cores

Let **A** and **B** be σ-structures.

B is a *core* iff every homomorphism $\mathbf{B} \to \mathbf{B}$ is bijective
(ie, every endomorphism of **B** is an automorphism).

Example

$= \mathbf{B}$

$= \mathbf{B}'$

Primitive Positive Logic | Cores

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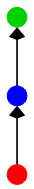
$\mathbf{B}$ is a *core of* $\mathbf{A}$ if (i) $\mathbf{B}$ is a core, (ii) $\mathbf{B}$ is a substructure of $\mathbf{A}$, (iii) $\mathbf{B} \leftrightarrow \mathbf{A}$.
Every finite structure $\mathbf{A}$ has a unique core up to isomorphism, $\text{core}(\mathbf{A})$.

Example

 $= \mathbf{B}$

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Primitive Positive Logic | Cores

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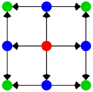
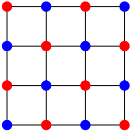
Example

A =  $\leftrightarrow$  = **B** = core(**A**)

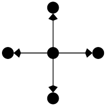
A$'$ =  $\leftrightarrow$  = **B**$'$ = core(**A**$'$)

Primitive Positive Logic | Notation

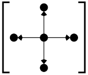
Natural correspondence between PP-sentences and relational structures.

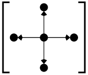
Example (Canonical Structure of a Query)

$$\mathbf{C}[\exists x_1 \exists x_2 \exists x_3 \exists x_4 \exists x_5 (Ex_3x_1 \wedge Ex_3x_2 \wedge Ex_3x_4 \wedge Ex_3x_5)] =$$



Example (Canonical Query of a Structure)

F [] $= Ex_3x_1 \wedge Ex_3x_2 \wedge Ex_3x_4 \wedge Ex_3x_5$

Q [] $= \exists x_1 \exists x_2 \exists x_3 \exists x_4 \exists x_5 (Ex_3x_1 \wedge Ex_3x_2 \wedge Ex_3x_4 \wedge Ex_3x_5)$

Primitive Positive Logic | Characterization

A combinatorial characterization of k-variable expressibility for PP-logic.

Theorem (Dalmau et al.)
Let $\phi \in \mathrm{PP}_\sigma$. *Then,* $\phi \in \mathrm{PP}^k$-Express *if and only if* $\mathrm{tw}(\mathrm{core}(\mathbf{C}[\phi])) < k$.

Primitive Positive Logic | Grid Revisited

Example (Grid Revisited)

$$\exists x_1 \ldots \exists x_9 (\bigwedge_{i=2,4,6,8} Ex_5 x_i \wedge \bigwedge_{i=1,3} Ex_2 x_i \wedge \bigwedge_{i=1,7} Ex_4 x_i \wedge \bigwedge_{i=3,9} Ex_6 x_i \wedge \bigwedge_{i=7,9} Ex_8 x_i) \in \text{PP}^2\text{-Express}$$

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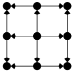
tw (core ()) < 2

Primitive Positive Logic | Grid Revisited

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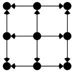
tw (core ()) < 2

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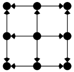
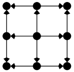
tw (core ()) < 2 ≤ tw () = 3

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Primitive Positive Logic | Classification

A complexity classification of k-variable expressibility for PP-logic.

Theorem (Dalmau et al.)
PP^k_σ-EXPRESS *is* NP-*complete for all* $k \geq 2$ *and all* $\sigma \supseteq \{E\}$.

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Proof (Sketch).
The upper bound follows from the characterization. A reduction from the graph k-colorability problem gives the lower bound for $k \geq 3$ (extra work required for $k = 2$). □

Outline

Existential Positive Logic | Implicants

Let $\phi \in$ EP (recall EP is the class of existential positive sentences).

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An irredundant disjunctive form of ϕ is obtained as follows:

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where the ϕ'_i are PP-formulas,
$\{\tau_j \mid j \in J\} \subseteq$ PP and $\tau_j \not\models \tau_{j'}$ for all $j, j' \in J, j \neq j'$.

Existential Positive Logic | Characterization

A combinatorial characterization of EP^k-Express.

Theorem (B, Chen)

Let $\phi \in \text{EP}_\sigma$. *Then,* $\phi \in \text{EP}^k_\sigma$-Express *if and only if* $\text{tw}(\text{core}(\mathbf{C}[\tau])) < k$, *for all implicants* τ *in an irredundant disjunctive form of* ϕ.

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Proof (Sketch).

Combine the combinatorial characterization of k-expressibility in PP-logic and the following combinatorial characterization of equivalence in EP-logic: If $\phi' = \bigvee_{i \in [m]} \phi'_i$ and $\phi'' = \bigvee_{j \in [n]} \phi''_j$ are irredundant disjunctive forms of $\phi \in \mathrm{EP}$, then there exists a bijection $\pi \colon [m] \to [n]$ such that, for all $i \in [m]$, $\mathbf{C}[\phi'_i] \leftrightarrow \mathbf{C}[\phi''_{\pi(i)}]$. □

Existential Positive Logic | Classification

A complexity classification of EP^k-Express.

Theorem (B, Chen)

EP^k_σ-Express *is* Π^p_2*-complete for all* $k \geq 3$ *and all* $\sigma \supseteq \{U_n \mid n \in \mathbb{N}\} \cup \{E\}$*,*
and for all $k \geq 6$ *and all* $\sigma \supseteq \{E\}$*.*

Existential Positive Logic | Classification

A complexity classification of EP^k-EXPRESS.

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Proof (Sketch).

The upper bound follows from the characterization. A reduction from a Π^p_2-complete quantified version of the graph k-colorability problem gives the lower bound for all $k \geq 3$ (extra work required if $\sigma = \{E\}$). □

Existential Positive Logic | Reduction

Let $\mathbf{K}_k = ([k], [k]^2 \setminus \{(i,i) \mid i \in [k]\})$.

Existential Positive Logic | Reduction

Let $\mathbf{K}_k = ([k], [k]^2 \setminus \{(i,i) \mid i \in [k]\})$.

Reduction from the following Π_2^p-complete problem:

Problem Π_2-k-COLORABILITY

Instance $\psi = \forall y_1 \ldots \forall y_m \exists x_1 \ldots \exists x_n F[\mathbf{G}]$, where $\mathbf{G}$ is a graph and $G = \{y_1, \ldots, y_m, x_1, \ldots, x_n\} \cap [k] = \emptyset$.

Question $\mathbf{K}_k \models \psi$?

Note,
$\mathbf{K}_k \models \psi$
$\iff \mathbf{K}_k, f \models \exists x_1 \ldots \exists x_n F[\mathbf{G}]$ for all $f\colon \{y_1, \ldots, y_m\} \to [k]$
$\iff$ each partial k-coloring $f\colon \{y_1, \ldots, y_m\} \to [k]$ extends to a k-coloring of $\mathbf{G}$.

Existential Positive Logic | Reduction

$\psi = \forall y_1 \ldots \forall y_m \exists x_1 \ldots \exists x_n F[\mathbf{G}]$.

Reduction maps ψ to χ in $\mathrm{EP}_{\{E, U_1, \ldots, U_k, U_{y_1}, \ldots, U_{y_m}\}}$ defined as follows:

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- $\mathrm{matrix}(\chi) = F[\mathbf{G} \cup \mathbf{K}_k^k] \wedge \bigwedge_{i \in [m]} \bigvee_{j \in [k]} F[\mathbf{L}_{y_i \mapsto j} \cup \mathbf{M}_{y_i \mapsto j}]$;
- $\chi = \exists 1 \ldots \exists k \exists y_1 \ldots \exists y_m \exists x_1 \ldots \exists x_n \mathrm{matrix}(\chi)$,

where

Existential Positive Logic | Reduction

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- $\mathbf{K}_k^k = \mathbf{C}[F[\mathbf{K}_k] \wedge U_1 1 \wedge \cdots \wedge U_k k]$;
- $\mathbf{L}_{y_i \mapsto j} = \mathbf{C}[U_{y_i} j]$;
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Existential Positive Logic | Reduction

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Claim

$\mathbf{K}_k \models \psi \iff \chi \in \mathrm{EP}^k$-EXPRESS.

Existential Positive Logic | Reduction

$$\text{matrix}(\chi) = F[\mathbf{G} \cup \mathbf{K}_k^k] \wedge \bigwedge_{i \in [m]} \bigvee_{j \in [k]} F[\mathbf{L}_{y_i \mapsto j} \cup \mathbf{M}_{y_i \mapsto j}]$$

Existential Positive Logic | Reduction

By distributing $\wedge$ over $\vee$,

$$\begin{aligned} \mathrm{matrix}(\chi) &= F[\mathbf{G} \cup \mathbf{K}_k^k] \wedge \bigwedge_{i \in [m]} \bigvee_{j \in [k]} F[\mathbf{L}_{y_i \mapsto j} \cup \mathbf{M}_{y_i \mapsto j}] \\ &\equiv \bigvee_{f\colon \{y_1, \ldots, y_m\} \to [k]} F[\mathbf{G} \cup \mathbf{K}_k^k \cup \bigcup_{i \in [m]} (\mathbf{L}_{y_i \mapsto f(y_i)} \cup \mathbf{M}_{y_i \mapsto f(y_i)})] \end{aligned}$$

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By the caracterization, suffices to show the following:

1. The disjunctive form $\bigvee_{f\colon \{y_1,\ldots,y_m\} \to [k]} Q[\mathbf{H}_f]$ of χ is irredundant.

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2. $\mathbf{K}_k, f \models \exists x_1 \dots x_n F[\mathbf{G}] \iff \mathrm{tw}(\mathrm{core}(\mathbf{H}_f)) < k$,
   for all $f\colon \{y_1, \dots, y_m\} \to [k]$.

Existential Positive Logic | Idea Item 1

The disjunctive form $\bigvee_{f\colon \{y_1,\ldots,y_m\}\to[k]} Q[\mathbf{H}_f]$ of χ is irredundant.

Example ($k = 3$, $m = 2$)

$f(y_1) = f'(y_1) = f'(y_2) = 2, f(y_2) = 1.$

$\mathbf{H}_f$ $\mathbf{H}_{f'}$

Existential Positive Logic | *Idea Item* 1

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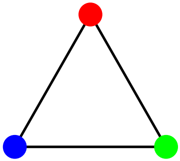
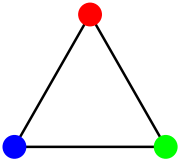
$U_1^{\mathbf{H}_f} = U_1^{\mathbf{H}_{f'}} = \{\bullet\}, U_2^{\mathbf{H}_f} = U_2^{\mathbf{H}_{f'}} = \{\bullet\}, U_3^{\mathbf{H}_f} = U_3^{\mathbf{H}_{f'}} = \{\bullet\}.$



$\mathbf{H}_f$



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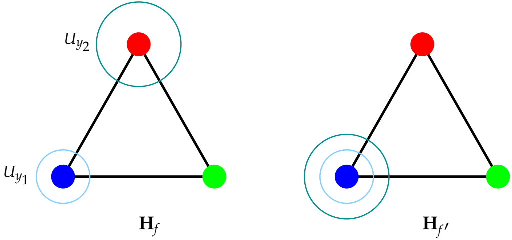
$U_{y_1}^{\mathbf{H}_f} = U_{y_1}^{\mathbf{H}_{f'}} = U_{y_2}^{\mathbf{H}_{f'}} = \{\bullet\}, U_{y_2}^{\mathbf{H}_f} = \{\bullet\}.$



Existential Positive Logic | *Idea Item* 1

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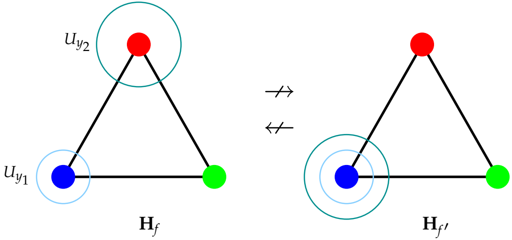
$U_{y_1}^{\mathbf{H}_f} = U_{y_1}^{\mathbf{H}_{f'}} = U_{y_2}^{\mathbf{H}_{f'}} = \{\textcolor{blue}{\bullet}\}, U_{y_2}^{\mathbf{H}_f} = \{\textcolor{red}{\bullet}\}.$
$\mathbf{H}_f \not\to \mathbf{H}_{f'}$ and $\mathbf{H}_{f'} \not\to \mathbf{H}_f$, ie, $Q[\mathbf{H}_f] \not\models Q[\mathbf{H}_{f'}]$ and $Q[\mathbf{H}_{f'}] \not\models Q[\mathbf{H}_f]$.



Existential Positive Logic | Sketch Item 2

$\mathbf{K}_k, f \models \exists x_1 \ldots x_n F[\mathbf{G}] \iff \mathrm{tw}(\mathrm{core}(\mathbf{H}_f)) < k.$

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Primitive Positive Logic | *Picture* 1

$\mathrm{tw}(\mathrm{core}(\mathbf{H}_f)) < k \Longrightarrow \mathrm{core}(\mathbf{H}_f) \to \mathbf{K}_k$, ie, $\mathrm{core}(\mathbf{H}_f)$ is k-colorable.

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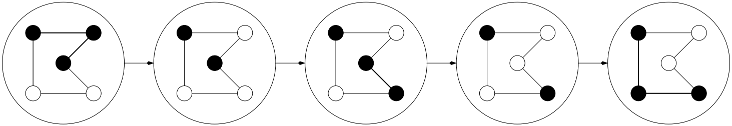

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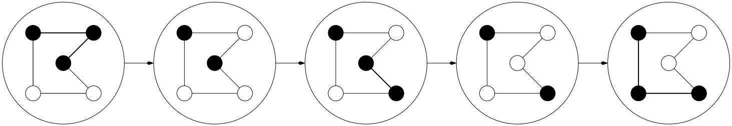

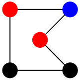

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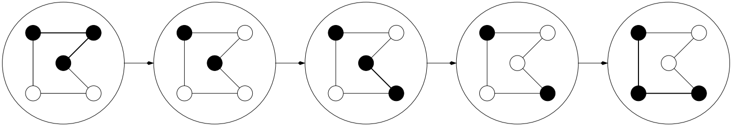

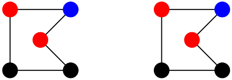

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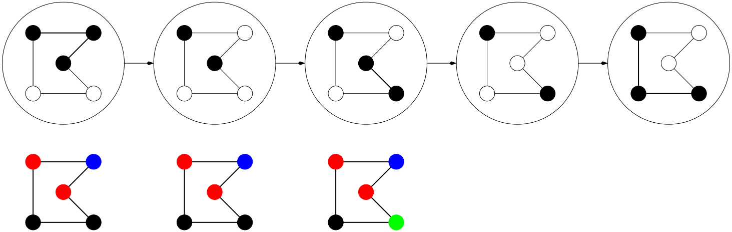

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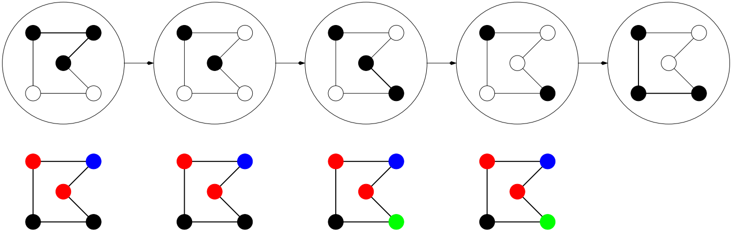

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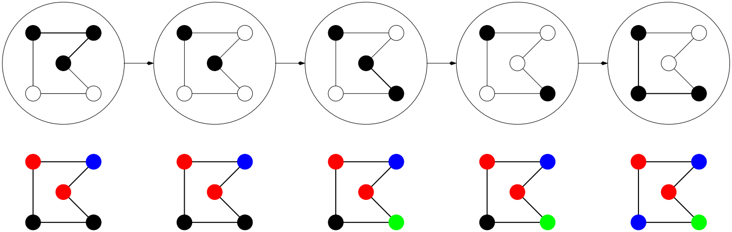

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Existential Positive Logic | *Sketch Item* 2

$\mathbf{K}_k, f \models \exists x_1 \ldots x_n F[\mathbf{G}] \iff \mathrm{tw}(\mathrm{core}(\mathbf{H}_f)) < k.$

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implies $\mathbf{H}_f \to \mathbf{K}_k^k$ via homomorphism acting as h on G and identically on $[k]$.
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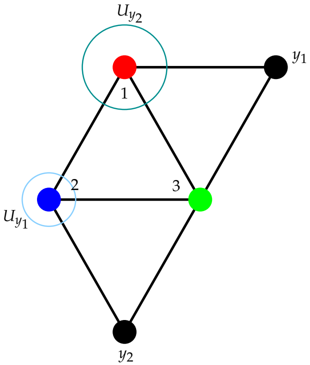
Since $\mathbf{K}_k, h \models \bigcup_{i \in [m]} (\mathbf{L}_{y_i \mapsto f(y_i)} \cup \mathbf{M}_{y_i \mapsto f(y_i)})$,
we have that h extends f (Picture 2).

Existential Positive Logic | *Picture* 2

$\mathbf{H}_f \to \mathbf{K}_k$ via $h \Longrightarrow h$ extends f.

Example ($k = 3$, $m = 2$)

$f(y_1) = 2$ and $f(y_2) = 1$. Thus $U_{y_1}^{\mathbf{H}_f} = \{2\}$, $U_{y_2}^{\mathbf{H}_f} = \{1\}$,
$E^{\mathbf{H}_f} \supseteq \{(y_1, 1), (y_1, 3), (1, y_1), (3, y_1), (y_2, 2), (y_2, 3), (2, y_2), (3, y_2)\}$.



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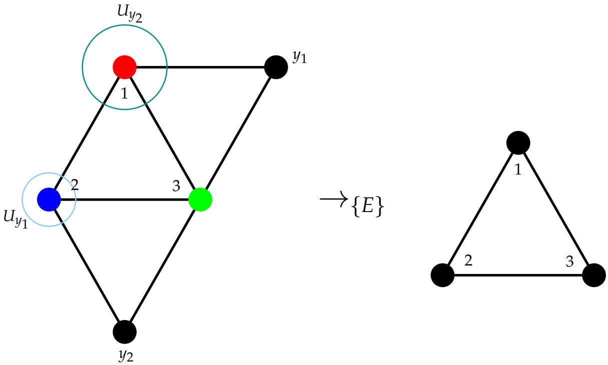
Assume $\mathbf{H}_f \to \mathbf{K}_k$ via h.



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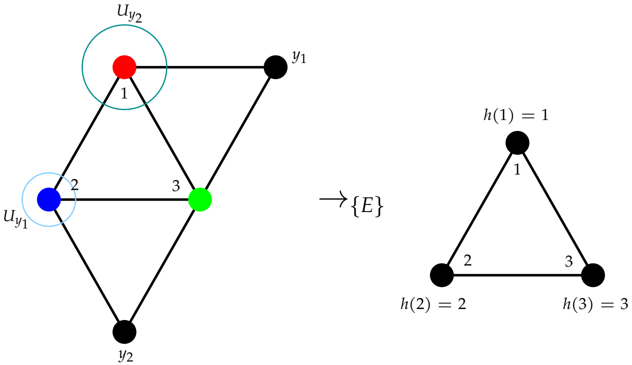

Existential Positive Logic | *Picture* 2

$\mathbf{H}_f \to \mathbf{K}_k$ via $h \Longrightarrow h$ extends f.

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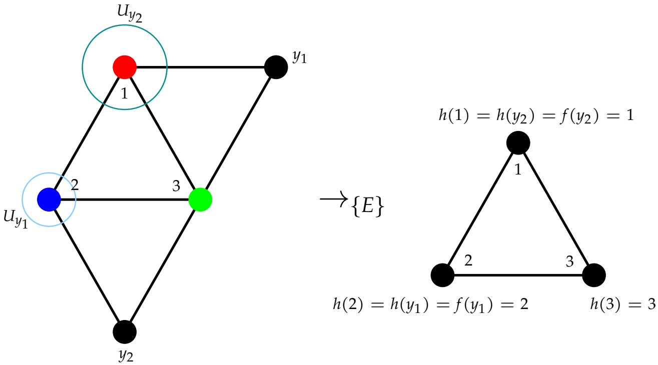
Then, $h(y_j) = f(y_j)$ for all $j \in [2]$, ie, h extends f.



Existential Positive Logic | Sketch Item 2

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we have that h extends f (Picture 2).
Moreover, $\mathbf{K}_k, h \models F[\mathbf{G}]$.

Outline

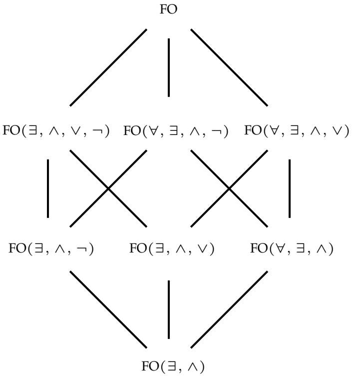
Research Motivation

Previous Work

Our Result

Other Results and Open Problems

First-Order Logic Fragments | Expressibility



S	FO(S)-EXPRESS
$\{\forall, \exists, \wedge, \vee, \neg\}$	undecidable [folklore]
$\{\forall, \exists, \wedge, \vee\}$	undecidable, $k \geq 3$ [B, Chen]
$\{\forall, \exists, \wedge, \neg\}$	open
$\{\exists, \wedge, \vee, \neg\}$	open
$\{\exists, \wedge, \neg\}$	open
$\{\forall, \exists, \wedge\}$	open
$\{\exists, \wedge, \vee\}$	Π_2^p-complete, $k \geq 3$ [B, Chen]
$\{\exists, \wedge\}$	NP-complete, $k \geq 2$ [Dalmau et al.]

FO(S) denotes equality-free relational FO-sentences in prefix negation form, using logical symbols in S.

First-Order Logic Fragments | Entailment and Equivalence

Understanding entailment/equivalence
helps in understanding expressibility.

As a byproduct, we obtained a (fairly complete) complexity classification of entailment/equivalence wrt:

- all existential fragments S of FO;
- all relational vocabularies σ;

thus refining known Π_2^p-completeness of $\mathrm{FO}_\sigma(\exists, \wedge, \neg)$ and $\mathrm{FO}_\sigma(\exists, \wedge, \vee)$.

σ	$\mathrm{FO}_\sigma(\exists, \wedge)$	$\mathrm{FO}_\sigma(\exists, \wedge, \neg)$	$\mathrm{FO}_\sigma(\exists, \wedge, \vee)$	$\mathrm{FO}_\sigma(\exists, \wedge, \vee, \neg)$
unary, $\lvert\sigma\rvert \leq 1$	P	P	P	coDP-hard, in $\mathrm{P}^{\mathrm{NP}[\mathrm{const}]}$
unary, finite, $\lvert\sigma\rvert > 1$	P	P	coDP-hard, in $\mathrm{P}^{\mathrm{NP}[\mathrm{const}]}$	coDP-hard, in $\mathrm{P}^{\mathrm{NP}[\mathrm{const}]}$
unary infinite	P	P	Π_2^p-complete	Π_2^p-complete
$R \in \sigma$, $\mathrm{ar}(R) \geq 2$	NP-complete	Π_2^p-complete	Π_2^p-complete	Π_2^p-complete

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σ	$\mathrm{FO}_\sigma(\exists, \wedge)$	$\mathrm{FO}_\sigma(\exists, \wedge, \neg)$	$\mathrm{FO}_\sigma(\exists, \wedge, \vee)$	$\mathrm{FO}_\sigma(\exists, \wedge, \vee, \neg)$
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unary, finite, $\lvert\sigma\rvert > 1$	P	P	coDP-hard, in $\mathrm{P}^{\mathrm{NP}[\mathrm{const}]}$	coDP-hard, in $\mathrm{P}^{\mathrm{NP}[\mathrm{const}]}$
unary infinite	P	P	Π_2^p-complete	Π_2^p-complete
$R \in \sigma$, $\mathrm{ar}(R) \geq 2$	NP-complete	Π_2^p-complete	Π_2^p-complete	Π_2^p-complete

The complexity of $\mathrm{FO}_\sigma(\forall, \exists, \wedge)$ and $\mathrm{FO}_\sigma(\forall, \exists, \wedge, \neg)$ is open.

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Thank you for your attention!