Local Consistency and *MV*-Algebras

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Outline

1. Motivation
   - Soft Constraint Satisfaction Problems
   - Commutative Bounded Residuated Lattices
2. Local Consistency
   - *k*-Hyperarc Consistency
   - Enforcing Algorithm
   - Lattice Orders and Nonidempotent Combinations
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   - Komori-Grigolia Variety
   - Combinatorial Representations
4. Conclusion

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Constraint Satisfaction Problems

Problem: CSP

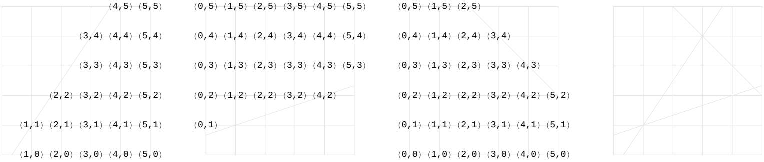
Instance: (X, D, P) where:

- *(i)* X is a finite set of *variables*;
- *(ii)* D is a finite set of *values* (aka *domain*);
- *(iii)* $P = \{C_1, \ldots, C_q\}$ is a finite set of *constraints*, that is, pairs $(\mathbf{x}_i, R_i)$ having $\mathbf{x}_i \in X^m$ as *scope* and $R_i \subseteq D^m$ as *relation*.

Question: Is there an *assignment* $f\colon X \to D$ *satisfying* all constraints, that is, such that $f(\mathbf{x}_i) \in R_i$ for all $i \in \{1, \ldots, m\}$?

CSP | Example

$(\{x_1, x_2\}, \{0, \ldots, 5\}, \{((x_1, x_2), R_1), ((x_1, x_2), R_2), ((x_1, x_2), R_3)\})$,
$R_1, R_2, R_3 \subseteq \{0, \ldots, 5\}^2$ as follows:

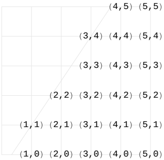


(a) R_1. (b) R_2. (c) R_3.

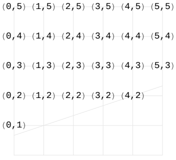
Is there $f\colon \{x_1, x_2\} \to \{0, \ldots, 5\}$ satisfying all constraints?

CSP | Example

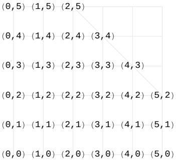
$(\{x_1, x_2\}, \{0, \ldots, 5\}, \{((x_1, x_2), R_1), ((x_1, x_2), R_2), ((x_1, x_2), R_3)\})$,
$R_1, R_2, R_3 \subseteq \{0, \ldots, 5\}^2$ as follows:



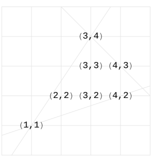
(a) R_1.



(b) R_2.



(c) R_3.



(d) f's.

Is there $f\colon \{x_1, x_2\} \to \{0, \ldots, 5\}$ satisfying all constraints? Yes.

Feasibility *vs.* Optimization

The *crisp* CSP is a *feasibility* question (any satisfying assignment is equally likely).

The *soft* CSP is an *optimization* question: each constraint maps assignments to a *valuation structure*, that is, a *bounded* poset equipped with a suitable *combination* operator; the task is to find an assignment such that the combination of its images under all the constraints is *maximal* in the poset.

Valuation Structure | Example

$\mathbf{A} = (\{0, \ldots, 10\}, \bot = 0 < \cdots < 10 = \top, \min)$. min:

(*i*) associative, commutative (no precedence, no order);

(*ii*) monotone over $\leq$ (more constraints, worst solutions);

(*iii*) $\min\{x, \bot\} = \bot$ (unsatisfied constraints);

(*iv*) $\min\{x, \top\} = x$ (trivial constraints).

Soft Constraints | Example

Suppose $f\colon \{x_1, x_2\} \to \{0, \ldots, 5\}$ pays $f(x_1) + f(x_2)$ euro...

				(4,5)	(5,5)
			(3,4)	(4,4)	(5,4)
			(3,3)	(4,3)	(5,3)
		(2,2)	(3,2)	(4,2)	(5,2)
	(1,1)	(2,1)	(3,1)	(4,1)	(5,1)
	(1,0)	(2,0)	(3,0)	(4,0)	(5,0)

(a) Crisp R_1.

(0,5)	(1,5)	(2,5)	(3,5)	(4,5)	(5,5)
(0,4)	(1,4)	(2,4)	(3,4)	(4,4)	(5,4)
(0,3)	(1,3)	(2,3)	(3,3)	(4,3)	(5,3)
(0,2)	(1,2)	(2,2)	(3,2)	(4,2)	(5,2)
(0,1)	(1,1)	(2,1)	(3,1)	(4,1)	(5,1)
(0,0)	(1,0)	(2,0)	(3,0)	(4,0)	(5,0)

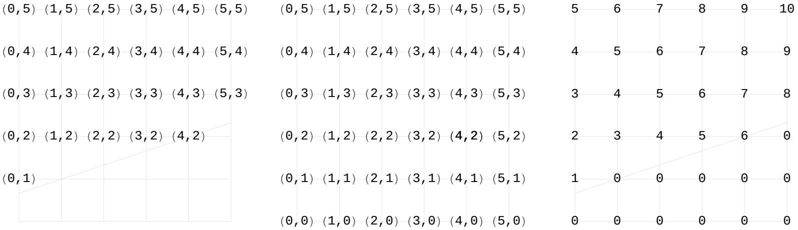
(b) Soft R_1 domain.

0	0	0	0	9	10
0	0	0	7	8	9
0	0	0	6	7	8
0	0	4	5	6	7
0	2	3	4	5	6
0	1	2	3	4	5

(c) Soft R_1 image.

Figure: $R_1\colon \{0, \ldots, 5\}^2 \to \{0, \ldots, 10\}$.

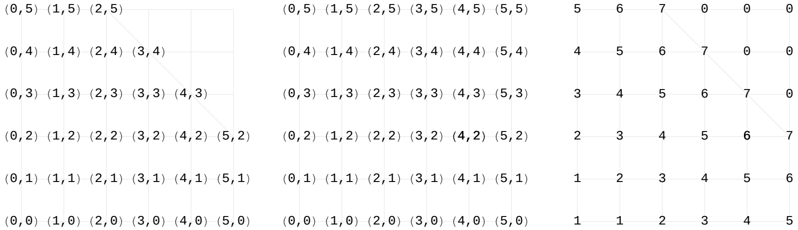
Soft Constraints | Example



(a) Crisp R_2. (b) Soft R_2 domain. (c) Soft R_2 image.

Figure: $R_2 \colon \{0,\ldots,5\}^2 \to \{0,\ldots,10\}$.

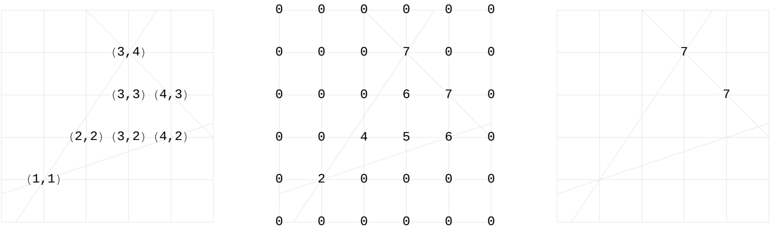
Soft Constraints | Example



(a) Crisp R_3. (b) Soft R_3 domain. (c) Soft R_3 image.

Figure: $R_3 \colon \{0, \ldots, 5\}^2 \to \{0, \ldots, 10\}$.

Combination and Maximization | Example



(a) $R_1 \cap R_2 \cap R_3$. (b) $\min_{\mathbf{a}}\{R_i(\mathbf{a})\}$. (c) Optimal $\mathbf{a}$'s.

Figure: $\min_{\mathbf{a} \in \{0,\ldots,5\}^2}\{R_1(\mathbf{a}), R_2(\mathbf{a}), R_3(\mathbf{a})\}$.

$\ldots$ $f(x_i) = 3, f(x_j) = 4$ maximize the venue $f(x_1) + f(x_2)$.

Definition

Definition (Soft CSP)

A *soft CSP* is a tuple $\mathbf{P} = (X, D, P, \mathbf{A})$ with:

(*i*) *variables* $X = \{1, \ldots, n\} = [n]$;

(*ii*) finite *domains* $D = (D_i)_{i \in [n]}$ where i ranges over D_i;

(*iii*) *valuation structure* $\mathbf{A} = (A, \leq, \odot, \top, \bot)$ st $(A, \leq, \top, \bot)$ is a bounded poset, $(A, \odot, \top)$ is a commutative monoid, $\odot$ is monotone over $\leq$ (that is, $x \leq y$ implies $z \odot x \leq z \odot y$);

(*iv*) P finite multiset of *constraints* of the form

$$C_Y : \prod_{i \in Y} D_i \to A,$$

where $Y \subseteq X$ is the *scope* of C_Y.

Definition

Notation ($Y \subseteq X$): $I(Y) = \prod_{i \in Y} D_i$; $t|_Y$ projects $t \in I(X)$ onto Y.

Definition (Solution, Inconsistence, Equivalence)

Any $t \in I(X)$ such that $\bigodot_{C_Y \in P} C_Y(t|_Y)$ is maximal wrt $\leq$ in

$$S(\mathbf{P}) = \{ \bigodot_{C_Y \in P} C_Y(t|_Y) \mid t \in I(X)\} \subseteq A$$

is a *solution* to $\mathbf{P}$, and $\mathbf{P}$ is *inconsistent* if $S(\mathbf{P}) = \{\bot\}$.
$\mathbf{P} = (X, D, P, \mathbf{A})$ is *equivalent* to $\mathbf{P}' = (X, D, P', \mathbf{A})$
iff for every $t \in I(X)$,

$$\bigodot_{C_Y \in P} C_Y(t|_Y) = \bigodot_{C_Y \in P'} C_Y(t|_Y).$$

Soft CSP

Problem: SOFT-CSP

Instance: $(X, D, P, \mathbf{A})$

Goal: Find $t \in I(X)$ maximizing $\bigodot_{C_Y \in P} C_Y(t|_Y)$.

Enforcing Algorithms

An *enforcing* algorithm enforces in a polynomial-time a *local consistency* property over a given a soft CSP.

Either the input problem is found locally (hence, globally) inconsistent, or it is transformed into an *equivalent* problem, possibly inconsistent but *easier* (with a smaller solution space).

Despite their incompleteness as inconsistency test, enforcing algorithms are useful as subprocedures in exhaustive search methods (eg *branch and bound*).

Valuation Structures

The generalization of local consistency notions and techniques from the crisp to the soft setting plays a central role in the algorithmic investigation of soft CSPs.

The *minimal* valuation structure has been *specialized* to implement consistency techniques (eg *fair* valuation structures, commutative *idempotent* semirings).

Valuation Structures

The generalization of local consistency notions and techniques from the crisp to the soft setting plays a central role in the algorithmic investigation of soft CSPs.

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Question: Are there *natural* valuation structures?

Logical Structures

Fact

A CSP is a soft CSP $(X, D, P, \mathbf{A})$ *where:*

(*i*) $D = (D_i)_{i \in X}$ *with* $|\{D_i \mid i \in X\}| = 1$;

(*ii*) $\mathbf{A} = (\{0, 1\}, 0 < 1, \min, 1, 0)$.

In the crisp CSP, **A** is a reduct of the Boolean algebra **2**, the algebraic counterpart of classical *two-valued* logic.

Proposal: Consider algebraic counterparts of nonclassical *many-valued* logics as valuation structures for the soft CSP.

Residuated Lattices

In Boolean logic the relation between *conjunction*, $\wedge$, and *implication*, $\rightarrow$, is given by the *residuation* equivalences,

$$x \wedge y \leq z \text{ iff } x \leq y \rightarrow z \text{ iff } y \leq x \rightarrow z,$$

which imply many of the properties of $\wedge$ and $\rightarrow$ (commutativity of $\wedge$, distributivity of $\wedge$ over $\vee$, left-distributivity of $\rightarrow$ over $\vee$, and right-distributivity of $\rightarrow$ over $\wedge$).

The prominent approach in generalizing Boolean logic relies upon generalizing Boolean conjunction, by means of a binary operation, $\odot$, called *fusion*, and imposing the residuation equivalences with $\wedge$ replaced by $\odot$.

Residuated Lattices

Definition (Commutative Bounded Residuated Lattice, *CBRL*)

A *(commutative bounded) residuated lattice* is an algebra $(A, \vee, \wedge, \odot, \to, \top, \bot)$ of type $(2, 2, 2, 2, 0, 0)$ st:

(*i*) $(A, \odot, \top)$ is a commutative monoid;

(*ii*) $(A, \vee, \wedge, \top, \bot)$ is a bounded lattice;

(*iii*) *residuation* holds, that is $x \odot y \leq z$ if and only if $y \leq x \to z$.

The monotonicity of fusion over the order follows.

Residuated Lattices

What additional structure is required to implement local consistency techniques?

Residuated Lattices

What additional structure is required to implement local consistency techniques? *Divisibility* is necessary, *prelinearity* is auxiliary...

Definition (*GBL*-algebra, *BL*-algebra)

A *GBL-algebra* is a *CBRL* where *divisibility* holds, that is, $x \wedge y = x \odot (x \to y)$. A *BL-algebra* is a *GBL*-algebra where *prelinearity* holds, that is, $(x \to y) \vee (y \to x) = \top$.

BL- and *GBL*-algebras have a natural logical interpretation, respectively *Hájek*'s logic and the intersection of Hájek's and intuitionistic logic.

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k-Hyperarc Consistency

A soft CSP is *k-hyperarc consistent* if it is possible to extend any *consistent* assignment of a variable i to an assignment of any other $\leq k - 1$ variables, constrained by i, avoiding additional costs [BG06, CS04, LS04].

If the valuation structure has a logical interpretation, enforcing local consistency coincides with performing logical inferences, aiming to a refutation.

Definition

Notation ($Y \subseteq X$, $i \in Y$, $a \in D_i$, $t \in l(Y \setminus \{i\})$):
$(t \cdot a) = t' \in l(Y)$ st $t'|_{\{i\}} = a$ and $t'|_{Y \setminus \{i\}} = t$.

Definition (*k*-Hyperarc Consistency)

$\mathbf{P} = (X, D, P, \mathbf{A})$ soft CSP, $Y \subseteq X$ st $2 \leq |Y| \leq k$ and $C_Y \in P$. Y is *k-hyperarc consistent* if for each $i \in Y$ and each $a \in D_i$ such that $C_{\{i\}}(a) > \bot$, there exists $t \in l(Y \setminus \{i\})$ such that,

$$C_Y(t \cdot a) = \top.$$

$\mathbf{P}$ is *k-hyperarc consistent* if every $Y \subseteq X$ st $2 \leq |Y| \leq k$ and $C_Y \in P$ is k-hyperarc consistent.

Specification

Algorithm: k-HYPERARCCONSISTENCY

Input: A soft CSP $\mathbf{P} = (X, D, P, \mathbf{A})$, where $\mathbf{A}$ is *GBL*-algebra.

Output: $\perp$, or a k-hyperarc consistent soft CSP, equivalent to $\mathbf{P}$.

Pseudocode | 1

k-HYPERARCCONSISTENCY$((X, D, P, \mathbf{A}))$

$Q \leftarrow \{1, \ldots, n\}$
while $Q \neq \emptyset$ **do**
  $i \leftarrow$ POP(Q)
  foreach $Y \subseteq X$ such that $2 \leq |Y| \leq k$, $i \in Y$ and $C_Y \in P$ **do**
    domainShrink $\leftarrow$ PROJECT(Y, i)
    if $C_{\{i\}}(a) = \bot$ for each $a \in D_i$ **then**
      return $\bot$
    else if domainShrink **then**
      PUSH(Q, i)
    endif
  endforeach
endwhile
return $(X, D, P', \mathbf{A})$

Pseudocode | 2

PROJECT(Y, i)

domainShrink $\leftarrow$ **false**
foreach $a \in D_i$ such that $C_{\{i\}}(a) > \bot$ **do**
  $x \leftarrow$ a maximal element in $\{C_Y(t \cdot a) \mid t \in I(Y \setminus \{i\})\}$
  $C_{\{i\}}(a) \leftarrow C_{\{i\}}(a) \odot x$
  if $C_{\{i\}}(a) = \bot$ **then**
    domainShrink $\leftarrow$ **true**
  endif
  foreach $t \in I(Y \setminus \{i\})$ **do**
    $C_Y(t \cdot a) \leftarrow (x \rightarrow C_Y(t \cdot a))$
    $\triangleright$ by divisibility, $z \leq x$ implies $(y \odot x) \odot (x \rightarrow z) = y \odot z$
  endforeach
endforeach
return domainShrink

Correctness and Complexity

Lemma (Complexity)

Let $\mathbf{P} = (X, D, P, \mathbf{A})$ *be soft CSP with* $X = [n]$*,* $d = \max_{i \in [n]} |D_i|$ *and* $e = |P|$*. Then,* k-HYPERARCCONSISTENCY$(\mathbf{P})$ *runs in* $O(e^2 \cdot d^{k+1})$ *time.*

Lemma (Soundness)

Let $\mathbf{P} = (X, D, P, \mathbf{A})$ *be a soft CSP. Consider the output of* k-HYPERARCCONSISTENCY$(\mathbf{P})$*:*

(*i*) *if it is* $\bot$*, then* $\mathbf{P}$ *is inconsistent;*

(*ii*) *ow it is a* k*-hyperarc consistent soft CSP equivalent to* $\mathbf{P}$*.*

Lattice Orders and Nonidempotent Combinations

$Y \subseteq X$, $t, t' \in I(Y)$, **A** *GBL*-algebra.

- $C_Y(t) \leq C_Y(t')$ says that t' is preferred to t (the distance between $C_Y(t)$ and $C_Y(t')$ gives the degree of such preference, ranging over **A**'s *depth*).
- $C_Y(t) \parallel C_Y(t')$ says that t' and t are incomparable (**A**'s *width* gives the number of simultaneous rankings supported by **A**).
- $\wedge$'s and $\vee$'s serve to embed consistency techniques over residuated lattices inside branch and bound methods (tentative).
- $C_Y(t) \odot C_Y(t) < C_Y(t)$ says that repetitions matter.

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Prelinearity

k-HYPERARCCONSISTENCY works without prelinearity.

The bonus of prelinearity is the *representability* of finite algebras in locally finite subvarieties of *BL*-algebras.

Definition (*MV*-Algebras)

An *MV-algebra* is a *BL*-algebra where *involutiveness* holds, that is, $(x \to \bot) \to \bot = x$.

Generic *MV*-Algebra

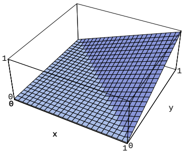
For every $x, y \in [0, 1]$, let:

(*i*) $x \odot y = \max\{0, x+y-1\}$;
(*ii*) $x \to y = \min\{1, y+1-x\}$;
(*iii*) $\bot = 0$;
(*iv*) $x \wedge y = x \odot (x \to y)$;
(*v*) $x \vee y = (x \to y) \to y$;
(*vi*) $\top = \bot \to \bot$.

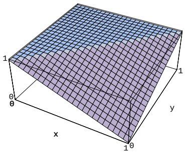
Fact

(*i*) $[0, 1]_{MV} = ([0, 1], \vee, \wedge, \odot, \to, \top, \bot)$ *is an MV-algebra;*
(*ii*) $[0, 1]_{MV}$ *generates the variety of MV-algebras.*

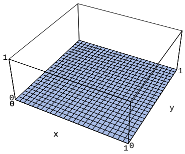
Standard *MV*-Operations



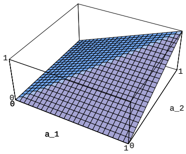
(a) $x \odot y$.



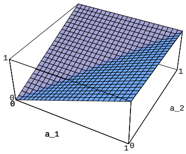
(b) $x \rightarrow y$.



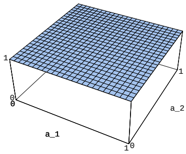
(c) $\bot$.



(d) $x \wedge y$.



(e) $x \vee y$.



(f) $\top$.

Komori-Grigolia Varieties

For every $m \geq 1$, let $L_m = \{0, 1/m, \ldots, (m-1)/m, 1\} \subseteq [0,1]$.

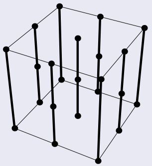
Fact

$L_m = (L_m, \vee|_{L_m}, \wedge|_{L_m}, \odot|_{L_m}, \rightarrow|_{L_m}, \top|_{L_m}, \bot|_{L_m})$ *is an MV-algebra.*

MV_m, the variety generated by L_m (*Komori-Grigolia*).

Theorem (Free n-Generated MV_m-Algebra, $F_n(MV_m)$)

Let m be prime. The free n-generated MV-algebra in MV_m *is the direct product of* 2^n *chains* L_1 *and* $(m+1)^n - 2^n$ *chains* L_m.



(a) $F_2(MV_2)$.

Combinatorial Representations

Fact

Let m be prime. $F_n(MV_m)$ is the algebra of $(m+1)^n$-dimensional integer vectors having the first 2^n coordinates ranging over L_1 and the last $(m+1)^n - 2^n$ coordinates ranging over L_m, equipped with standard MV-operations defined coordinatewise.

Features of $F_n(MV_m)$'s lattice reduct:

(*i*) size $2^{2^n} \cdot (m+1)^{(m+1)^n - 2^n}$;

(*ii*) depth $2^n + m((m+1)^n - 2^n)$;

(*iii*) width $\geq \binom{(m+1)^n}{\lfloor (m+1)^n/2 \rfloor}$ by Sperner's lemma.

Combinatorial Representations | Examples

$F_2(MV_3)$ has domain $\{(a_1, \ldots, a_4, b_1, \ldots, b_{12}) \mid a_i \in L_1, b_j \in L_3\}$ of size $268, 435, 456$, depth 40 and width ≥ 12870.

$F_1(MV_2)$'s contains the 12 3-dimensional vectors (a, b, c) st $a, c \in \{0, 1\}, b \in \{0, 1/2, 1\}$. $F_1(MV_2)$'s lattice reduct has depth 4 and width $4 > \binom{3}{\lfloor 3/2 \rfloor} = 3$.

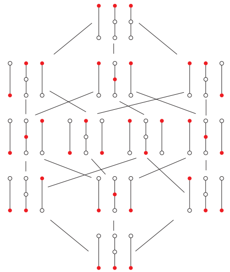


Figure: $F_1(MV_2)$.

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Summary

We presented certain subvarieties of commutative bounded residuated lattices as *natural* valuation structures for soft CSP's.

These structures constitute the algebraic counterparts of nonclassical *many-valued* logics, and provide a uniform *logical* interpretation of enforcing procedures.

Divisibility and *prelinearity* allow for a sound implementation and a concrete representation of useful techniques of local consistency.

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