Combinatorics of Interpolation in Gödel Logic

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Outline

Gödel Logic
- Gödel Logic
- Free Gödel Algebra
- Interpolation Properties

Ongoing Work
- Basic Logic
- Deductive Interpolation
- Construction Sketch

Outline

Gödel Logic

Gödel (propositional) logic, G, is:

- intuitionistic logic plus $((x \to y) \vee (y \to x))$, the intermediate logic of linear Kripke frames;
- Hájek's basic logic plus $(x \to (x \odot x))$, the many-valued logic of "minimum and its residual",

$$[0,1] = ([0,1], \wedge = \odot = \min, \vee = \max, x \to y, \bot = 0, \top = 1)$$

where $x \to y$ equals 1 if $x \le y$ and y otherwise.

Free Gödel Algebra

Definition (Gödel Algebras)

Gödel algebras are algebras in the variety generated by $[0, 1]$.

Fact

The free X-generated Gödel algebra, G_X, is (isomorphic to) the Lindenbaum algebra of the X-variate fragment of Gödel logic.

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G_X "supports" the investigation of (finite) consequence relations and interpolation properties in Gödel logic.

G_X has a nice combinatorial representation (X finite).

Free X-generated Gödel Algebra | *Construction*

Step 1: Construction of forest F_X:

$t = 0$: the subsets of X are the maximal elements in F_X at $t = 0$;

$t = i + 1$: if R is maximal in F_X at $t = i$, then there exists S st S covers R and S is maximal in F_X at $t = i + 1$ iff:
$X = \bigcup_{T \leq R} T$ and $S = \{1\}$, or
$X \neq \bigcup_{T \leq R} T$ and $\emptyset \neq S \subseteq X \setminus \bigcup_{T \leq R} T$.

Ex.: $F_{\{x,y,z\}}$ at $t = 0$.

0xyz 0xy 0xz 0yz 0x 0y 0z 0

Free X-generated Gödel Algebra | Construction

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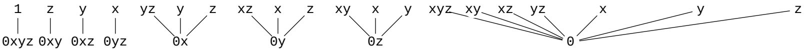
Ex.: $F_{\{x,y,z\}}$ at $t = 1$.



Free X-generated Gödel Algebra | Construction

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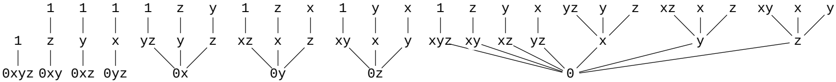
Ex.: $F_{\{x,y,z\}}$ at $t = 2$.



Free X-generated Gödel Algebra | Construction

Step 1: Construction of forest F_X:

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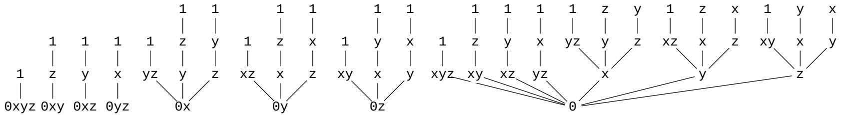
$X = \bigcup_{T \leq R} T$ and $S = \{1\}$, or
$X \neq \bigcup_{T \leq R} T$ and $\emptyset \neq S \subseteq X \setminus \bigcup_{T \leq R} T$.

Ex.: $F_{\{x,y,z\}}$ at $t = 3$.



Free X-generated Gödel Algebra | Construction

Step 1: Construction of forest F_X:

- *$t = 0$:* the subsets of X are the maximal elements in F_X at $t = 0$;
- *$t = i + 1$:* if R is maximal in F_X at $t = i$, then there exists S st S covers R and S is maximal in F_X at $t = i + 1$ iff:
  $X = \bigcup_{T \leq R} T$ and $S = \{1\}$, or
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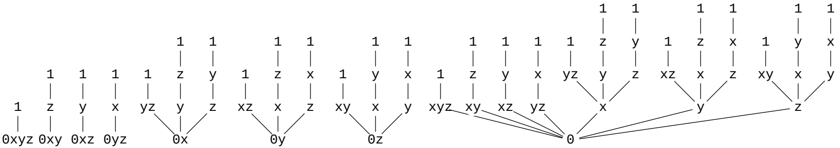
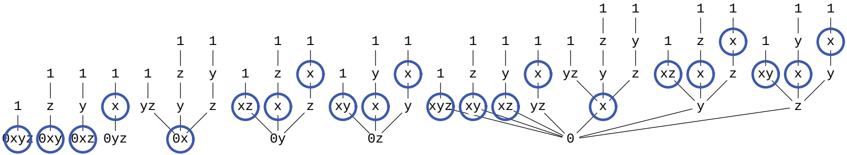
Ex.: $F_{\{x,y,z\}}$ at $t \geq 4$.



Free X-generated Gödel Algebra | Construction

Step 2: The generator $x \in X$ is the maximal antichain in F_X "mentioning x" over each maximal chain in F_X.

Ex.: $X = \{x, y, z\}$. For each maximal chain in $F_{\{x,y,z\}}$, the generator x picks the point containing x.



Free X-generated Gödel Algebra | Construction

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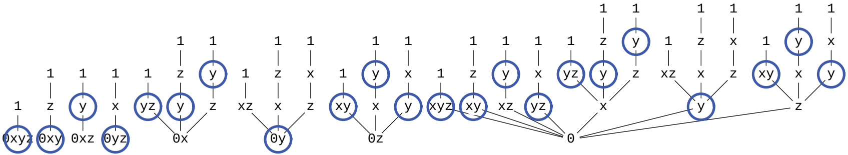
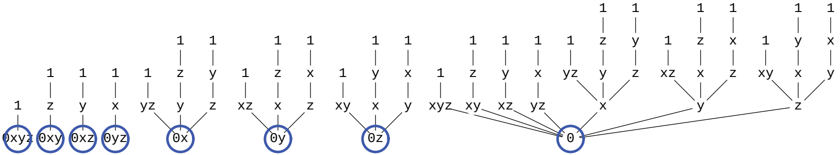
Ex.: $X = \{x, y, z\}$. For each maximal chain in $F_{\{x,y,z\}}$, the generator y picks the point containing y.



Free X-generated Gödel Algebra | *Construction*

Step 3: The op's over maximal antichains in F_X are defined "maxchainwise" by the corr. operations in $[0, 1]$.

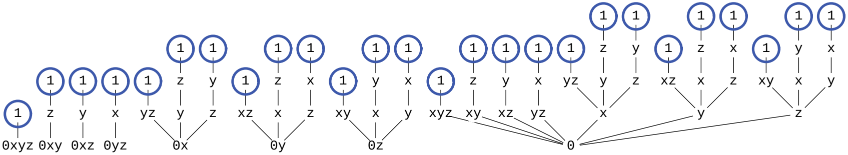
Ex.: $X = \{x, y, z\}$. As $\perp = 0$ in $[0, 1]$, $\perp$ picks the point containing 0 for each maximal chain in $F_{\{x,y,z\}}$.



Free X-generated Gödel Algebra | Construction

Step 3: Equip the maximal antichains in F_X with operations defined “maxchainwise” by the corr. generic operations.

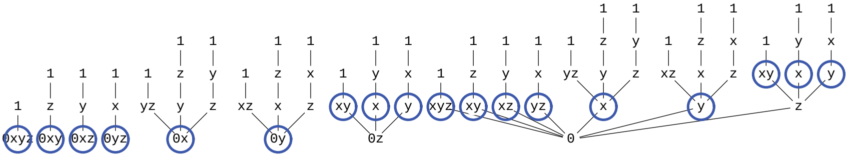
Ex.: $X = \{x, y, z\}$. As $\top = 1$ in $[0, 1]$, $\top$ picks the point containing 1 for each maximal chain in $F_{\{x,y,z\}}$.



Free X-generated Gödel Algebra | Construction

Step 3: Equip the maximal antichains in F_X with operations defined "maxchainwise" by the corr. generic operations.

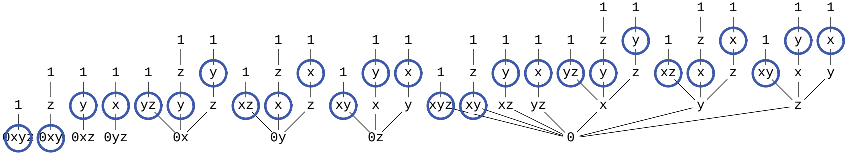
Ex.: $X = \{x, y, z\}$. As $x \wedge y = \min\{x, y\}$ in $[0, 1]$, $x \wedge y$ picks the minimum of x and y.



Free X-generated Gödel Algebra | Construction

Step 3: Equip the maximal antichains in F_X with operations defined "maxchainwise" by the corr. generic operations.

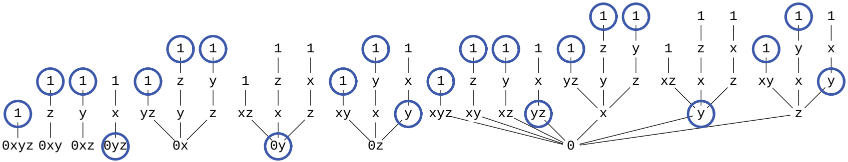
Ex.: $X = \{x, y, z\}$. As $x \vee y = \max\{x, y\}$ in $[0, 1]$, $x \vee y$ picks the maximum of x and y.



Free X-generated Gödel Algebra | Construction

Step 2: Equip the maximal antichains in F_X with operations defined "maxchainwise" by the corr. generic operations.

Ex.: $X = \{x, y, z\}$. As $x \to y$ is 1 if $x \leq y$ and y ow in $[0, 1]$, $x \to y$ picks 1 if $x \leq y$ and y ow.



Interpolation Properties | Craig (CIP)

$X \cap Y = Z$ and $X \cup Y = W$.

Definition

G has the CIP iff, for all r_X and t_Y st $\vdash_G r \to t$,
there exists s_Z st $\vdash_G r \to s$ and $\vdash_G s \to t$.

Theorem ([BV99])

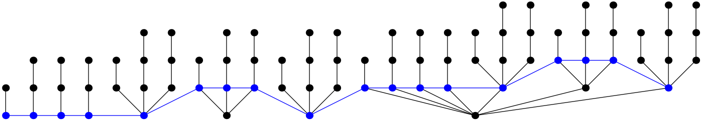
G has the CIP.

Corollary

*For all $A_X \in G_X$ and $C_Y \in G_Y$ st $A_W \leq C_W$ in G_W,
there exists $B_Z \in G_Z$ st $A_W \leq B_W \leq C_W$ in G_W.*

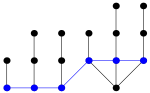
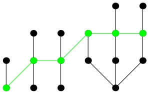
CIP | *Sampling with* $X = \{x, z\}$ *and* $Y = \{y, z\}$

$A_{\{x,z\}} \in G_{\{x,z\}}$ and $C_{\{y,z\}} \in G_{\{y,z\}}$

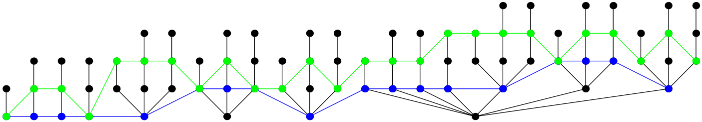


are such that $A_{\{x,y,z\}} \leq C_{\{x,y,z\}}$ in $G_{\{x,y,z\}}$. Hence, ...

CIP | Sampling with $X = \{x, z\}$ and $Y = \{y, z\}$

$A_{\{x,z\}} \in G_{\{x,z\}}$ and $C_{\{y,z\}} \in G_{\{y,z\}}$



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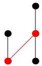
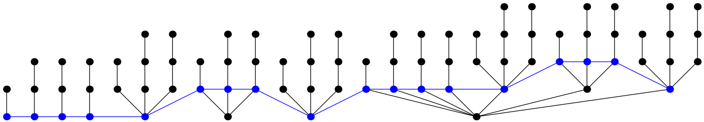


…there exists $B_{\{z\}} \in G_{\{z\}}$



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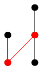

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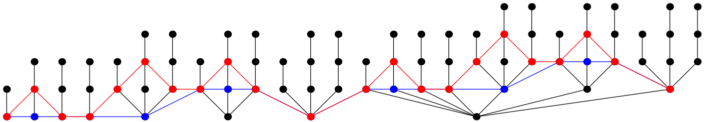

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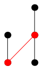

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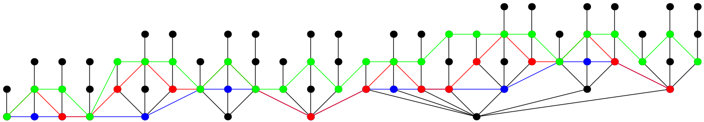

such that $A_{\{x,y,z\}} \leq B_{\{x,y,z\}} \leq C_{\{x,y,z\}}$ in $G_{\{x,y,z\}}$.

Interpolation Properties | Deductive (DIP)

$X \cap Y = Z$ and $X \cup Y = W$.

Definition

G has the DIP iff, for all r_X and t_Y st $r \vdash_G t$, there exists s_Z st $r \vdash_G s$ and $s \vdash_G t$.

Theorem ([KO09])

G has the DIP.

Corollary

For all $A_X \in G_X$ and $C_Y \in G_Y$ st $A_W \cap \top \subseteq C_W \cap \top$ in G_W, there is $B_Z \in G_Z$ st $A_W \cap \top \subseteq B_W \cap \top \subseteq C_W \cap \top$ in G_W.

DIP | Sampling with $X = \{x, z\}$ and $Y = \{y, z\}$

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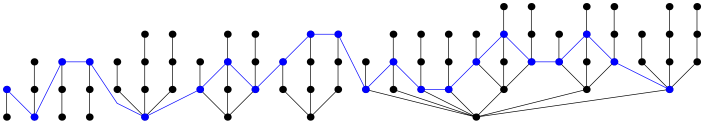

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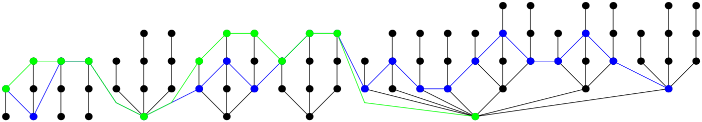

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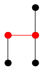

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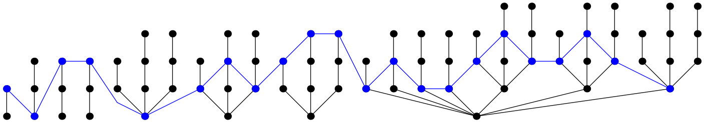

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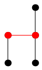

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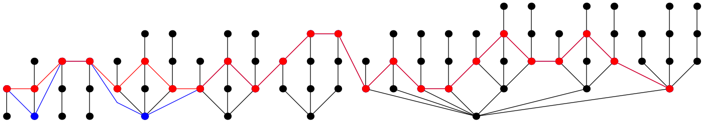

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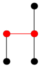

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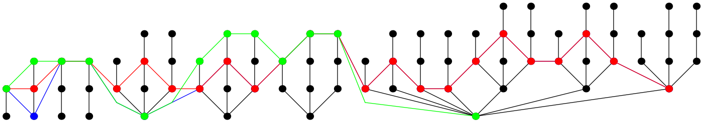

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- Construction Sketch

Basic Logic

(Hájek's) basic (propositional) logic, *BL*, is:

- the many-valued logic of all continuous triangular norms and their residuals;
- the substructural logic of commutative bounded integral divisible prelinear residuated lattices.

Deductive Interpolation

Fact ([M06])
BL has the DIP (not the CIP).

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Fact (Functional Representation [B08])
The elements of the Lindenbaum algebra of BL are suitable real functions, "described by combining the geometry of Łukasiewicz logic and the combinatorics of Gödel logic".

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Fact (Functional Representation [B08])
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Goal: Solve Problem 1 and Problem 2 in this setting.

Deductive Interpolation

$X \cap Y = Z$ and $X \cup Y = W$.

Fact ([M06])

For all r_X *and* t_Y *st* $r \vdash_{BL} t$, *there exists* s_Z *st* $r \vdash_{BL} s$ *and* $s \vdash_{BL} t$.

Fact

$r \vdash_{BL} t$ *iff* $r_W^{-1}(1) \subseteq t_W^{-1}(1) \subseteq [0,1]^W$.

Corollary

For all r_X *and* t_Y *st* $r_W^{-1}(1) \subseteq t_W^{-1}(1) \subseteq [0,1]^W$,
there exists s_Z *st* $r_W^{-1}(1) \subseteq s_W^{-1}(1) \subseteq t_W^{-1}(1) \subseteq [0,1]^W$.

Deductive Interpolation

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For all r_X and t_Y st $r \vdash_{BL} t$, there exists s_Z st $r \vdash_{BL} s$ and $s \vdash_{BL} t$.

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$r \vdash_{BL} t$ *iff* $r_W^{-1}(1) \subseteq t_W^{-1}(1) \subseteq [0,1]^W$.

Corollary

For all r_X and t_Y st $r_W^{-1}(1) \subseteq t_W^{-1}(1) \subseteq [0,1]^W$,
there exists s_Z st $r_W^{-1}(1) \subseteq s_W^{-1}(1) \subseteq t_W^{-1}(1) \subseteq [0,1]^W$.

Idea: Exploiting the functional representation of r_X and s_Z, construct a "strongest possible" interpolant s_Z, that is, a s_Z with smallest possible

$$[0,1]^X \supseteq s_X^{-1}(1) \supseteq r_X^{-1}(1).$$

Construction | Sampling with $X = \{x, z\}$ and $Y = \{y, z\}$.

Idea: Exploiting the functional representation of $r_{\{x,z\}}$ and $s_{\{z\}}$, construct a $s_{\{z\}}$ having the smallest $s^{-1}_{\{z\}}(1) \supseteq r^{-1}_{\{x,z\}}(1)$.

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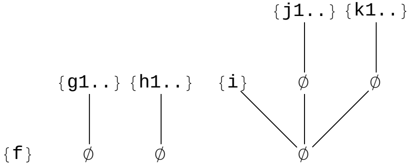

The given $r_{\{x,z\}} : [0,1]^{\{x,z\}} \to [0,1]$ decomposes into finitely many "Łukasiewicz functions" over a Gödel skeleton, satisfying certain constraints.

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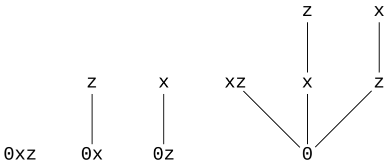

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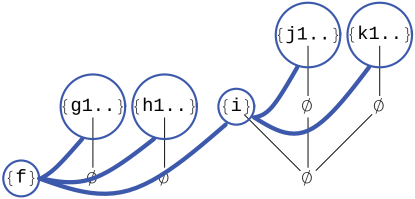
Construction | Sampling with $X = \{x, z\}$ and $Y = \{y, z\}$.

Idea: Exploiting the functional representation of $r_{\{x,z\}}$ and $s_{\{z\}}$, construct a $s_{\{z\}}$ having the smallest $s^{-1}_{\{z\}}(1) \supseteq r^{-1}_{\{x,z\}}(1)$.



The given $r_{\{x,z\}} : [0,1]^{\{x,z\}} \to [0,1]$ decomposes
into finitely many "Łukasiewicz functions"
over a Gödel skeleton,
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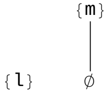

The target $s_{\{z\}} : [0, 1]^{\{z\}} \to [0, 1]$ decomposes
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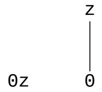

The target $s_{\{z\}} : [0,1]^{\{z\}} \to [0,1]$ decomposes
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Construction | *Sampling with* $X = \{x, z\}$ *and* $Y = \{y, z\}$.

Idea: Exploiting the functional representation of $r_{\{x,z\}}$ and $s_{\{z\}}$,
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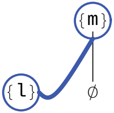

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Thanks!