Model Checking of Conjunctive Queries

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Outline

Outline

Model Checking | Example

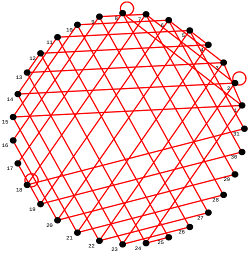


Figure: $G = (\{1, \ldots, 31\}, \{(a, b) \mid a + b \text{ perfect square}\})$.

Model Checking | Example

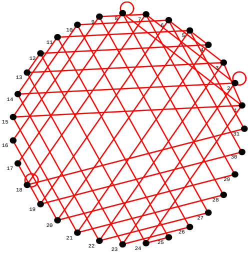
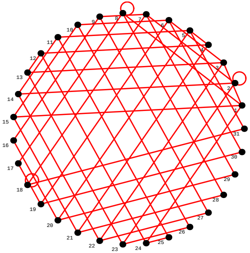


Figure: $G = (\{1, \ldots, 31\}, \{(a, b) \mid a + b \text{ perfect square}\})$. Is G Hamiltonian?

Model Checking | Example



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Problem HAMILTONICITY

Model Checking | Example

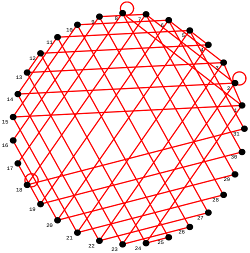

Figure: $G = (\{1, \ldots, 31\}, \{(a, b) \mid a + b \text{ perfect square}\})$. Is G Hamiltonian?

Problem HAMILTONICITY

Instance A *finite* relational $\{E\}$-structure $\mathbf{G} = (G, E^{\mathbf{G}})$ with $E^{\mathbf{G}} \subseteq G^2$.

Model Checking | Example

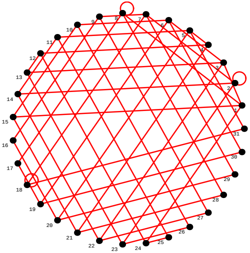

Figure: $G = (\{1, \ldots, 31\}, \{(a, b) \mid a + b \text{ perfect square}\})$. Is G Hamiltonian?

Problem	HAMILTONICITY
Instance	A *finite* relational $\{E\}$-structure $\mathbf{G} = (G, E^{\mathbf{G}})$ with $E^{\mathbf{G}} \subseteq G^2$.
Question	Is ϕ true in $\mathbf{G}$?

$$\phi = \exists X \exists Y (\text{“}X \text{ total order relation”} \wedge \text{“}Y \setminus \{(\max, \min)\} \text{ cover relation of } X\text{”} \\ \wedge \forall x \forall y (Yxy \rightarrow Exy)).$$

Model Checking | Logic

$\sigma = \{R_1, \ldots, R_k\}$ is a finite finitary relational signature, that is,
R_i is a finitary relation symbol of arity $\mathrm{ar}(R_i) \in \mathbb{N}$ for $i = 1, \ldots, k$.

$\mathbf{A} = (A, R_1^{\mathbf{A}}, \ldots, R_k^{\mathbf{A}})$ is a σ-structure, that is,
A is a *finite* nonempty set (universe),
$R_i^{\mathbf{A}} \subseteq A^{\mathrm{ar}(R_i)}$ is an $\mathrm{ar}(R_i)$-ary relation on A ($i = 1, \ldots, k$).

ϕ is a second-order σ-sentence.

Fact
For all σ-sentences ϕ and σ-structures $\mathbf{A}$,
$\mathbf{A} \models \phi$ xor $\mathbf{A} \not\models \phi$ (ϕ is true xor false in $\mathbf{A}$).

Model Checking | *Logic*

Example

$\sigma = \{E\}$ with $\mathrm{ar}(E) = 2$.

$\{\mathbf{A} \mid \mathbf{A}$ is $\{E\}$-structure$\}$ is the class of finite directed graphs.

Model Checking | Logic

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Model Checking | Logic

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$\exists x z_1 z_2 z_3 (Exz_1 \wedge Ez_1z_2 \wedge Ez_2z_3)$ is a *conjunctive query*,
or *primitive positive* first-order sentence ($\mathcal{PP} \subseteq \mathcal{FO}$).

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the class of finite simple (loopless undirected) graphs.

Model Checking | Problem

Problem $\textsc{ModelCheck}(\mathcal{L}, \mathcal{C})$,
where $\mathcal{L}$ is a class of σ-sentences,
and $\mathcal{C}$ is a class of σ-structures.

Instance A σ-sentence ϕ in $\mathcal{L}$ and a σ-structure $\mathbf{A} \in \mathcal{C}$.

Question $\mathbf{A} \models \phi$?

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Remark
ϕ *encoding has size* k, $\mathbf{A}$ *encoding has size* $n = (|A| + 1) + \sum_{R \in \sigma} |A|^{\mathrm{ar}(R)}$.

Model Checking | Combined Complexity

How hard is $\textsc{ModelCheck}(\mathcal{FO}, \mathcal{C})$ computationally,
where $\mathcal{FO}$ is the class of first-order σ-sentences,
and $\mathcal{C}$ is the class of finite σ-structures?

Model Checking | Combined Complexity

How hard is MODELCHECK($\mathcal{FO}, \mathcal{C}$) computationally,
where $\mathcal{FO}$ is the class of first-order σ-sentences,
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Theorem
The combined complexity of $\textsc{ModelCheck}(\mathcal{FO}, \mathcal{C})$ *is PSPACE-complete.*

Proof (Idea).
Wts $\{(\phi, \mathbf{A}) \in \mathcal{FO} \times \mathcal{C} \mid \mathbf{A} \models \phi\}$ *is PSPACE-complete.*
Checking $\mathbf{A} \models \phi$ *is feasible in space* $O(kn)$.
A quantified Boolean formula $Q_1 x_1 \cdots Q_n x_n \phi(x_1, \ldots, x_n)$ *is true iff*
$\mathbf{A} \models Q_1 x_1 \cdots Q_n x_n \phi'$ *where* $\mathbf{A} = (\{0, 1\}, P^{\mathbf{A}})$, $P^{\mathbf{A}} = \{1\}$, *and*
$\phi' = \phi[x_i / Px_i \mid i = 1, \ldots, n]$. □

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Feasible (and useful) restrictions to $\textsc{ModelCheck}(\mathcal{FO}, \mathcal{C})$?

Model Checking | Restricting MODELCHECK($\mathcal{FO}, \mathcal{C}$)

Restrictions to MODELCHECK($\mathcal{FO}, \mathcal{C}$) $= \{(\phi, \mathbf{A}) \in \mathcal{FO} \times \mathcal{C} \mid \mathbf{A} \models \phi\}$:

Effects on complexity:

Model Checking | *Restricting* MODELCHECK$(\mathcal{FO}, \mathcal{C})$

Restrictions to MODELCHECK$(\mathcal{FO}, \mathcal{C}) = \{(\phi, \mathbf{A}) \in \mathcal{FO} \times \mathcal{C} \mid \mathbf{A} \models \phi\}$:

1 Fix $\phi \in \mathcal{FO}$ as a parameter, ie study $\{\mathbf{A} \mid \mathbf{A} \models \phi\} \subseteq \mathcal{C}$.

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3. Study ModelCheck$(\mathcal{L}, \mathcal{S})$ with $\mathcal{L} \subseteq \mathcal{FO}$ or $\mathcal{S} \subseteq \mathcal{C}$.

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Effects on complexity:

1 *Data* complexity of $\textsc{ModelCheck}(\mathcal{FO}, \mathcal{C})$ is $O(kn^k)$ time.

2 *Expression* complexity of $\textsc{ModelCheck}(\mathcal{FO}, \mathcal{C})$ is PSPACE-complete.

1&3 Data complexity of $\textsc{ModelCheck}(\mathcal{FO}, \mathcal{T})$ is $O(f(k)n)$ time, where $\mathcal{T}$ is the class of finite (simple) trees.

Model Checking | Restricting $\textsc{ModelCheck}(\mathcal{FO}, \mathcal{C})$

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2&3 Expression complexity of $\textsc{ModelCheck}(\mathcal{PP}, \mathcal{B})$ is polytime, where $\mathcal{PP} \subseteq \mathcal{FO}$ is the class of conjunctive queries, and $\mathcal{B}$ is the class of finite (simple) bipartite graphs.

Model Checking | *Restricting* MODELCHECK($\mathcal{FO}, \mathcal{C}$)

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1&3 Data complexity of MODELCHECK($\mathcal{FO}, \mathcal{T}$) is $O(f(k)n)$ time, where $\mathcal{T}$ is the class of finite (simple) trees. *Not this talk.*

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Outline

(Nonuniform) CSP | Problem

Problem $\textsc{ModelCheck}(\mathcal{PP}, \mathcal{C})$,
where $\mathcal{PP} \subseteq \mathcal{FO}$ is the class of conjunctive queries (over σ),
and $\mathcal{C}$ is the class of σ-structures.

Instance A σ-sentence ϕ in $\mathcal{PP}$ and a σ-structure $\mathbf{A} \in \mathcal{C}$.

Question $\mathbf{A} \models \phi$?

Remark
$\mathrm{CSP}(\mathbf{A})$ *is a shortcut for* $\{\phi \mid \mathbf{A} \models \phi\} \subseteq \mathcal{PP}$.

CSP | Hardness

Theorem

The expression complexity of $\textsc{ModelCheck}(\mathcal{PP}, \mathcal{C})$ *is NP-complete.*

Proof (Idea).

Checking $\mathbf{A} \models \phi$ *with* $\phi \in \mathcal{PP}$ *is in NP (in fact, wrt to both k and n) for all* $\mathbf{A} \in \mathcal{C}$*, then the (combined so) expression complexity of* $\textsc{ModelCheck}(\mathcal{PP}, \mathcal{C})$ *is in NP.*

$3\textsc{Col} \leq_p^m \{\phi \mid \mathbf{C}_3 \models \phi\}$, *where* $\mathbf{C}_3 = (C_3, I^{\mathbf{C}_3})$,

$C_3 = \{\textcolor{red}{\bullet}, \textcolor{blue}{\bullet}, \textcolor{green}{\bullet}\}$, $I^{\mathbf{C}_3} = \{(\textcolor{red}{\bullet}, \textcolor{blue}{\bullet}), (\textcolor{red}{\bullet}, \textcolor{green}{\bullet}), (\textcolor{blue}{\bullet}, \textcolor{red}{\bullet}), (\textcolor{blue}{\bullet}, \textcolor{green}{\bullet}), (\textcolor{green}{\bullet}, \textcolor{red}{\bullet}), (\textcolor{green}{\bullet}, \textcolor{blue}{\bullet})\}$. $\square$

CSP | Hardness

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The expression complexity of MODELCHECK($\mathcal{PP}, \mathcal{C}$) *is NP-complete.*

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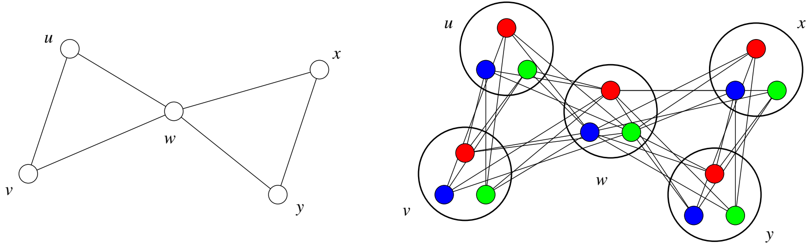

Figure: $G \mapsto \phi_G = \exists u \exists v \exists w \exists x \exists y (Iwu \wedge Iwv \wedge Iwx \wedge Iwy \wedge Iuv \wedge Ixy)$.

CSP | Tractability

Fact

CSP($\mathbf{C}_2$) *is in P, where* $\mathbf{C}_2 = (C_2, I^{\mathbf{C}_2})$,
$C_2 = \{\bullet, \bullet\}$, $I^{\mathbf{C}_2} = \{(\bullet, \bullet), (\bullet, \bullet)\}$.

Proof.

$\{\phi \mid \mathbf{C}_2 \models \phi\} \leq_p^m$ 2Col. $\square$

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CSP($\mathbf{C}_2$) *is in P, where* $\mathbf{C}_2 = (C_2, I^{\mathbf{C}_2})$,
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Proof.

$\{\phi \mid \mathbf{C}_2 \models \phi\} \leq_p^m$ 2COL. $\square$

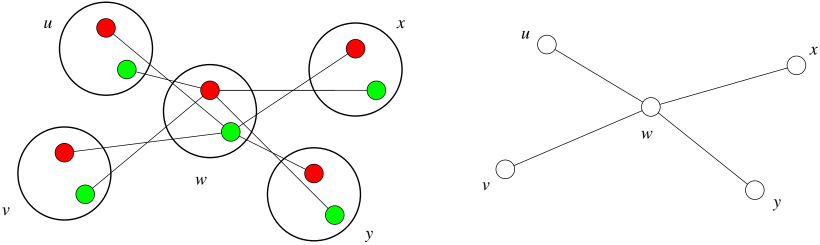

Figure: $\exists u \exists v \exists w \exists x \exists y (Iwu \wedge Iwv \wedge Iwx \wedge Iwy) = \phi \mapsto G_\phi$.

CSP | Dichotomy

Hard problem.

*Ie, avoids NPI, nonobvious if P≠NP.

CSP | Dichotomy

Hard problem. Find some useful easy cases…

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CSP | Dichotomy

Hard problem. Find some useful easy cases...

Theorem (Hell and Nešetřil)

Let $\mathcal{C}$ be the class of finite connected simple graphs and $\mathbf{A} \in \mathcal{C}$.
Then $\mathrm{CSP}(\mathbf{A})$ is in P if $\mathbf{A}$ is bipartite, and NP-complete otherwise.

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CSP | Dichotomy

Hard problem. Find some useful easy cases…

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Let $\mathcal{C}$ be the class of finite connected simple graphs and $\mathbf{A} \in \mathcal{C}$.
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…or better classify *all* easy cases.

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CSP | Dichotomy

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Let $\mathcal{C}$ be the class of finite connected simple graphs and $\mathbf{A} \in \mathcal{C}$.
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CSP | Dichotomy

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No reasonable line of attack until Jeavons (late 90s) discovers *polymorphisms*, a pertinent classification criterion for finite structures.

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CSP | Polymorphisms

If **A** "admits" only "trivial polymorphisms",
then CSP(**A**) is NP-complete.

Example (Trivial Polymorphisms ⇒ Hardness)

$\mathbf{C}_3$ admits only trivial polymorphisms, ie, *projection* polymorphisms,
$f \in \mathrm{Pol}(I^{\mathbf{C}_3})$ iff $f(x_1, \ldots, x_i, \ldots, x_n) = x_i$ for some $i \in [n]$.

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Theorem (Jeavons)

The complexity of $\mathrm{CSP}(\mathbf{A})$ *is characterized by the algebra* $\mathbb{A} = (A, \mathrm{Pol}(\mathbf{A}))$*:*

1. $\mathrm{Pol}(\mathbf{A}_1) \subseteq \mathrm{Pol}(\mathbf{A}_2)$ *implies* $\mathrm{CSP}(\mathbf{A}_2) \leq_m^p \mathrm{CSP}(\mathbf{A}_1)$;
2. $\mathrm{Pol}(\mathbf{A}_1) = \mathrm{Pol}(\mathbf{A}_2)$ *implies* $\mathrm{CSP}(\mathbf{A}_1) \equiv_m^p \mathrm{CSP}(\mathbf{A}_2)$.

CSP | Polymorphisms | Classification

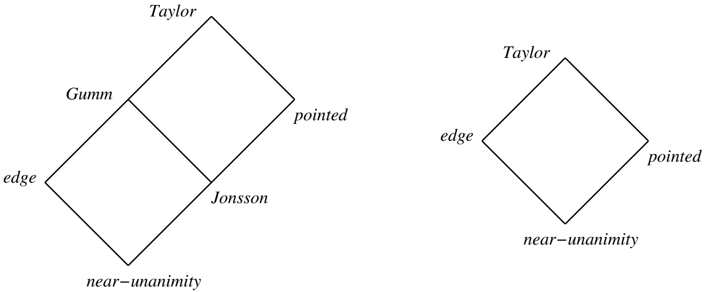
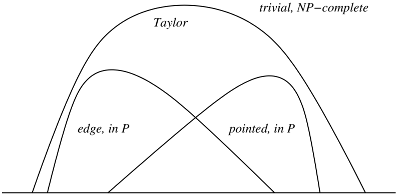


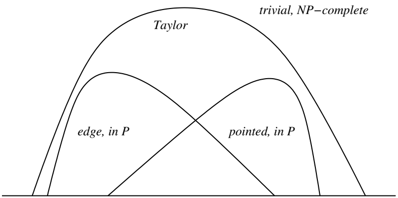
Figure: Nontrivial polimorphisms classified in 6 blocks, orderer by increasing triviality (left). Modulo Valeriote conjecture (right).

Taylor polymorphisms are maximally trivial among nontrivial polymorphisms.

CSP | Polymorphisms | Complexity



CSP | Polymorphisms | Complexity



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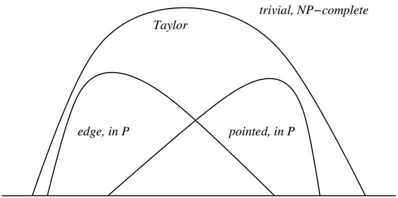

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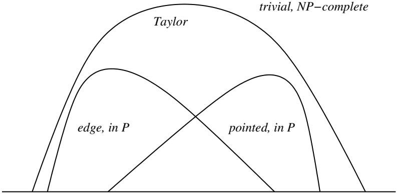

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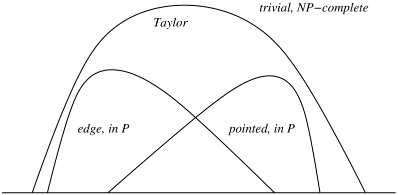

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CSP($\mathbf{A}$) is conjectured in P in the uncovered case (BJK conjecture).

Tractability | Algorithms

Known tractable cases of the CSP($\mathbf{A}$) rely on two algorithms:

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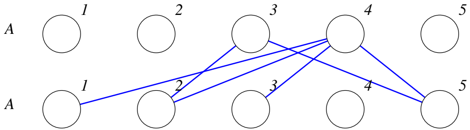
Tractability | Semilattice | Local Consistency

$t\colon A^2 \longrightarrow A$ is a *semilattice* operation if
$t(x,x) = x$, $t(x,y) = t(y,x)$, and $t(x,t(y,z)) = t(t(x,y),z)$.

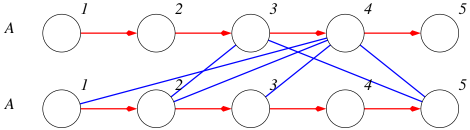
Theorem
If $\mathbf{A}$ *admits a semilattice polymorphism, then* $\mathbf{A}$ *has width* 1
(ie, the 1*-consistency algorithm decides* $\mathrm{CSP}(\mathbf{A})$ *in polytime).*

Tractability | *Semilattice* | *Local Consistency*

$\mathbf{A} = (A, E^{\mathbf{A}})$ where $A = \{1, 2, 3, 4, 5\}$ and
$E^{\mathbf{A}} = \{(3, a), (a, 3), (4, b), (b, 4) \mid a = 2, 5, b = 1, 2, 3, 5\} \subseteq A^2$:



$E^{\mathbf{A}}$ admits the semilattice polymorphism $t(x, y) = \max\{x, y\}$
($x \leq y$ iff $x \rightarrow y$):

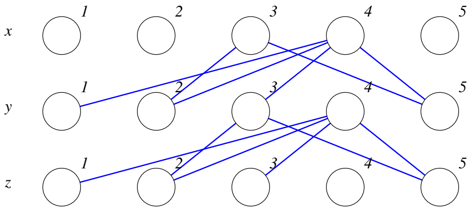


The 1-consistency algorithm decides $\mathrm{CSP}(\mathbf{A})$.

Tractability | Semilattice | Local Consistency

Example

Run 1-consistency on instance $\phi = \exists x \exists y \exists z (Exy \wedge Eyz)$ to CSP($\mathbf{A}$):

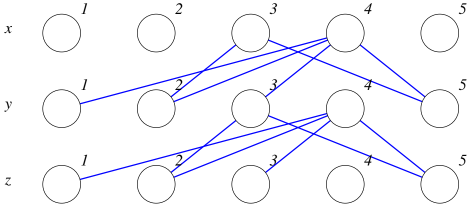


Tractability | Semilattice | Local Consistency

Example (Cont'd)

Initializing 1-consistency on instance $\phi = \exists x \exists y \exists z (Exy \wedge Eyz)$ to CSP($\mathbf{A}$) . . .

$\{x\}$



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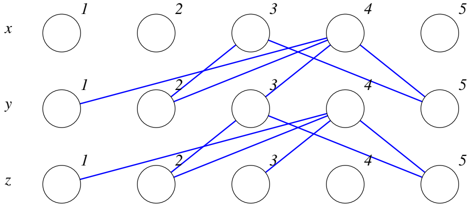
Initializing 1-consistency on instance $\phi = \exists x \exists y \exists z(Exy \wedge Eyz)$ to CSP$(\mathbf{A})\ldots$

$\{x\} \rightsquigarrow \phi$ has no constraints acting on x individually



Tractability | Semilattice | Local Consistency

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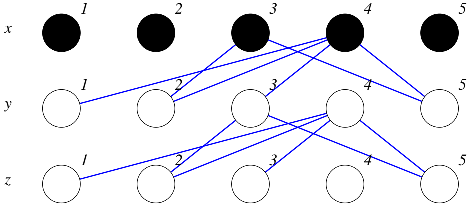

Tractability | Semilattice | Local Consistency

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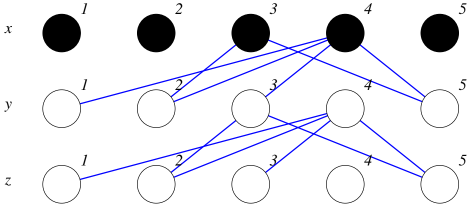

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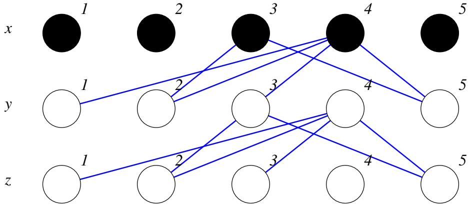

Tractability | Semilattice | Local Consistency

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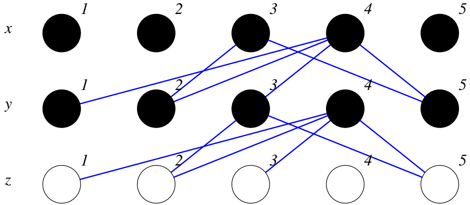

Tractability | Semilattice | Local Consistency

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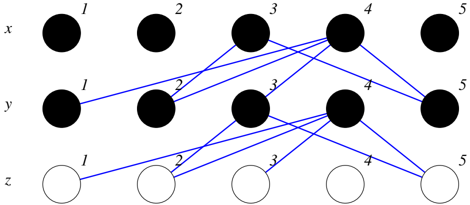

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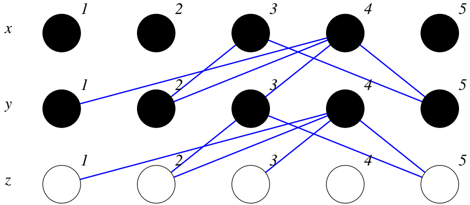

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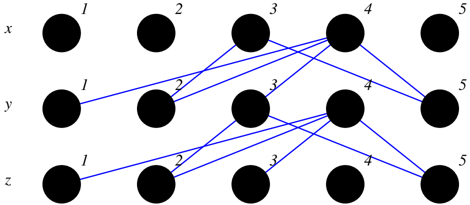

Tractability | Semilattice | Local Consistency

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Iterating 1-consistency on instance $\phi = \exists x \exists y \exists z (Exy \wedge Eyz)$ to $\mathrm{CSP}(\mathbf{A}) \ldots$

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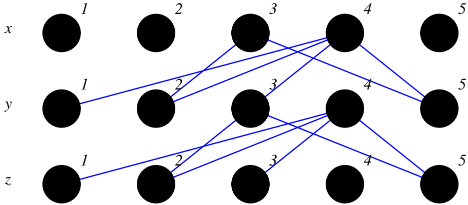

Tractability | Semilattice | Local Consistency

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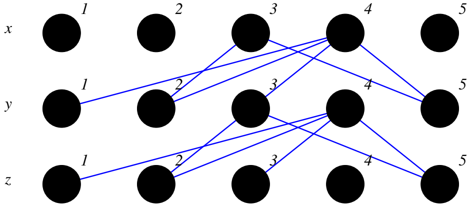
$\{x, y\} \rightsquigarrow \phi$ has constraint Exy

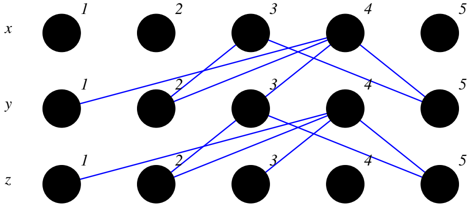


Tractability | Semilattice | Local Consistency

Example (Cont'd)

Iterating 1-consistency on instance $\phi = \exists x \exists y \exists z(Exy \wedge Eyz)$ to CSP($\mathbf{A}$) . . .

$\{x, y\} \rightsquigarrow \phi$ has constraint $Exy \rightsquigarrow (x, y) \mapsto E^{\mathbf{A}}$

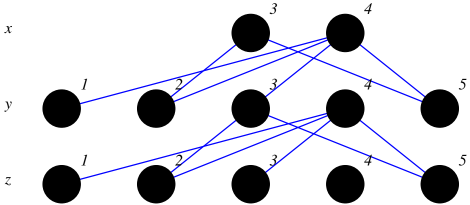


Tractability | Semilattice | Local Consistency

Example (Cont'd)

Iterating 1-consistency on instance $\phi = \exists x \exists y \exists z (Exy \wedge Eyz)$ to $\mathrm{CSP}(\mathbf{A})\ldots$

$\{x, y\} \rightsquigarrow \phi$ has constraint $Exy \rightsquigarrow (x, y) \mapsto E^{\mathbf{A}} \rightsquigarrow x \not\mapsto \{1, 2, 5\}$:

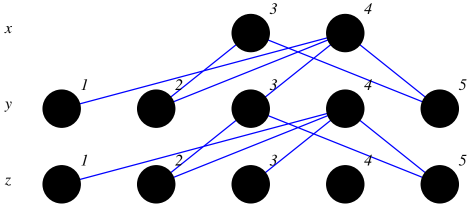


Tractability | Semilattice | Local Consistency

Example (Cont'd)

Iterating 1-consistency on instance $\phi = \exists x \exists y \exists z (Exy \wedge Eyz)$ to $\mathrm{CSP}(\mathbf{A})\ldots$

$\{x, z\}$

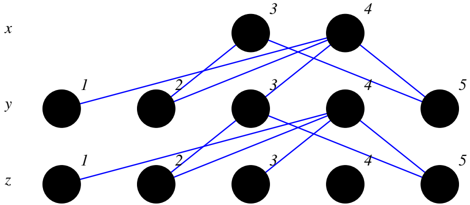


Tractability | Semilattice | Local Consistency

Example (Cont'd)

Iterating 1-consistency on instance $\phi = \exists x \exists y \exists z (Exy \wedge Eyz)$ to CSP($\mathbf{A}$) . . .

$\{x, z\} \rightsquigarrow \phi$ has no constraints on x and z

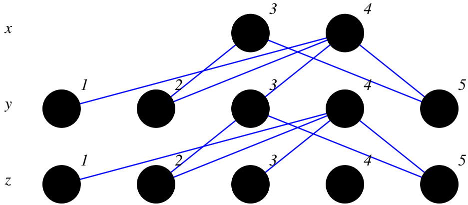


Tractability | Semilattice | Local Consistency

Example (Cont'd)

Iterating 1-consistency on instance $\phi = \exists x \exists y \exists z (Exy \wedge Eyz)$ to CSP($\mathbf{A}$) . . .

$\{x, z\} \rightsquigarrow \phi$ has no constraints on x and $z \rightsquigarrow$ no changes:

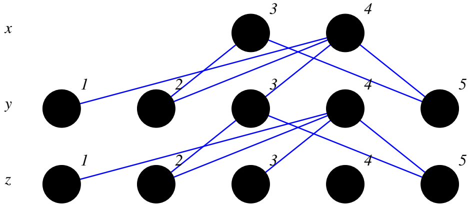


Tractability | Semilattice | Local Consistency

Example (Cont'd)

Iterating 1-consistency on instance $\phi = \exists x \exists y \exists z (Exy \wedge Eyz)$ to CSP($\mathbf{A}$) . . .

$\{y, x\}$

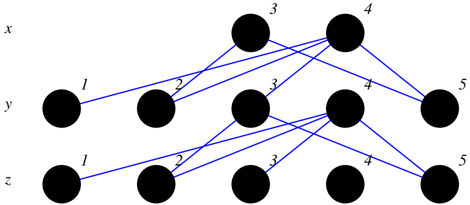


Tractability | Semilattice | Local Consistency

Example (Cont'd)

Iterating 1-consistency on instance $\phi = \exists x \exists y \exists z (Exy \wedge Eyz)$ to $\mathrm{CSP}(\mathbf{A})\ldots$

$\{y, x\} \rightsquigarrow \phi$ has constraint Exy



Tractability | Semilattice | Local Consistency

Example (Cont'd)

Iterating 1-consistency on instance $\phi = \exists x \exists y \exists z(Exy \wedge Eyz)$ to CSP($\mathbf{A}$) . . .

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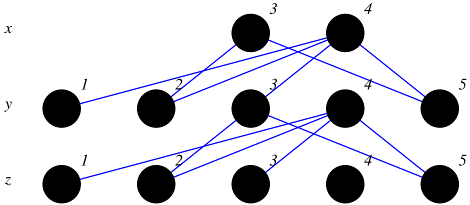
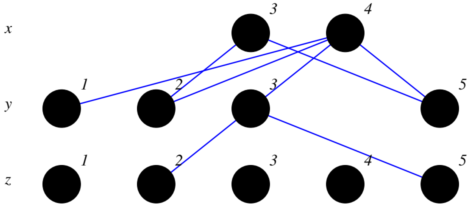


Tractability | Semilattice | Local Consistency

Example (Cont'd)

Iterating 1-consistency on instance $\phi = \exists x \exists y \exists z (Exy \wedge Eyz)$ to $\mathrm{CSP}(\mathbf{A}) \ldots$

$\{y, x\} \rightsquigarrow \phi$ has constraint $Exy \rightsquigarrow (x, y) \mapsto E^{\mathbf{A}} \rightsquigarrow y \not\mapsto \{4\}$:

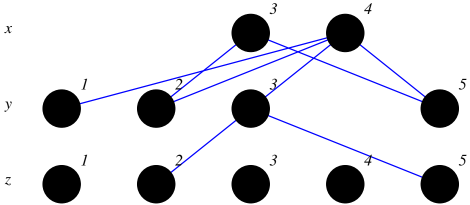


Tractability | Semilattice | Local Consistency

Example (Cont'd)

Iterating 1-consistency on instance $\phi = \exists x \exists y \exists z (Exy \wedge Eyz)$ to $\mathrm{CSP}(\mathbf{A}) \ldots$

$\{y, z\}$

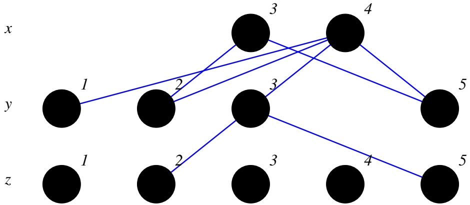


Tractability | Semilattice | Local Consistency

Example (Cont'd)

Iterating 1-consistency on instance $\phi = \exists x \exists y \exists z(Exy \wedge Eyz)$ to $\mathrm{CSP}(\mathbf{A}) \ldots$

$\{y, z\} \rightsquigarrow \phi$ has constraint Eyz

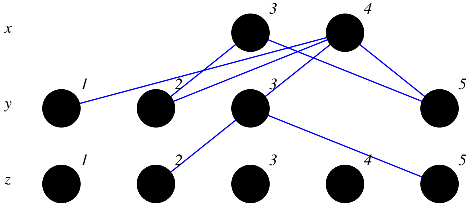


Tractability | Semilattice | Local Consistency

Example (Cont'd)

Iterating 1-consistency on instance $\phi = \exists x \exists y \exists z (Exy \wedge Eyz)$ to $\mathrm{CSP}(\mathbf{A}) \ldots$

$\{y, z\} \rightsquigarrow \phi$ has constraint $Eyz \rightsquigarrow (y, z) \mapsto E^{\mathbf{A}}$

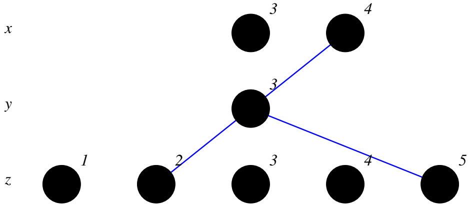


Tractability | Semilattice | Local Consistency

Example (Cont'd)

Iterating 1-consistency on instance $\phi = \exists x \exists y \exists z(Exy \wedge Eyz)$ to $\mathrm{CSP}(\mathbf{A}) \ldots$

$\{y, z\} \rightsquigarrow \phi$ has constraint $Eyz \rightsquigarrow (y, z) \mapsto E^{\mathbf{A}} \rightsquigarrow y \not\mapsto \{1, 2, 5\}$:

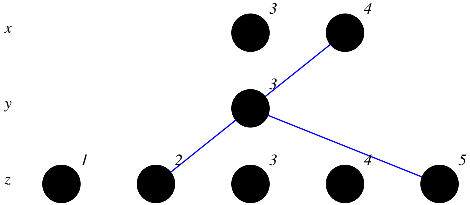


Tractability | Semilattice | Local Consistency

Example (Cont'd)

Iterating 1-consistency on instance $\phi = \exists x \exists y \exists z(Exy \wedge Eyz)$ to CSP($\mathbf{A}$) . . .

$\{z, x\}, \{z, y\}, \{x, y\}$

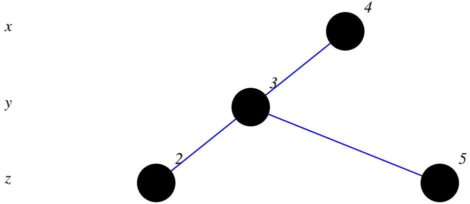


Tractability | Semilattice | Local Consistency

Example (Cont'd)

Iterating 1-consistency on instance $\phi = \exists x \exists y \exists z (Exy \wedge Eyz)$ to CSP($\mathbf{A}$) ...

$\{z, x\}, \{z, y\}, \{x, y\} \rightsquigarrow$ stabilizes at $A_x = \{4\}$, $A_y = \{3\}$, $A_z = \{2, 5\}$:



Tractability | Semilattice | Local Consistency

Example (Cont'd)

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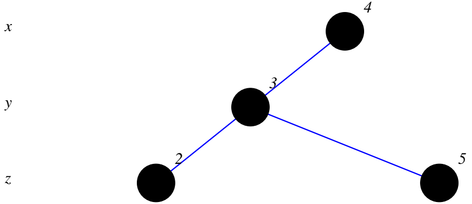

$A_x, A_y, A_z \neq \emptyset$

Tractability | Semilattice | Local Consistency

Example (Cont'd)

Iterating 1-consistency on instance $\phi = \exists x \exists y \exists z (Exy \wedge Eyz)$ to CSP($\mathbf{A}$) . . .

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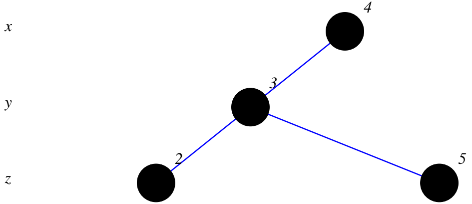

$A_x, A_y, A_z \neq \emptyset \rightsquigarrow$ return "Yes!"

Tractability | Semilattice | Local Consistency

Example (Cont'd)

Iterating 1-consistency on instance $\phi = \exists x \exists y \exists z (Exy \wedge Eyz)$ to CSP($\mathbf{A}$) . . .

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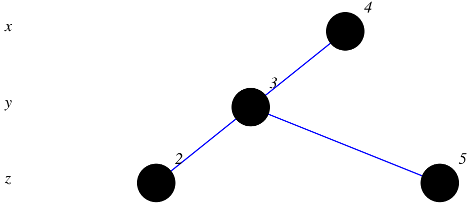

$A_x, A_y, A_z \neq \emptyset \rightsquigarrow$ return "Yes!" Correct?

Tractability | Semilattice | Local Consistency

Example (Cont'd)

$A_x = \{4\}, A_y = \{3\}, A_z = \{2, 5\}.$

Tractability | Semilattice | Local Consistency

Example (Cont'd)

$A_x = \{4\}, A_y = \{3\}, A_z = \{2, 5\}$.

$w \mapsto \bigvee A_w$ for $w \in \{x, y, z\}$ witnesses $\mathbf{A} \models \exists x \exists y \exists z (Exy \wedge Eyz)$.

Tractability | Semilattice | Local Consistency

Example (Cont'd)

$A_x = \{4\}, A_y = \{3\}, A_z = \{2, 5\}.$

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$$\left(\mathbf{A}, \begin{array}{l} x \mapsto 4 \\ y \mapsto 3 \end{array}\right) \models Exy \text{ iff } (4, 3) \in E^{\mathbf{A}}.$$

Tractability | Semilattice | Local Consistency

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Check:

Tractability | Semilattice | Local Consistency

Example (Cont'd)

$A_x = \{4\}, A_y = \{3\}, A_z = \{2, 5\}.$

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$$\left(\mathbf{A}, \begin{array}{l} x \mapsto 4 \\ y \mapsto 3 \end{array} \right) \models Exy \text{ iff } (4, 3) \in E^{\mathbf{A}}.$$

Check:

1. $x \mapsto 4$ extends to some $y \mapsto a \in A_y$ st $(4, a) \in E^{\mathbf{A}}$, and
   $y \mapsto 3$ extends to some $x \mapsto b \in A_x$ st $(b, 3) \in E^{\mathbf{A}}$, by 1-consistency;

Tractability | Semilattice | Local Consistency

Example (Cont'd)

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2.

$$\begin{array}{ccc} 4 & a & \in E^{\mathbf{A}} \\ b & 3 & \in E^{\mathbf{A}} \end{array}$$

Tractability | Semilattice | Local Consistency

Example (Cont'd)

$A_x = \{4\}, A_y = \{3\}, A_z = \{2, 5\}$.

$w \mapsto \bigvee A_w$ for $w \in \{x, y, z\}$ witnesses $\mathbf{A} \models \exists x \exists y \exists z (Exy \wedge Eyz)$.

$\left(\mathbf{A}, \begin{array}{c} x \mapsto 4 \\ y \mapsto 3 \end{array} \right) \models Exy$ iff $(4, 3) \in E^{\mathbf{A}}$.

Check:

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   $y \mapsto 3$ extends to some $x \mapsto b \in A_x$ st $(b, 3) \in E^{\mathbf{A}}$, by 1-consistency;
2. $$\begin{array}{ccc} t & t & \\ 4 & a & \in E^{\mathbf{A}} \\ b & 3 & \in E^{\mathbf{A}} \\ \| & \| & \\ 4 & 3 & \end{array}$$

   as t semilattice implies $t(\bigvee A_w, c) = t(c, \bigvee A_w) = \bigvee A_w$ for all $c \in A_w$ and $w \in \{x, y\}$,

Tractability | *Semilattice* | *Local Consistency*

Example (Cont'd)

$A_x = \{4\}, A_y = \{3\}, A_z = \{2, 5\}$.

$w \mapsto \bigvee A_w$ for $w \in \{x, y, z\}$ witnesses $\mathbf{A} \models \exists x \exists y \exists z (Exy \wedge Eyz)$.

$\left(\mathbf{A}, \begin{array}{l} x \mapsto 4 \\ y \mapsto 3 \end{array}\right) \models Exy$ iff $(4, 3) \in E^{\mathbf{A}}$.

Check:

1. $x \mapsto 4$ extends to some $y \mapsto a \in A_y$ st $(4, a) \in E^{\mathbf{A}}$, and $y \mapsto 3$ extends to some $x \mapsto b \in A_x$ st $(b, 3) \in E^{\mathbf{A}}$, by 1-consistency;
2. $$\begin{array}{ccc} t & t & \\ 4 & a & \in E^{\mathbf{A}} \\ b & 3 & \in E^{\mathbf{A}} \\ \| & \| & \Downarrow \\ 4 & 3 & \in E^{\mathbf{A}} \end{array}$$

   as t semilattice implies $t(\bigvee A_w, c) = t(c, \bigvee A_w) = \bigvee A_w$ for all $c \in A_w$ and $w \in \{x, y\}$, and t is a polymorphism of $E^{\mathbf{A}}$.

Tractability | Semilattice | Local Consistency

Theorem

If $\mathbf{A}$ *admits a semilattice polymorphism* t*, then* $\mathbf{A}$ *has width* 1
(ie, the 1*-consistency algorithm decides* $\mathrm{CSP}(\mathbf{A})$ *in polytime).*

Proof (Idea).

ϕ *instance of* $\mathrm{CSP}(\mathbf{A})$ *over variables* $\{x_1, \ldots, x_n\}$*. Run* 1*-consistency.*
If $A_{x_i} = \emptyset$ *for some* $i \in \{1, \ldots, n\}$*, then* $\mathbf{A} \not\models \phi$*.*
Otherwise $\mathbf{A} \models \phi$*, witnessed by* $x_i \mapsto \bigvee A_{x_i} \in A$
(A *is a complete join semilattice under* $a \leq b$ *iff* $t(a, b) = b$*,*
so every $S \subseteq A$ *has a least upper bound in* A*).* $\square$

Tractability | Mal'tsev | Dalmau Algorithm

$t\colon A^3 \to A$ is a *Mal'tsev* operation if $t(x,y,y) = t(y,y,x) = x$.

Theorem

If $\mathbf{A}$ *admits a Mal'tsev polymorphism,*
then the solution space of $\mathrm{CSP}(\mathbf{A})$ *admits a compact representation*
(and Dalmau algorithm decides $\mathrm{CSP}(\mathbf{A})$ *in polytime).*

Tractability | Mal'tsev | Dalmau Algorithm

Example

$\mathbf{A} = (A, R^{\mathbf{A}})$ where $A = \{0, 1, 2, 3, 4\}$, $R^{\mathbf{A}} = \{(a, b, c) \mid \mathbf{Z}_5 \models a + b = c\} \subseteq A^3$.

$R^{\mathbf{A}}$ admits the Mal'tsev polymorphism $t(x, y, z) = x - y + z$ (over $\mathbf{Z}_5$):

$$\begin{array}{ccccccc} a_1 & + & a_2 & = & a_3 & \in R^{\mathbf{A}} \\ b_1 & + & b_2 & = & b_3 & \in R^{\mathbf{A}} \\ c_1 & + & c_2 & = & c_3 & \in R^{\mathbf{A}} \end{array}$$

Tractability | Mal'tsev | Dalmau Algorithm

Example

$\mathbf{A} = (A, R^{\mathbf{A}})$ where $A = \{0, 1, 2, 3, 4\}$, $R^{\mathbf{A}} = \{(a, b, c) \mid \mathbf{Z}_5 \models a + b = c\} \subseteq A^3$.

$R^{\mathbf{A}}$ admits the Mal'tsev polymorphism $t(x, y, z) = x - y + z$ (over $\mathbf{Z}_5$):

$$\begin{array}{ccccccc}
a_1 & + & a_2 & = & a_3 & & \in R^{\mathbf{A}} \\
b_1 & + & b_2 & = & b_3 & & \in R^{\mathbf{A}} \\
c_1 & + & c_2 & = & c_3 & & \in R^{\mathbf{A}} \\
\| & & \| & & \| & & \\
(a_1 - b_1 + c_1) & & (a_2 - b_2 + c_2) & & (a_3 - b_3 + c_3) & &
\end{array}$$

Tractability | Mal'tsev | Dalmau Algorithm

Example

$\mathbf{A} = (A, R^{\mathbf{A}})$ where $A = \{0, 1, 2, 3, 4\}$, $R^{\mathbf{A}} = \{(a, b, c) \mid \mathbf{Z}_5 \models a + b = c\} \subseteq A^3$.

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$$\begin{array}{ccccccc}
\nrightarrow & & \nrightarrow & & \nrightarrow & & \\
a_1 & + & a_2 & = & a_3 & & \in R^{\mathbf{A}} \\
b_1 & + & b_2 & = & b_3 & & \in R^{\mathbf{A}} \\
c_1 & + & c_2 & = & c_3 & & \in R^{\mathbf{A}} \\
\| & & \| & & \| & & \Downarrow \\
(a_1 - b_1 + c_1) & + & (a_2 - b_2 + c_2) & = & (a_3 - b_3 + c_3) & & \in R^{\mathbf{A}}
\end{array}$$

Tractability | Mal'tsev | Dalmau Algorithm

Example

$\mathbf{A} = (A, R^{\mathbf{A}})$ where $A = \{0, 1, 2, 3, 4\}$, $R^{\mathbf{A}} = \{(a, b, c) \mid \mathbf{Z}_5 \models a + b = c\} \subseteq A^3$.

$R^{\mathbf{A}}$ admits the Mal'tsev polymorphism $t(x, y, z) = x - y + z$ (over $\mathbf{Z}_5$):

$$\begin{array}{ccccccc}
t & & t & & t & & \\
a_1 & + & a_2 & = & a_3 & & \in R^{\mathbf{A}} \\
b_1 & + & b_2 & = & b_3 & & \in R^{\mathbf{A}} \\
c_1 & + & c_2 & = & c_3 & & \in R^{\mathbf{A}} \\
\Vert & & \Vert & & \Vert & & \Downarrow \\
(a_1 - b_1 + c_1) & + & (a_2 - b_2 + c_2) & = & (a_3 - b_3 + c_3) & & \in R^{\mathbf{A}}
\end{array}$$

Dalmau algorithm decides $\mathrm{CSP}(\mathbf{A})$ in polytime.

Tractability | Mal'tsev | Dalmau Algorithm

Example (Cont'd)

$\phi = \exists x \exists y \exists z (Rxzx \wedge Rxyy)$ instance of CSP$(\mathbf{A})$ with 2 constraints.

$$C_1 = \left[\begin{array}{ccccc|c} 0 & 1 & 2 & 3 & 4 & \\ a & a & a & a & a & a \in \{0,1,2,3,4\} \\ 0 & 0 & 0 & 0 & 0 & \end{array} \right] \text{ solutions to } Rxzx.$$

$$C_2 = \left[\begin{array}{ccccc|c} 0 & 0 & 0 & 0 & 0 & \\ 0 & 1 & 2 & 3 & 4 & a \in \{0,1,2,3,4\} \\ a & a & a & a & a & \end{array} \right] \text{ solutions to } Rxyy.$$

$$S_2 = C_1 \cap C_2 = \left[\begin{array}{c|c} 0 & \\ a & a \in \{0,1,2,3,4\} \\ 0 & \end{array} \right] \text{ solutions to } Rxzx \wedge Rxyy.$$

$S_2 \neq \emptyset$.

$\mathbf{A} \models \phi$.

Tractability | Mal'tsev | Dalmau Algorithm

ϕ instance of CSP($\mathbf{A}$) with m constraints and k variables,

$$\phi = \exists x_1 \cdots \exists x_k(\phi_1(x_1, \ldots, x_k) \wedge \cdots \wedge \phi_m(x_1, \ldots, x_k)).$$

$C_i \subseteq A^{\{x_1, \ldots, x_k\}}$ solutions ith constraint ($i \in \{1, \ldots, m\}$).
$S_i = C_1 \cap \cdots \cap C_i$ solutions first i constraints ($i \in \{0, 1, \ldots, m\}$).
S_m solutions of ϕ (ie, any $\mathbf{a} \in S_m$ witnesses $\mathbf{A} \models \phi$).

NAIVEALGORITHM(ϕ)

$S_0 = A^{\{x_1, \ldots, x_k\}}$
for $i = 1, \ldots, m$
  $S_i = S_{i-1} \cap C_i$
  if ($S_i = \emptyset$) **output** $\mathbf{A} \not\models \phi$
endfor
output $\mathbf{A} \models \phi$

Tractability | Mal'tsev | Dalmau Algorithm

ϕ instance of CSP($\mathbf{A}$) with m constraints and k variables,

$$\phi = \exists x_1 \cdots \exists x_k (\phi_1(x_1, \ldots, x_k) \wedge \cdots \wedge \phi_m(x_1, \ldots, x_k)).$$

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$S_i = C_1 \cap \cdots \cap C_i$ solutions first i constraints ($i \in \{0, 1, \ldots, m\}$).
S_m solutions of ϕ (ie, any $\mathbf{a} \in S_m$ witnesses $\mathbf{A} \models \phi$).

NaiveAlgorithm(ϕ)
$S_0 = A^{\{x_1, \ldots, x_k\}}$
for $i = 1, \ldots, m$
  $S_i = S_{i-1} \cap C_i$
  if ($S_i = \emptyset$) **output** $\mathbf{A} \not\models \phi$
endfor
output $\mathbf{A} \models \phi$

Correct,

Tractability | Mal'tsev | Dalmau Algorithm

ϕ instance of $\mathrm{CSP}(\mathbf{A})$ with m constraints and k variables,

$$\phi = \exists x_1 \cdots \exists x_k(\phi_1(x_1, \ldots, x_k) \wedge \cdots \wedge \phi_m(x_1, \ldots, x_k)).$$

$C_i \subseteq A^{\{x_1,\ldots,x_k\}}$ solutions ith constraint ($i \in \{1, \ldots, m\}$).
$S_i = C_1 \cap \cdots \cap C_i$ solutions first i constraints ($i \in \{0, 1, \ldots, m\}$).
S_m solutions of ϕ (ie, any $\mathbf{a} \in S_m$ witnesses $\mathbf{A} \models \phi$).

NAIVEALGORITHM(ϕ)
$S_0 = A^{\{x_1,\ldots,x_k\}}$ ▶ size $O(2^k)$
for $i = 1, \ldots, m$
  $S_i = S_{i-1} \cap C_i$ ▶ size $O(2^k)$
  if ($S_i = \emptyset$) **output** $\mathbf{A} \not\models \phi$
endfor
output $\mathbf{A} \models \phi$

Correct, but not polytime!

Tractability | Mal'tsev | Dalmau Algorithm

ϕ instance of $\mathrm{CSP}(\mathbf{A})$ with m constraints and k variables,

$$\phi = \exists x_1 \cdots \exists x_k (\phi_1(x_1, \ldots, x_k) \wedge \cdots \wedge \phi_m(x_1, \ldots, x_k)).$$

$C_i \subseteq A^{\{x_1, \ldots, x_k\}}$ solutions ith constraint.
$S'_i = C_1 \cap \cdots \cap C_i$ *compact representation* of solutions first i constraints.
S'_m *compact representation* of solutions to ϕ.

```
DALMAUALGORITHM(φ)
    S'_0 = (A^{x_1,...,x_k})'
    for i = 1, ..., m
        S'_i = S'_{i-1} ∩ C_i
        if (S'_i = ∅) output A ⊭ φ
    endfor
    output A ⊨ φ
```

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  if ($S'_i = \emptyset$) **output** $\mathbf{A} \not\models \phi$
endfor
output $\mathbf{A} \models \phi$

Correct, and (with clever implementation) polytime!

Tractability | Mal'tsev | Dalmau Algorithm

If a k-ary relation $R \subseteq A^k$ on a set A admits a Mal'tsev polymorphism t, then R admits a *compact representation* R', that is a $R' \subseteq R$ such that:

1. $|R'| \leq 2|A|k = O(k)$ versus $|R| \leq |A|^k = O(2^k)$;
2. R is equal to $t(R')$, the smallest relation containing R' closed under t ($\mathbf{a}, \mathbf{b}, \mathbf{c} \in t(R')$ implies $(t(\mathbf{a}_1, \mathbf{b}_1, \mathbf{c}_1), \ldots, t(\mathbf{a}_k, \mathbf{b}_k, \mathbf{c}_k)) \in t(R')$), ie, each $\mathbf{a} \in R$ is derivable from R' using the Mal'tsev polymorphism t of R.

R' and t provide a poly(k) size encoding for R!

R' constructed by: For all $(i, a, b) \in \{1, \ldots, k\} \times A \times A$,
there exist $\mathbf{a}, \mathbf{b} \in R$ such that $\mathbf{a}_j = \mathbf{b}_j$ for all $j < i$, $\mathbf{a}_i = a$, $\mathbf{b}_i = b$,
iff there exist $\mathbf{a}', \mathbf{b}' \in R'$ such that $\mathbf{a}'_j = \mathbf{b}'_j$ for all $j < i$, $\mathbf{a}'_i = a$, $\mathbf{b}'_i = b$.

Tractability | Mal'tsev | Dalmau Algorithm

Example ($k = 3$, $A = \{0, 1, 2, 3, 4\}$)

$R = \{0, 1, 2, 3, 4\}^3$ admits Mal'tsev polymorphism $t(x, y, z) = x - y + z$,

$$t(x, y, y) = t(y, y, x) = x.$$

Define $R' = \{(a, 0, 0), (0, a, 0), (0, 0, a) \mid a \in \{0, 1, 2, 3, 4\}\} \subseteq R$.

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$$\begin{array}{cccl} 2 & 0 & 0 & \in R' \\ 0 & 0 & 0 & \in R' \\ 0 & 3 & 0 & \in R' \end{array}$$

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$$\begin{array}{cccc}
t & t & t & \\
2 & 0 & 0 & \in R' \\
0 & 0 & 0 & \in R' \\
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\| & \| & \| & \\
2 & 3 & 0 &
\end{array}$$

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0 & 3 & 0 & \in R' & & 0 & 0 & 1 & \in R' \\
\| & \| & \| & & & \| & \| & \| & \\
2 & 3 & 0 & & & 2 & 3 & 1 & \in R
\end{array}$$

Outline

Related Problems

1. If a finite digraph admits Gumm polymorphisms, then it admits edge polymorphisms (Valeriote conjecture)?
2. Classify the complexity of the CSP: P/NP-complete?
3. Classify the complexity of the *quantified* CSP: P/NP-complete/PSPACE-complete?
4. Classify the complexity of the *valued* CSP.

Containment of Conjunctive Queries

Problem PPDEFCONT($\mathcal{C}$),
where $\mathcal{C}$ is the class of finite σ-structures.

Instance Two conjunctive queries ϕ_1 and ϕ_2, over σ,
with free variables $x_1, \ldots, x_n$, and a σ-structure $\mathbf{A}$.

Question $\mathbf{A} \models \phi_1 \subseteq \phi_2$, ie,
for all $\mathbf{b} \in A^{\{x_1,\ldots,x_n\}}$, $\mathbf{A}, \mathbf{b} \models \phi_1$ implies $\mathbf{A}, \mathbf{b} \models \phi_2$?

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Theorem (B, Chen and Valeriote)

Let $\mathcal{C}$ be the class of finite σ-structures and $\mathbf{A} \in \mathcal{C}$.
Then $\text{PPDefCont}(\mathbf{A}) = \{(\phi_1, \phi_2) \mid \mathbf{A} \models \phi_1 \subseteq \phi_2\}$ *is:*

1. Π_2^p*-complete, if* $\mathbf{A}$ *omits Taylor polymorphisms;*
2. *coNP-complete, if* $\mathbf{A}$ *admits Taylor but omits Gumm polymorphisms;*
3. *in P, if* $\mathbf{A}$ *admits edge polymorphisms.*

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Remark

Complete trichotomy classification modulo Valeriote and BJK conjecture.

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Danke für Ihre Aufmerksamkeit.