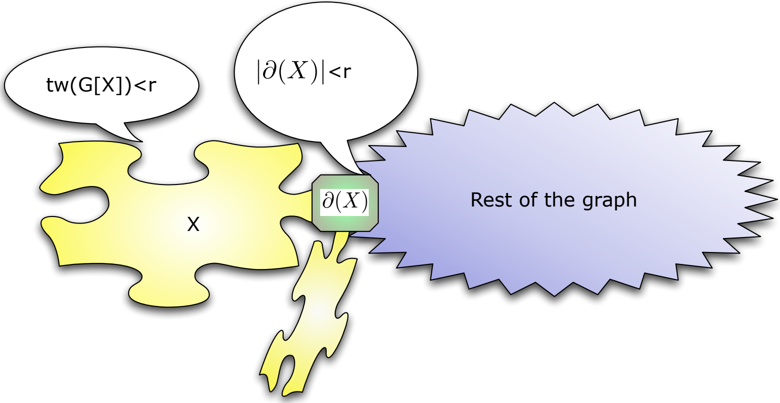
FEDOR V. FOMIN

Protrusions in Graphs and their Applications

Vienna

Worker 2011

Protrusion

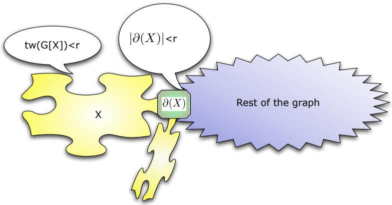


Protrusion: Definition

Given a graph G and $S \subseteq V(G)$, we define $\partial_G(S)$ as the set of vertices in S that have a neighbor in $V(G) \setminus S$.

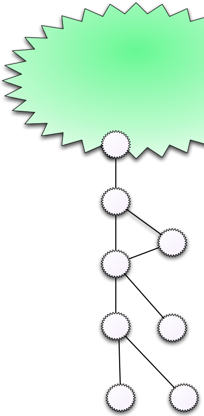
Definition

[r-**protrusion**] Given a graph G, we say that a set $X \subseteq V(G)$ is an *r-protrusion* of G if $\mathrm{tw}(G[X]) \leq r$ and $|\partial(X)| \leq r$.

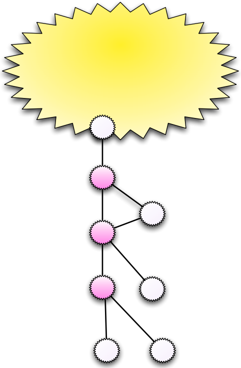


Example

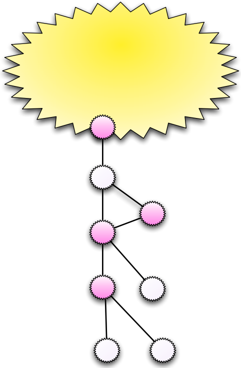
Graph G with a 2-protrusion. Does G have a vertex cover of size k?



Example

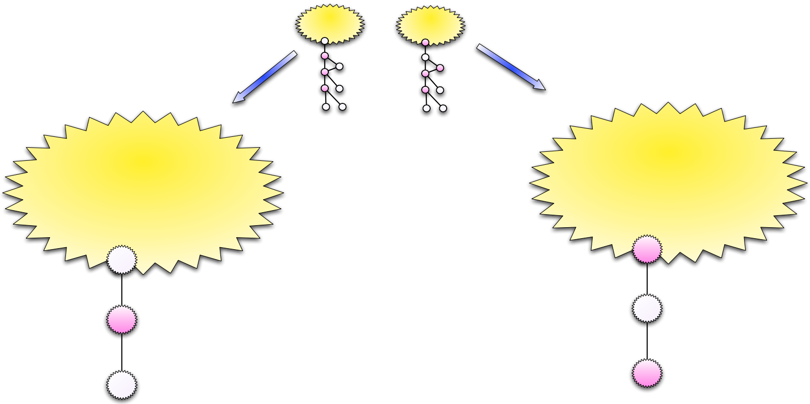


A vertex cover in G can look like that



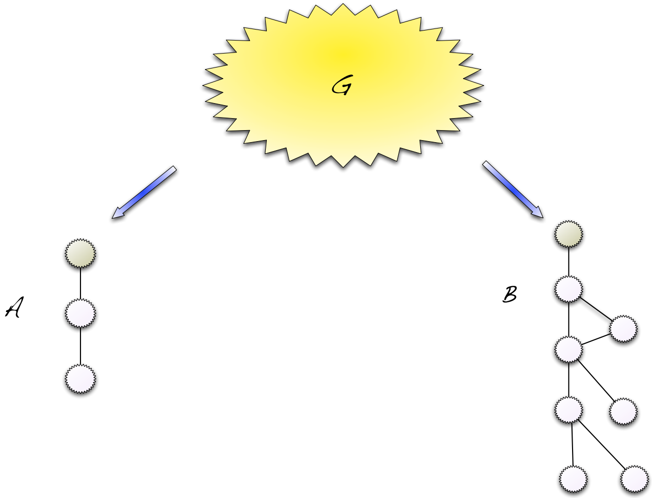
or like that

Example

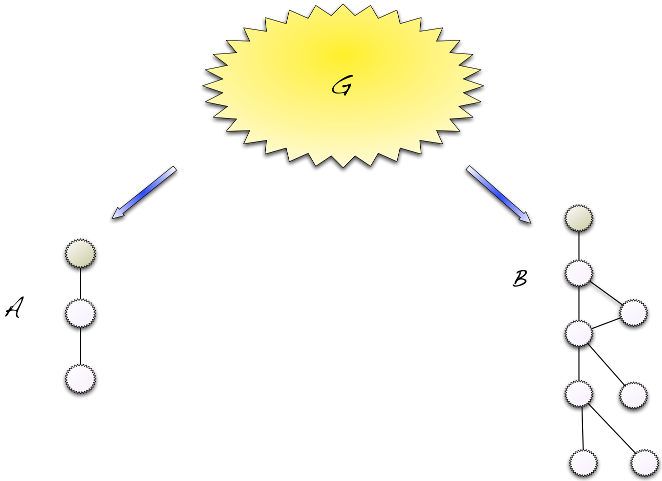


New graph has a vertex cover of size k−2 if and only if G has a vertex cover of size k

Or a bit differently



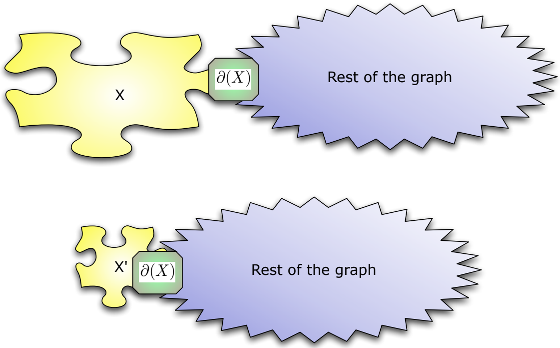
For any graph G, G+A has a vertex cover of size k if and only if G+B has a vertex cover of size k+2



For any graph G, G+A has a vertex cover of size k <=> G+B has a vertex cover of size k+2

How protrusions work for parameterized problem Π

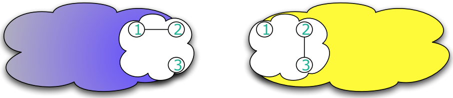
If the size of protrusion X is larger than some constant x (depending only on Π), it is possible to replace X by a protrusion X' of size $x' < x$ such that the solution for Π remains the "same" on the new graph.



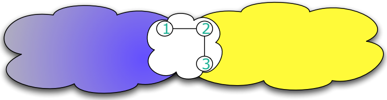
[t-Boundaried Graphs]

A *t-boundaried graph* is a graph $G = (V, E)$ with t distinguished vertices, uniquely labeled from 1 to t.

Gluing G_1 and G_2: $G_1 \oplus G_2$ the t-boundaried graph obtained by taking the disjoint union of G_1 and G_2 and identifying each vertex of $\partial(G_1)$ with the vertex of $\partial(G_2)$ with the same label.



G_1 G_2



$G_1 \oplus G_2$

Equivalence relation $G_1 \equiv_\Pi G_2$

For a parameterized problem Π and two t-boundaried graphs G_1 and G_2, we say that $G_1 \equiv_\Pi G_2$ if there exists a constant c such that for every t-boundaried graph G and for every integer k,

- $(G_1 \oplus G, k) \in \Pi$ if and only if $(G_2 \oplus G, k + c) \in \Pi$.

Finite Integer Index [Bodlaender and van Antwerpen-de Fluiter, 2001]

A parameterized problem Π has finite integer index in a graph class $\mathcal{G}$ if for every t there exists a finite set $\mathcal{S}$ of t-boundaried graphs such that $\mathcal{S} \subseteq \mathcal{G}$ and for any t-boundaried graph G_1 there exists $G_2 \in \mathcal{S}$ such that $G_2 \equiv_\Pi G_1$.

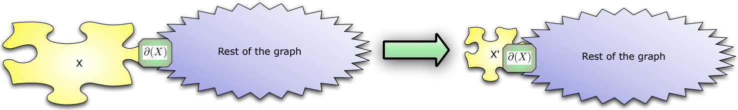
Problems with Finite Integer Index

Dominating Set, r-Dominating Set, q-Threshold Dominating Set, Efficient Dominating Set, Vertex Cover, Connected r-Dominating Set, Connected Vertex Cover, Minimum-Vertex Feedback Edge Set, Vertex-$\mathcal{H}$-Covering, Minimum Maximal Matching, Connected Dominating Set, Vertex-$\mathcal{S}$-Covering, Clique-Transversal, Almost-Outerplanar, Feedback Vertex Set, Cycle Domination, Edge Dominating Set, Independent Set, Induced d-Degree Subgraph, r-Scattered Set, Min Leaf Spanning Tree, Induced Matching, Triangle Packing, Cycle Packing, Maximum Full-Degree Spanning Tree, Vertex-$\mathcal{H}$-Packing, Vertex-$\mathcal{S}$-Packing...

Why protrusions work:

Lemma (Bodlaender, Fomin, Lokshtanov, Penninks, Saurabh, Thilikos, 2009)

Let Π be a problem that has finite integer index. Then there exists a computable function $\gamma : \mathbb{N} \to \mathbb{N}$ and an algorithm that, given an instance (G, k) and an r-protrusion X of G of size at least $\gamma(r)$, runs in $O(|X|)$ time and outputs an instance (G^*, k^*) such that $|V(G^*)| < |V(G)|$, $k^* \leq k$, and $(G^*, k^*) \in \Pi$ if and only if $(G, k) \in \Pi$.



Some history

Finite Integer Index defined by Bodlaender and van Antwerpen-de Fluiter (2001) and de Fluiter (1997)

Similar to the notion of finite state [Abrahamson and Fellows 1993; Borie et al. 1992; Courcelle 1990]

Talk overview:

- Compactness
- Bidimensionality
- Hitting forbidden minors

PART I: PLANAR GRAPHS and COMPACTNESS

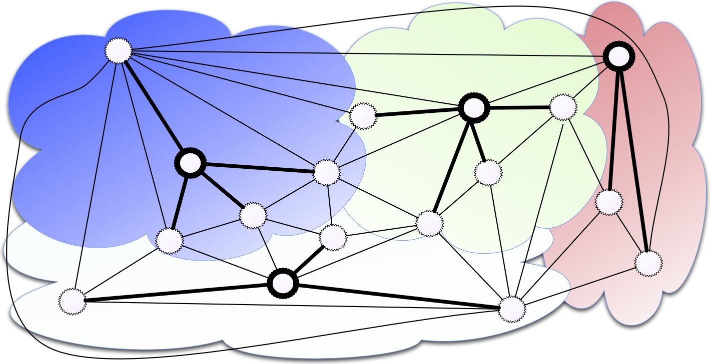
Let r be a (fixed) integer integer.

Claim: For any $a > 0$ there is b such that every planar graph

- covered by k balls of radius r
- with at least bk vertices

has a p-protrusion of size at least a, where p depends only from r.

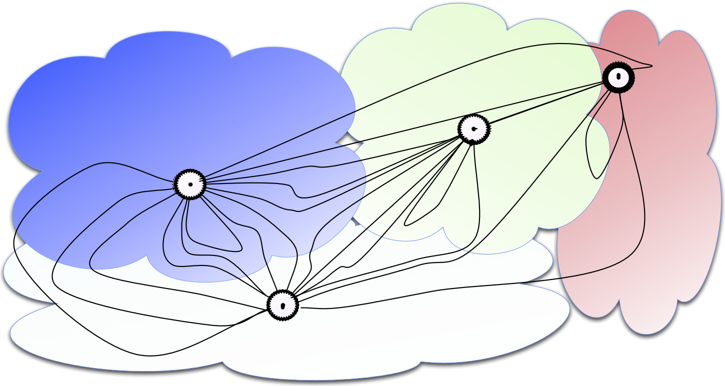
Sketch of the proof



- Triangulate a graph G

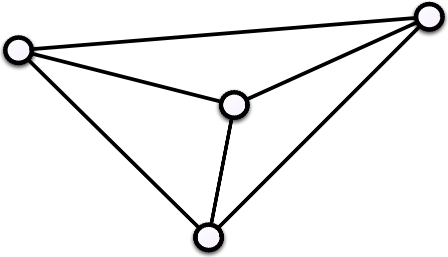
- For each ball, pick a BFS tree rooted in the centre of the ball

Sketch of the proof



- Contract edges of trees
- Remove "useless" parallel edges and loops

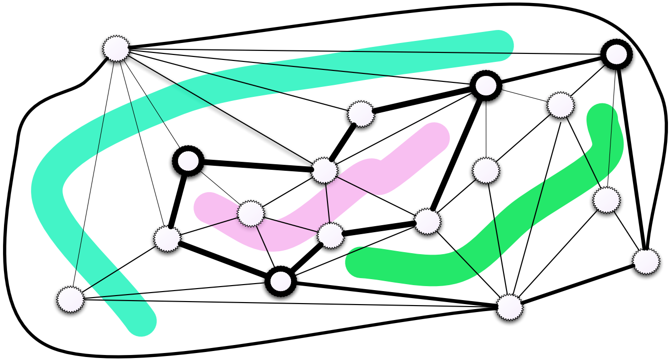
Sketch of the proof



- The number of faces in this graph is at most 3k

- Every face corresponds to a protrusion

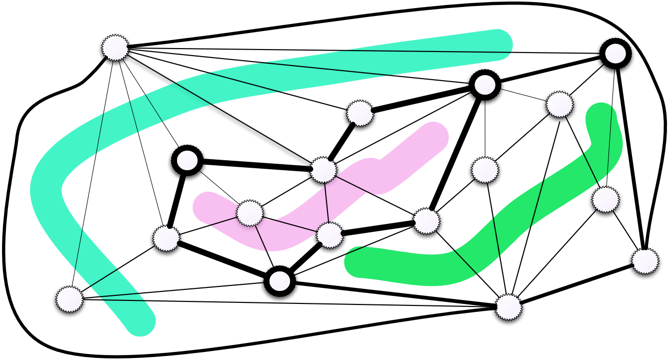
Sketch of the proof



- Each region is bounded by at most 12r vertices

- Each region is of diameter at most r, and hence of treewidth at most 3r

Sketch of the proof



- Each region is protrusion and every vertex is in a region
- There are at most 3k regions, thus if G has more than 3ak vertices, it has a 12r-protrusion of size more than a

Let $\mathcal{G}$ be the set of planar graphs.

Definition

A parameterized problem $\Pi \subseteq \mathcal{G} \times \mathbb{N}$ is *compact* if there exist an integer r such that for all $(G, k) \in \Pi$, there is a planar embedding of G and a set $S \subseteq V(G)$ such that

- $|S| \leq r \cdot k$, and
- $\mathbf{B}_G^r(S) = V(G)$.

$\mathbf{B}_G^r(S)$ — vertices at distance at most r from S in the vertex-face metrics of the graph.

Example: p-Dominating Set is compact for $r = 1$.

Theorem [Bodlaender, FF, Lokshtanov, Penninks, Saurabh, Thilikos, 2009]:

Let Π be a compact problem with FII. Then Π admits a linear kernel on planar graphs.

Proof

Let (G, k) be an instance of Π.

- Π is compact, hence G can be covered by kr balls, each of radius r.
- Pick up a constant a to be larger than the maximum size of a graph from the set of representatives $(\Pi, 12r)$, t-boundaried with $t \leq 12r$.
- If G has more than $b \cdot k$ vertices, it has a protrusion of size larger than a. Replace protrusion by a graph of size at most a.

An extension

Definition

A parameterized problem $\Pi \subseteq \mathcal{G} \times \mathbb{N}$ is *quasi-compact* if there exist an integer r such that for all $(G, k) \in \Pi$, there is a planar embedding of G and a set $S \subseteq V(G)$ such that $|S| \leq r \cdot k$ and $\mathrm{tw}(G \setminus \mathbf{B}_G^r(S)) \leq r$.

$\mathbf{B}_G^r(S)$ — vertices at distance at most r from S in the vertex-face metrics of the graph.

An extension

Definition

A parameterized problem $\Pi \subseteq \mathcal{G} \times \mathbb{N}$ is *quasi-compact* if there exist an integer r such that for all $(G, k) \in \Pi$, there is a planar embedding of G and a set $S \subseteq V(G)$ such that $|S| \leq r \cdot k$ and $\mathrm{tw}(G \setminus \mathbf{B}_G^r(S)) \leq r$.

$\mathbf{B}_G^r(S)$ — vertices at distance at most r from S in the vertex-face metrics of the graph.

Example: Feedback Vertex Set is quasi-compact for $r = 1$.

[Bodlaender, FF, Lokshtanov, Penninks, Saurabh, Thilikos, 2009]:

Problems with Quasi-compactness + FII admit linear kernels on planar graphs.

[Bodlaender, FF, Lokshtanov, Penninks, Saurabh, Thilikos, 2009]:

Problems with Quasi-compactness + FII admit linear kernels on planar graphs.

Can be extended to graphs of bounded genus

Problems that are Quasi-Compact and FII:

Dominating Set, r-Dominating Set, Vertex Cover, Connected r-Dominating Set, Connected Vertex Cover, Minimum-Vertex Feedback Edge Set, Minimum Maximal Matching, Connected Dominating Set, Almost Outerplanar, Feedback Vertex Set, Cycle Domination, Edge Dominating Set, Clique Transversal, different packing and covering problems...

PART II: Minor-free graphs and Bidimensionality

Another approach: Vertex Cover in planar graphs

Let G be a planar graph with vertex cover k

What we want: Show that there is a set S of size $O(k)$ such that every component of $G \setminus S$ is a protrusion

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Remark: This follows from the fact that VC is compact, but we want another proof

Another approach: Vertex Cover in planar graphs

Fact 1 The treewidth of a planar graph with vertex cover k is $O(\sqrt{k})$

Proof: Excluding grid arguments

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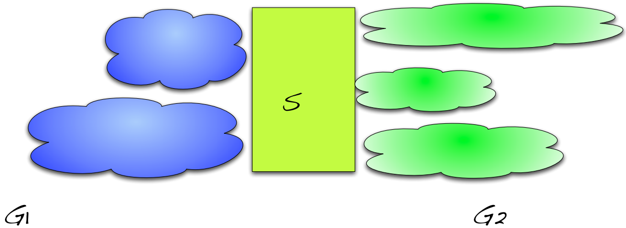
Fact 1 The treewidth of a planar graph with vertex cover k is $O(\sqrt{k})$

Proof: Excluding grid arguments

Fact 2 Graph of treewidth t has an $O(t)$ balanced separator

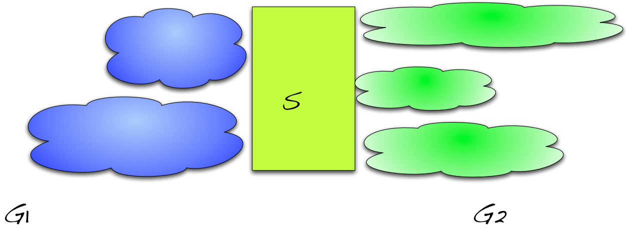
Fact 1 + **Fact 2**: Let G be a planar graph with vertex cover C of size k. There is a separator S of size at most $\alpha\sqrt{k}$ such that

- $|C \cap G_1| \leq \alpha k$ and $|C \cap G_2| \leq (1-\alpha)k$ for some $1/3 \leq \alpha \leq 1/2$.



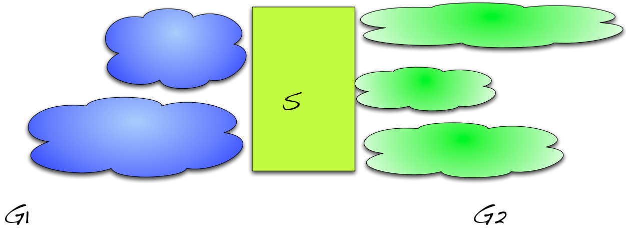
What we know about G_1:

- $(C \cap G_1) \cup S$ is a vertex cover in $G_1 \cup S$, and the size of this VC is at most $\alpha k + \beta\sqrt{k}$;
- $N(G_1) \subseteq S$



Apply arguments recursively for $G_1 \cup S$ and $G_2 \cup S$. We stop when for every component G_i, $(C \cap G_i) \cup N(G_i)$ is of constant size.

- Because $(C \cap G_i) \cup N(G_i)$ is a vertex cover of $G_i \cup N(G_i)$, the treewidth of G_i is constant
- Thus every G_i is a protrusion.



What about the size of set S?

$|S| = \mu(k)$

Recursive formula

$$\max_{1/3 \leq \alpha \leq 1/2} \{\mu\left(\alpha \cdot k + (\beta\sqrt{k})\right) + \mu\left((1-\alpha) \cdot k + (\beta\sqrt{k})\right) + (\beta\sqrt{k} + 1)\}$$

Possible to show that $\mu(k) = O(k)$.

What we have: There is a set S of size $O(k)$ such that every component of $G \setminus S$ is a protrusion

What we have: There is a set S of size $O(k)$ such that every component of $G \setminus S$ is a protrusion

We want more: If G has sufficiently many vertices, then G has sufficiently large protrusion

Claim

Let G be a planar graph with vertex cover k. If G has more than ak vertices, then G has a protrusion of size at leas b.

Proof: Planar hypergraph arguments.

Conclusion

Vertex cover has a linear kernel on planar graphs

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Vertex cover has a linear kernel on planar graphs

But where exactly did we use the properties of planarity and vertex cover?

Properties we use

- $\mathrm{tw}(G) = \sqrt{k}$
- A feasible solution on $G_1 \cup S$ can be formed from a general solution on G by adding S

Properties we use

- $\mathrm{tw}(G) = O(\sqrt{k})$: Holds for many problems on H-minor-free graphs
- A feasible solution on $G_1 \cup S$ can be formed from a general solution on G by adding S: Separability property, holds for many problems too

Bidimenstionality and Protrusions

FF, Lokshtanov, Saurabh, Thilikos, 2010:

Minor-bidimensionality + Separability on H-minor free graphs yields existence of large protrusions in "YES" instances of large size.

Bidimenstionality and Protrusions

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Minor-bidimensionality + Separability on H-minor free graphs yields existence of large protrusions in "YES" instances of large size.

Thus problems with Minor-bidimensionality + Separability + FII admit linear kernels on H-minor-free graphs.

PART III: Hitting Minors

Bizarre Problem

p-Treewidth-123-Deletion

Instance: A graph G and a non-negative integer k.

Parameter: k

Question: Does there exist $S \subseteq V(G)$, $|S| \leq k$, such that the treewidth of $G \setminus S$ is at most 123?

Solving Bizarre Problems

- The treewidth of a YES instance is at most $123 + k$.
- Compute (or approximate) treewidth and use dynamic programming.
- With some (very non-trivial) efforts, obtain the running time $2^{2^{O(k \log k)}} n^{O(1)}$

Solving Less Bizarre Problems

- p-Treewidth-0-Deletion aka p-Vertex Cover, is solvable in time $2^{O(k)}n^{O(1)}$;
- p-Treewidth-1-Deletion aka p-Feedback Vertex Set, is solvable in time $2^{O(k)}n^{O(1)}$

This bounds are tight unless ETH fails

Solving Less Bizarre Problems

- p-Treewidth-0-Deletion aka p-Vertex Cover, is solvable in time $2^{O(k)}n^{O(1)}$;
- p-Treewidth-1-Deletion aka p-Feedback Vertex Set, is solvable in time $2^{O(k)}n^{O(1)}$
- p-Treewidth-2-Deletion is solvable in time $2^{2^{O(k \log k)}}n^{O(1)}$!!??

We want to show that

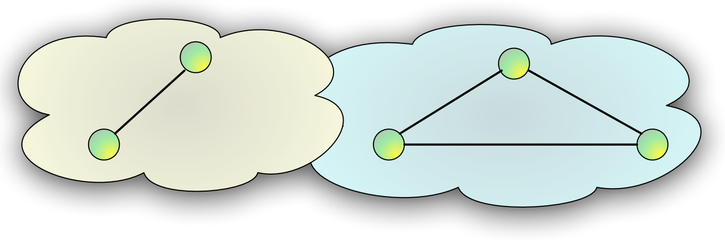
p-Treewidth-123-Deletion is solvable in time $2^{O(k \log k)} n^{O(1)}$

Problem

Let $\mathcal{F}$ be a set of graphs containing at least one planar graph.

p-Planar-$\mathcal{F}$-Deletion	
Instance:	A graph G and a non-negative integer k.
Parameter:	k
Question:	Does there exist $S \subseteq V(G)$, $|S| \leq k$, such that $G \setminus S$ contains no graph from $\mathcal{F}$ as a minor?

p-Planar-$\mathcal{F}$-Deletion: Examples



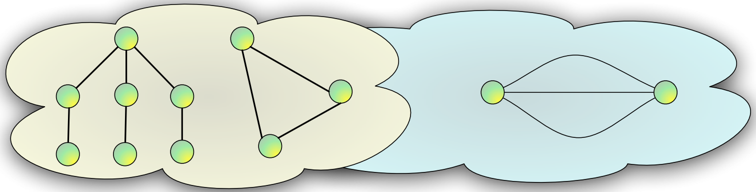
p-Vertex Cover:

$\mathcal{F} = \{K_2\}$

p-Feedback Vertex Set:

$\mathcal{F} = \{C_3\}$

p-Planar-$\mathcal{F}$-Deletion: Examples



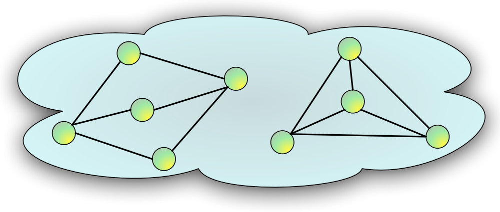
p-Pathwidth 1 Deletion Set

$\mathcal{F} = \{T_2, K_3\}$

p-Diamond Hitting Set

$\mathcal{F} = \{\theta_3\}$

p-Planar-$\mathcal{F}$-Deletion: Examples



p-Outerplanar Deletion Set

$\mathcal{F} = \{K_{2,3}, K_4\}$

p-Planar-$\mathcal{F}$-Deletion: Examples

p-Treewidth-123-Deletion

Theorem (FF, Lokshtanov, Misra, Saurabh, 2011)

p-$\textsc{Planar}$-$\mathcal{F}$-$\textsc{Deletion}$ *is solvable in time* $2^{O(k \log k)} n^2$.

Proof: Auxiliary problem

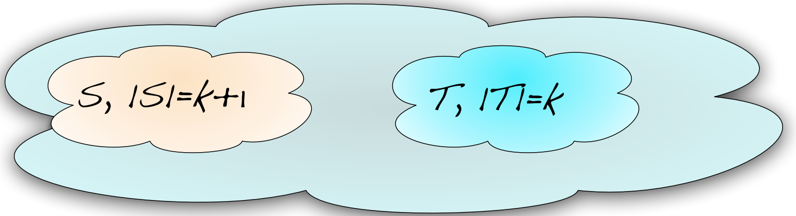
p-DISJOINT PLANAR $\mathcal{F}$-DELETION

Instance: A graph G, $k \geq 0$, and $S \subseteq V(G)$ of size at most $k+1$ such that $G[S]$ and $G \setminus S$ contains no graph from $\mathcal{F}$ as a minor?

Parameter: k

Question: Is there $T \subseteq V(G) \setminus S$, $|T| \leq k$, such that $G \setminus T$ has no graph from $\mathcal{F}$ as a minor?

p-Disjoint Planar $\mathcal{F}$-deletion



GIVEN:

- S is F-hitting set;
- G[S] has no minor from F

FIND:

- T is F-hitting set;
- T is disjoint from S

Claim

If we manage to solve p-$\textsc{Disjoint Planar}$ $\mathcal{F}$-$\textsc{deletion}$ in time $O^*(2^{k \log k})$, we also can solve p-$\textsc{Planar}$-$\mathcal{F}$-$\textsc{Deletion}$ in time $O^*(2^{k \log k})$.

Iterative compression

- Step of iterative compression for p-$\textsc{Planar-}\mathcal{F}\textsc{-Deletion}$:
- Given $\mathcal{F}$-hitting set S of size $k+1$, to find a $\mathcal{F}$-hitting set S^* of size $k+1$, for each partition X, Y of S, solve p-$\textsc{Disjoint Planar } \mathcal{F}\textsc{-deletion}$ with instance $(G \setminus Y, X, k - |Y|)$.
- Running time $O^*(2^{k \log k})$.

Lemma

p-Disjoint Planar $\mathcal{F}$-deletion *has a polynomial kernel*

Lemma

p-DISJOINT PLANAR $\mathcal{F}$-DELETION *has a polynomial kernel*

Remark: Lemma implies an $O^*(2^{k \log k})$ algorithm for p-DISJOINT PLANAR $\mathcal{F}$-DELETION.

To obtain kernel we need

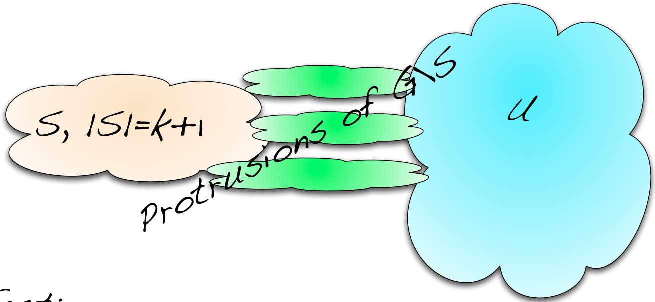
Fact

Let H be a planar graph. The treewidth of a H-minor-free graph G is at most $f(H)$.

Many big protrusions

Lemma

Let b, s, p be integers. Then there is d such that every graph G with at least $dbsp$ vertices and treewidth b has a partition of the vertex set into parts $V_1, \ldots, V_p$ and U such that each $G[V_i]$ is a $2(b+1)$-protrusion of size at least s.

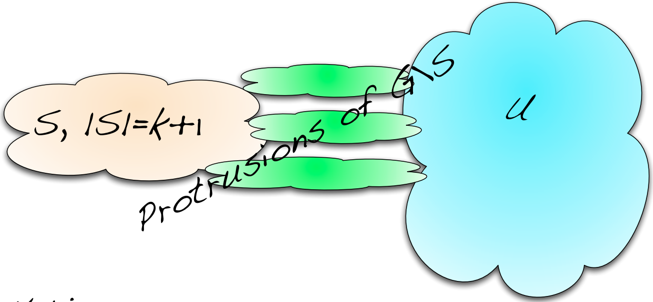


By Fact:

- $G\setminus S$ is of constant treewidth

By Lemma

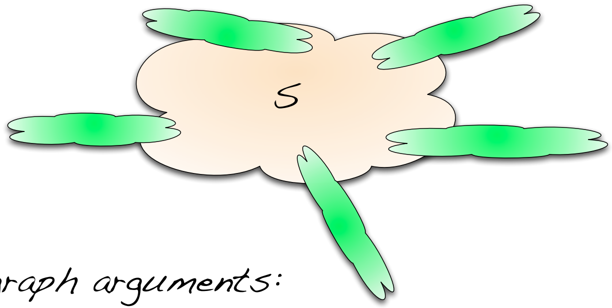
- There are many big protrusions in $G\setminus S$



However:

- Protrusion in G\S is not necessarily protrusion in G

What we want



Hypergraph arguments:

- A hypergraph with vertex set S, and hyperedges formed by the neighbourhoods of protrusions of G\S has a linear amount of large hyperedges

Fact

Let $\mathcal{H}$ be an n-vertex hypergraph (not necessarily simple) such that its incidence graph $I(\mathcal{H})$ does not contain K_h as a minor. Then the number of hyperedges of $\mathcal{H}$ of size at least h is at most $2^{s_h\sqrt{\log h}}h(h-1)n/2$.

In our case: $\mathcal{H}$ can be turned into H-minor-free hypergraph by removing at most k hyperedges, thus it has $O(k)$ hyperdges larger than some constant depending on $\mathcal{F}$ only.

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WE HAVE PROTRUSION!!!

Remark: In real life (and real proof) things are a more complicated because p-$\textsc{Disjoint Planar } \mathcal{F}\textsc{-deletion}$ is not FII, so we have to go through the annotated kernels and MSOL arguments.

Many thanks for joint searching of protrusions!!