Hierarchies of Polynomial Kernelization Hardness

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Excluding Small Kernels

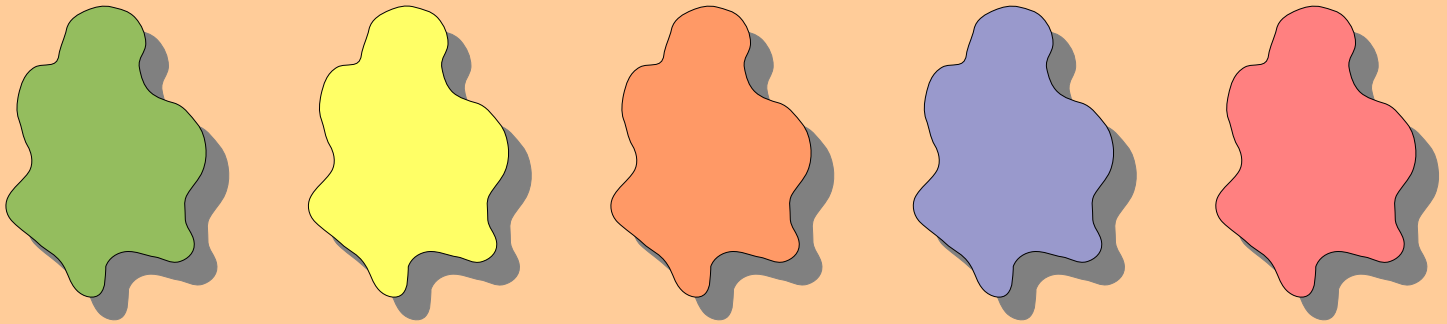
- a powerful framework for proving inexistence of polynomial kernels (unless PH collapses) [BDFH, FS '08]
- lower bounds for many problems (k-Path, Connected Vertex Cover, Steiner Tree, ...)
- in essence: compositional problems have no polynomial kernels

Compositional Problem

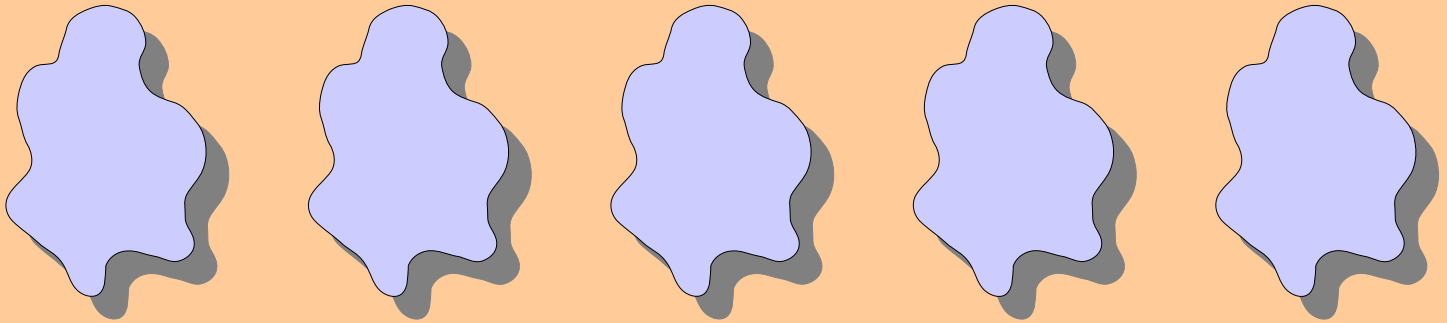
- we can in polynomial time encode the OR of many instances of the problem into a single instance (potentially large) with a small parameter
- example: k-Path

Compositional Problem

- example: k-Path
- does any of these graphs contain a path of length k?



- does this graph contain a path of length k?



Compositional Problem

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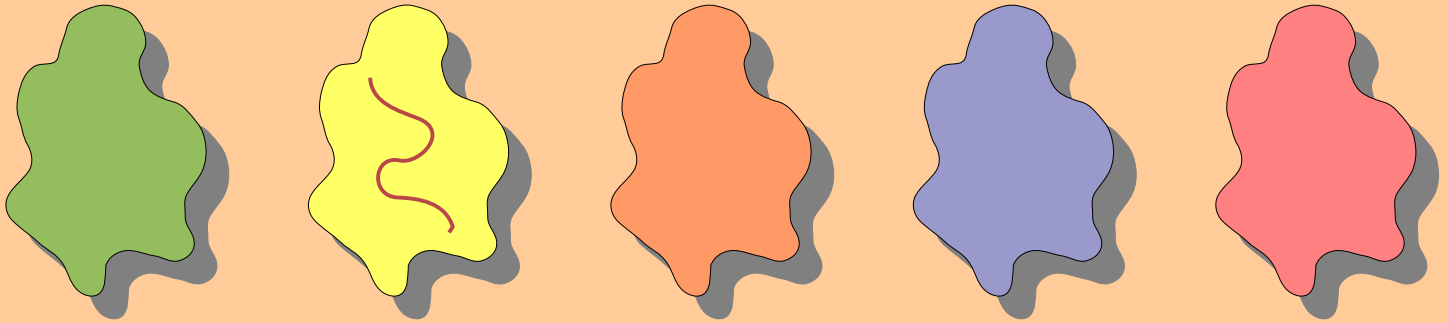

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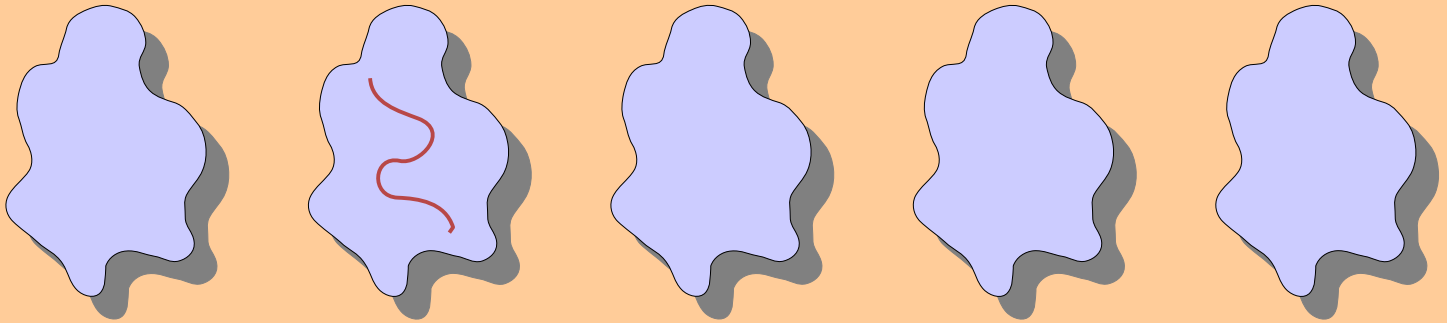


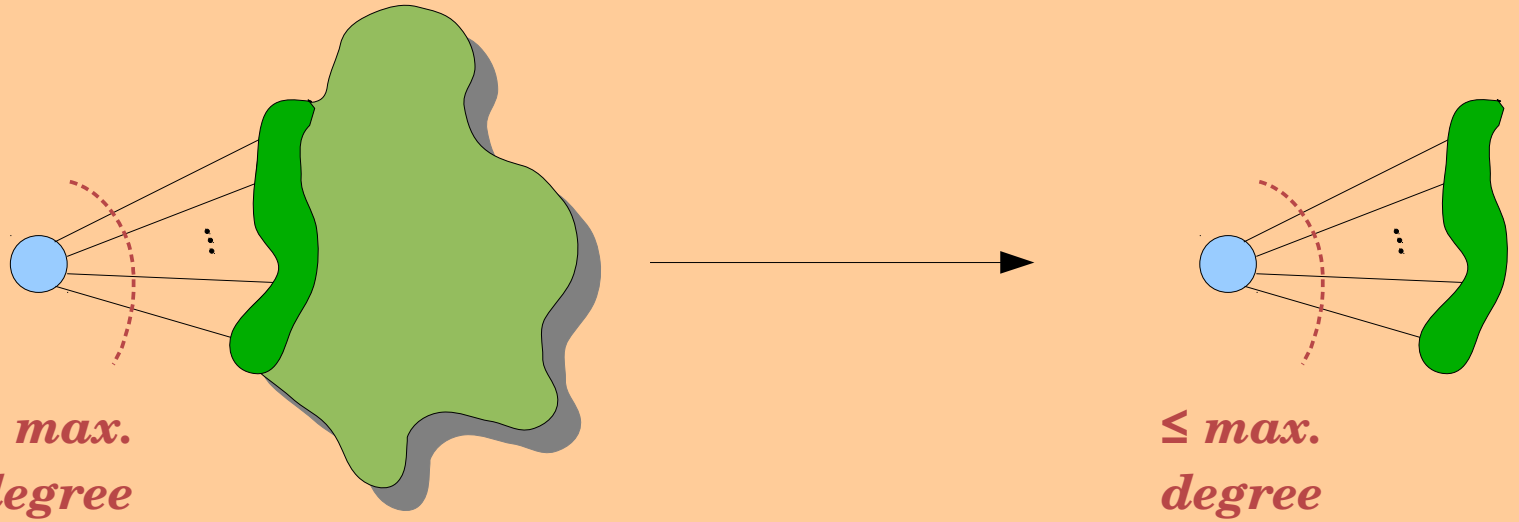
Excluding Small Kernels

- a powerful framework for proving inexistence of polynomial kernels (unless PH collapses) [BDFH, FS '08]

- but sometimes there exists a polynomial "cheating" kernel: an OR-kernel (Max-Leaf Out-Branching, Clique(Max. Degree)) or, more generally, a Turing Kernel
- and subexponential kernels could also be possibly useful in practice
- how to exclude such kernels?

A "Cheating" Kernel

- **Clique(Max. Degree) might not have a polynomial kernel...**
- **... but it has one if we localize the problem: i.e. require that a given vertex v is in the clique**



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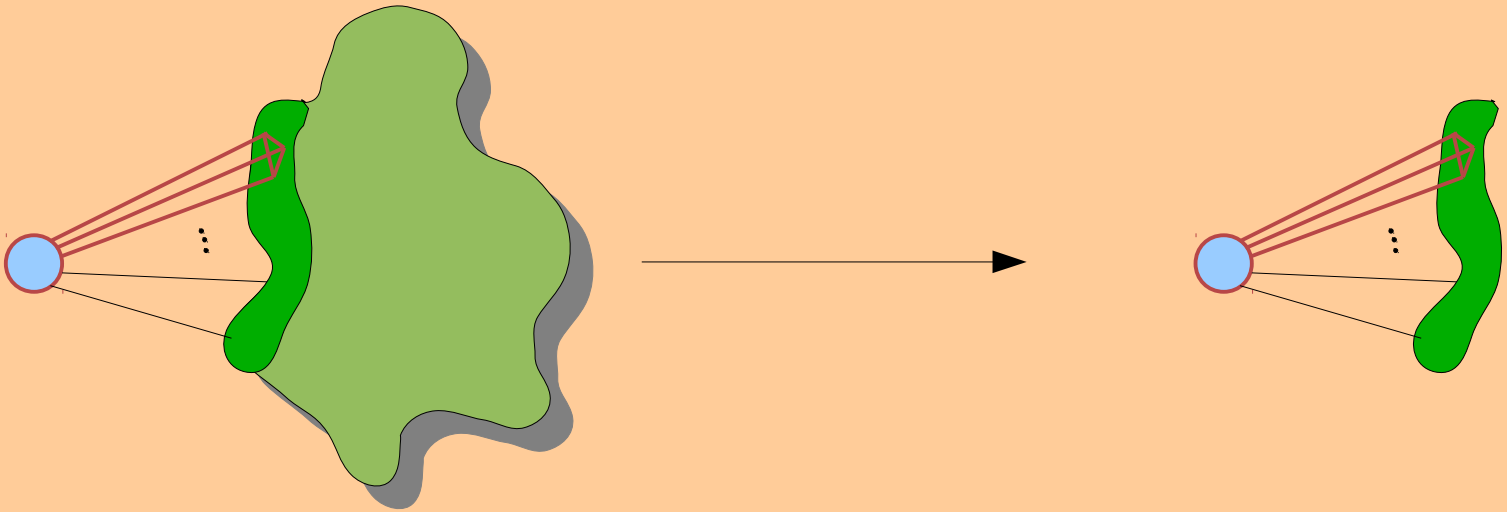
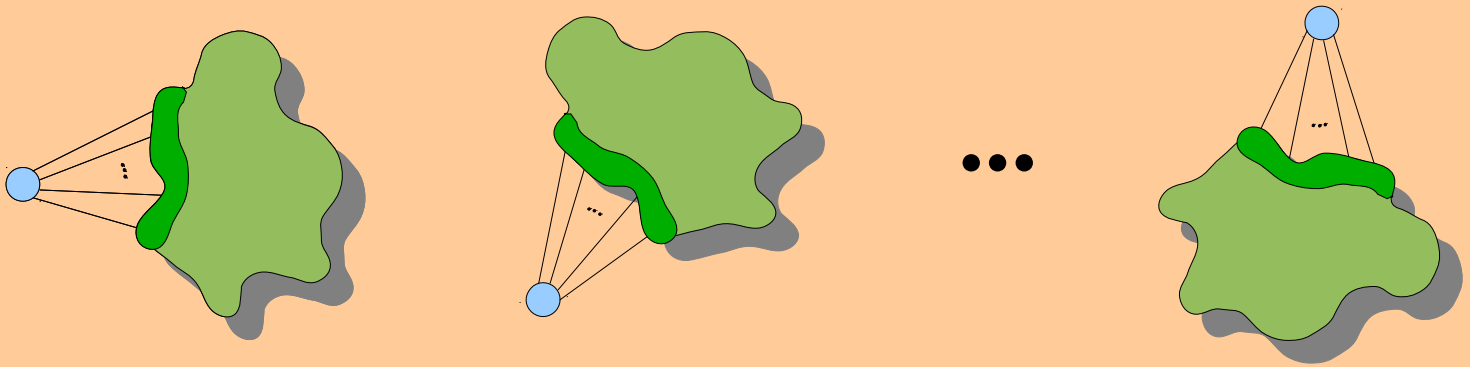


A "Cheating" Kernel

- Clique(Max. Degree) might not have a polynomial kernel...
- ... but it has one if we localize the problem: i.e. require that a given vertex v is in the clique
- ... and we can ask this question about v_1, v_2, v_3, ...



- this "kernel" is still very useful for FPT algorithms

Excluding "Cheating" Kernels

- seems difficult
- complexity theorist's approach:
  - find a good candidate for a hard problem
  - assuming it is indeed hard, prove hardness of other problems (so we need reductions preserving hardness)
  - PPT reductions (akin to standard FPT reductions, but changing the parameter at most polynomially) do the trick
  - this yields a class of problems (probably) not admitting "cheating" (Turing, subexponential) kernels

Hard Nuts to Kernelize

- candidate hard problems
  - FPT reparametrizations of basic non-FPT problems:
    - Clique (k log n)
  - NP problems verifiable in time polynomial in the witness size
    - k-Step Turing Machine Halting Problem
  - problems for which solution consistency is checked locally
    - Min Ones q-SAT
  - problems convenient to work with to use in future reductions ;)
    - (Exact) Hitting Set (m)

Hard Nuts to Kernelize

- candidate hard problems
  - Clique (k log n)
  - k-Step Turing Machine Halting Problem
  - Min Ones q-SAT
  - (Exact) Hitting Set (m)

- fortunately, we can prove they are all equivalent!

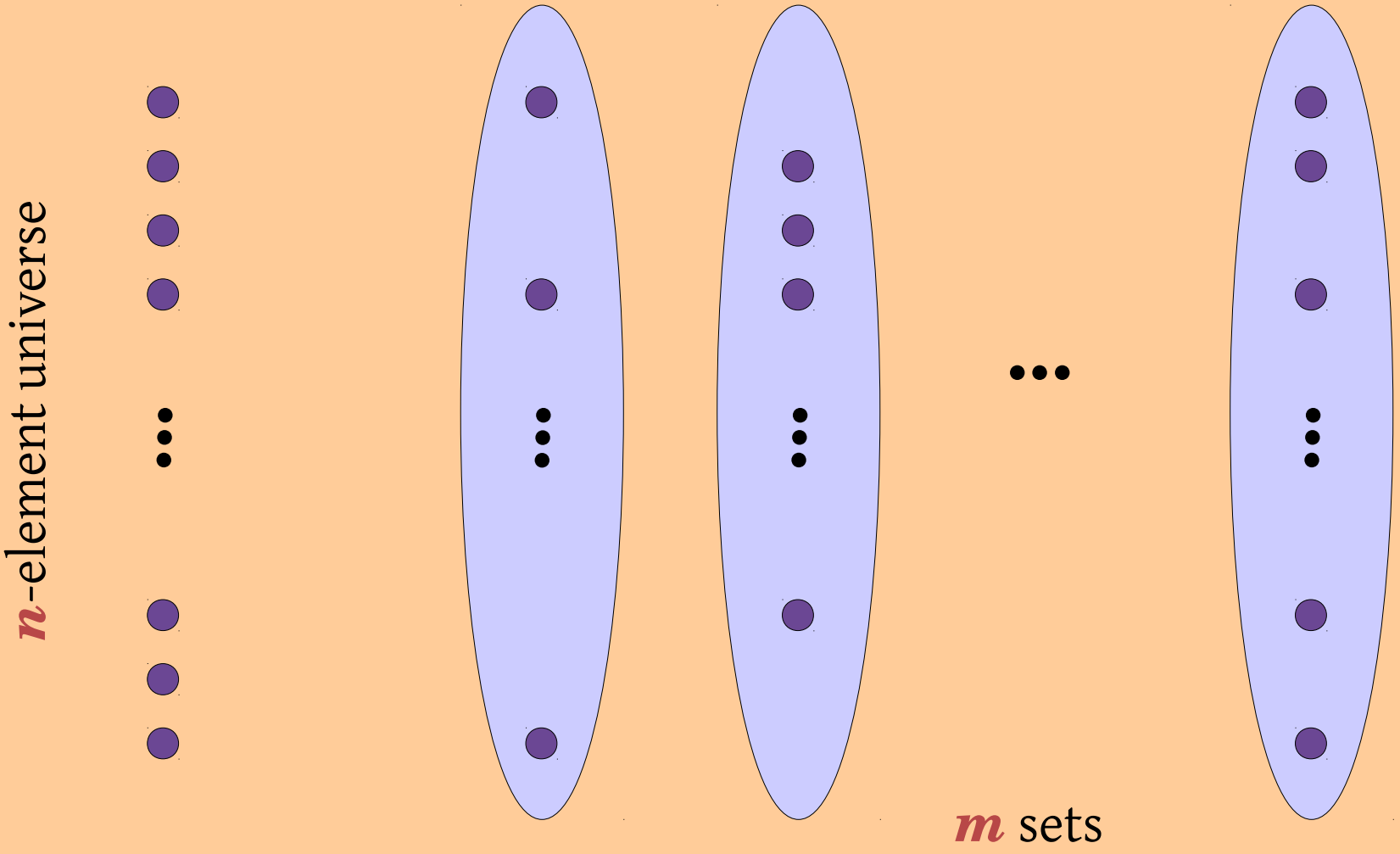
Formal Definition

- **WK(t)** = $[\text{Weft-t Weighted SAT } (k \log n)]_{\leq ppt}$
  - **WK(1)** is the class of problems that we discuss in this talk
- **MK(t)** = $[\text{Weft-t SAT } (n)]_{\leq ppt}$
  - **MK(2)** captures Hitting Set (n)

- $\text{PK} = \text{MK}(1) \subseteq \text{WK}(1) \subseteq \text{MK}(2) \subseteq \text{WK}(2) \subseteq \ldots \subseteq \text{EXPT}$

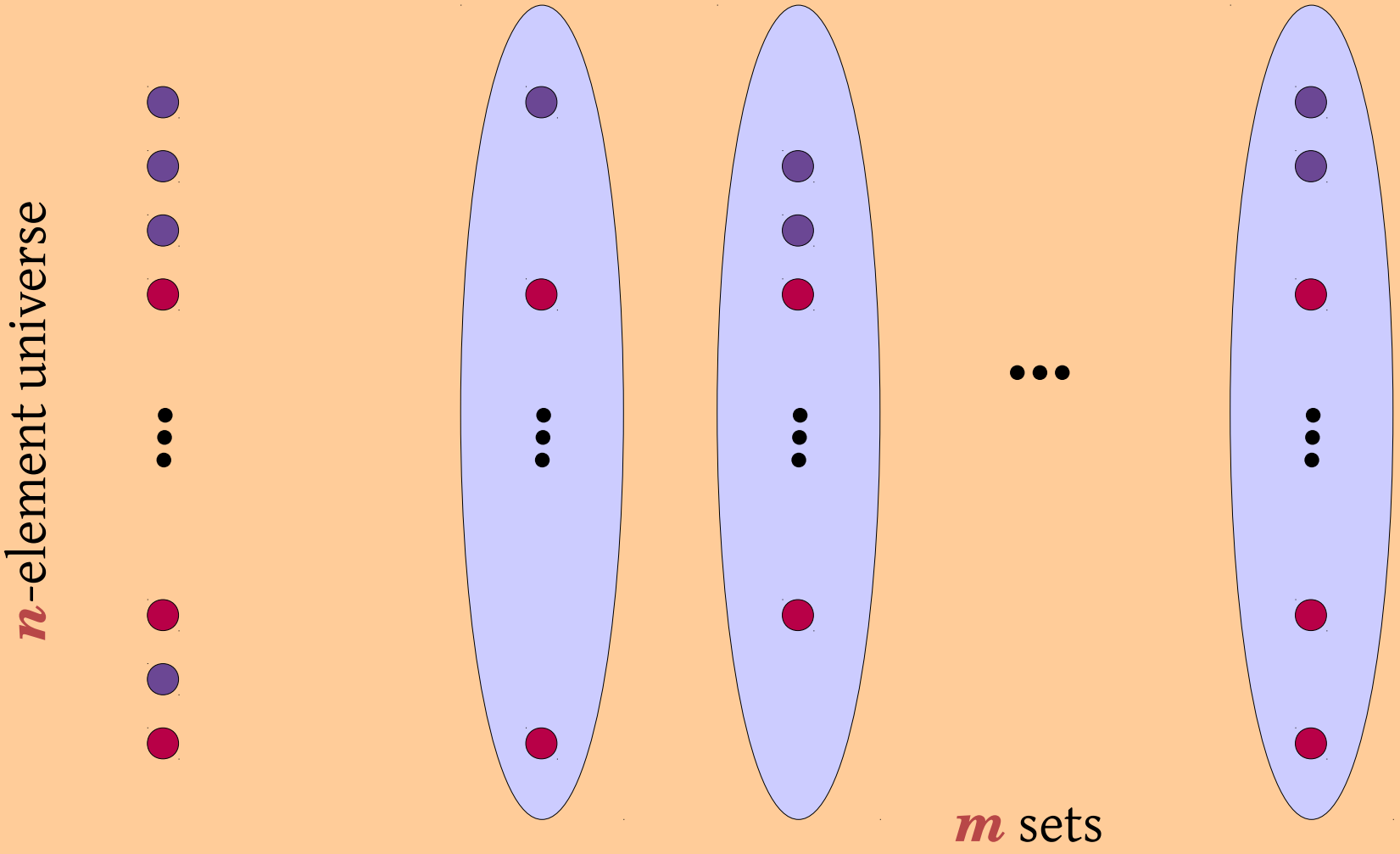
- **hierarchies for kernelizability analogous to W- and M-hierarchies for fixed-parameter tractability**

Hitting Set (m)



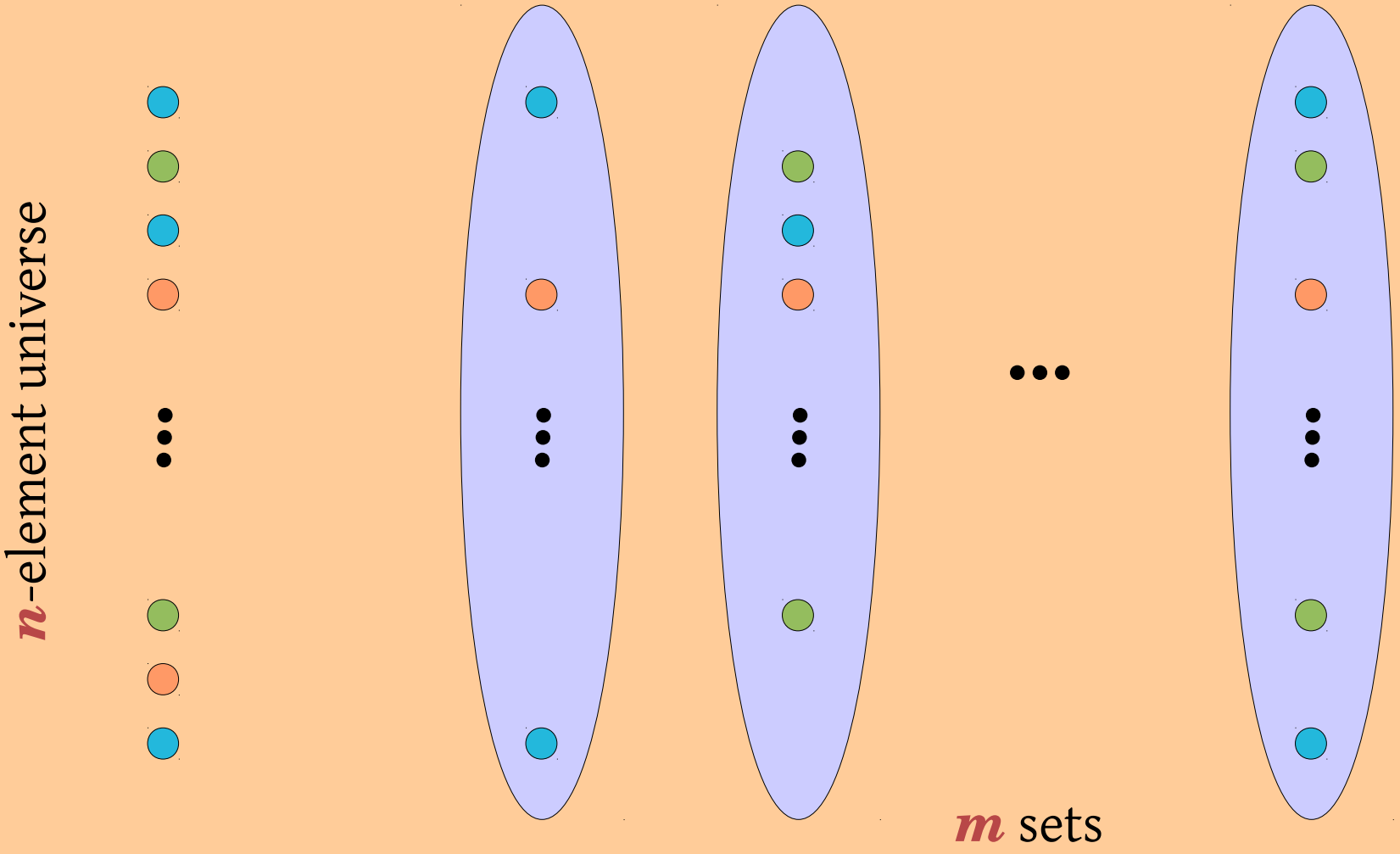
Hitting Set (m)

- **select k elements to hit all sets**



Multicolored Hitting Set (m)

- technical modification (elements come in k colors, select one of each)



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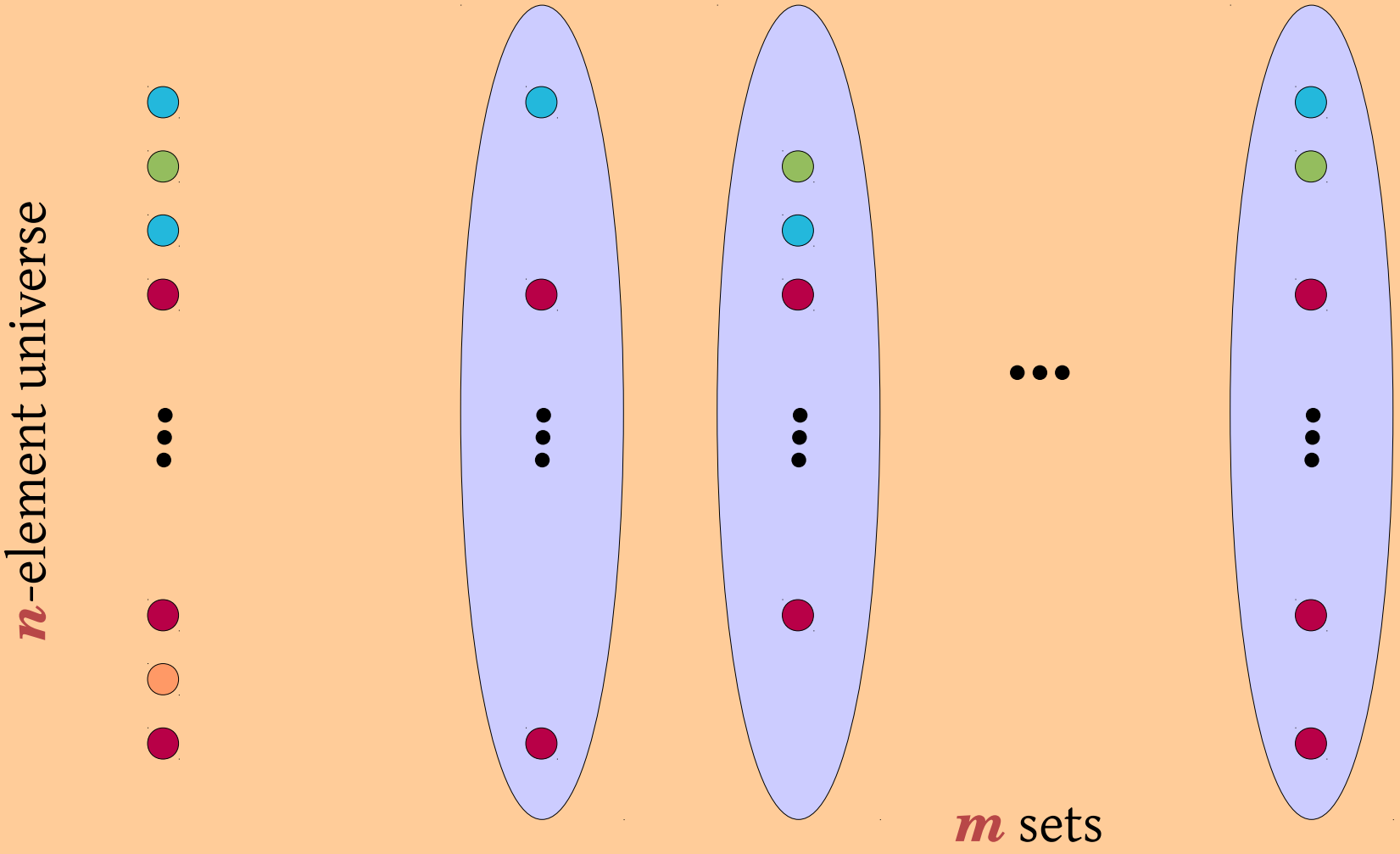


One Example of Equivalence

- (Multicolored) **Clique** (k log n) $\leq_{ppt}$ (Exact) **Hitting Set** (m)
  - we assign to vertices **binary IDs** (of length log n)
  - elements correspond to edges
  - sets require exactly one edge linking a given pair of colors... ($\binom{k}{2}$ sets)
  - ... and ensure that any two edges using a color use the same vertex ($\binom{k}{3}$ log n sets)

One Example of Equivalence

- **Hitting Set** $(m) \leq_{ppt}$ (Multicolored) **Clique** $(k \log n)$
  - w.l.o.g. we have at most exponential number of elements (n)
  - we create a graph with **m color classes** (one for every set)...
  - ...and $O(n)$ **vertices** (one **for every pair** (S,e) where $e \in S$)
  - a vertex (S,e) is selected to the clique iff S is hit by e
  - edges ensure that two sets are hit either by the same element, or by two elements from the symmetric difference

Other WK(1)-Hard Problems

- **Set Cover** (n), **Bipartite Dominating Set** (|T|)
- **Local Circuit SAT**
- **Disjoint Cycles**, **Disjoint Paths**
- **Chromatic Number** (|**Vertex Cover**|), **Weighted Feedback Vertex Set** (|**Vertex Cover**|)
- **Steiner Tree**, **Connected Vertex Cover**, **Capacitated Vertex Cover**
- **Small Subset Sum**
- **Tree-Contractibility**
- **3-SAT** (**Horn-backdoor**), **3-SAT** (**2-CNF-backdoor**)
- {0,1}-**CSP** (**width**)

Multicolored k-Path

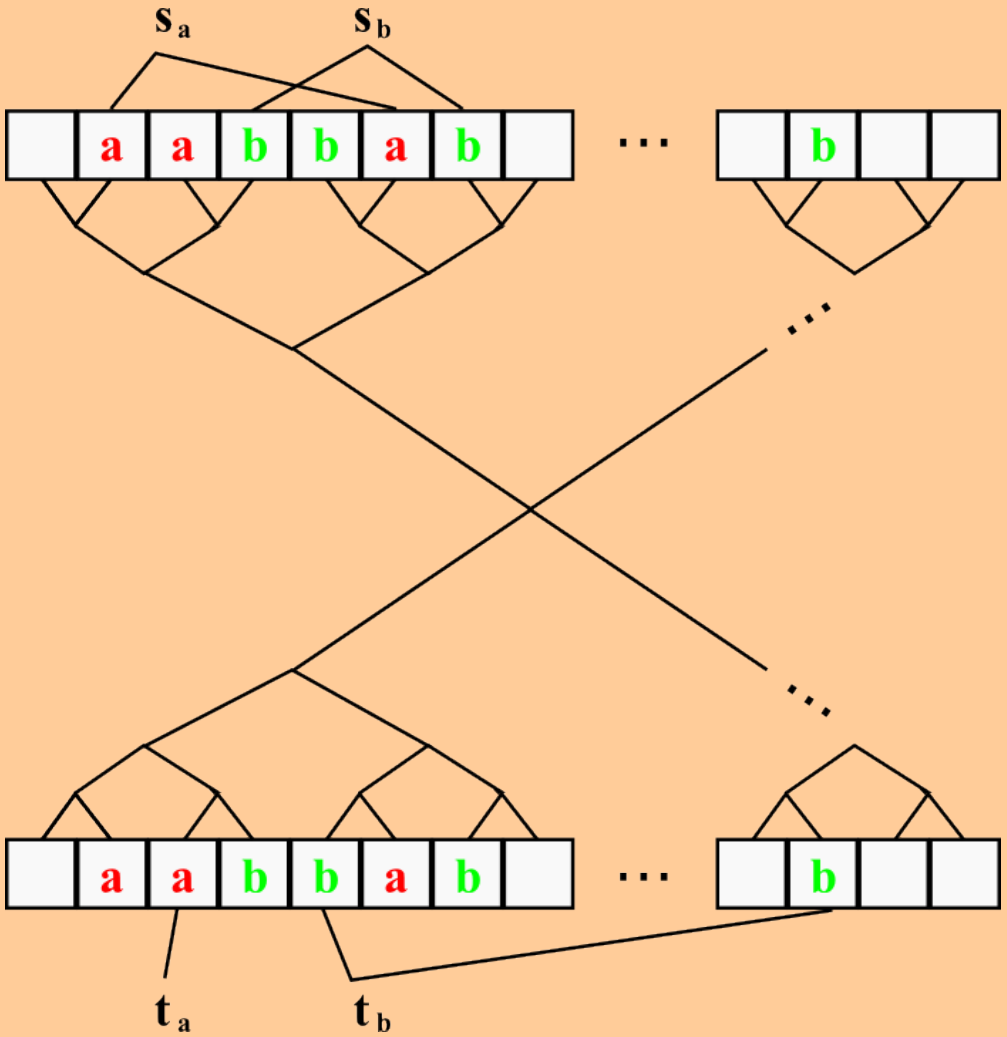
- **Disjoint Factors**
  - input: n-character string over k-ary alphabet
  - task: find k non-overlapping factors: subwords of the form a ... a for every symbol a
  - binary selection gadget: a ... a ... a
  - WK[1]-hard via a reduction from Hitting Set (m) – see blackboard
- **Multicolored k-Path**
  - Multicolored Directed k-Path – easy reduction from Disjoint Factors – see blackboard
  - we can go to the undirected case, but we still need colors

Disjoint Short Paths

- reduction from Non-Nested Factors...
  - a technical gap problem which can be proven to be WK[1]-hard
- ...to Disjoint Short Paths
  - we want to find a path of length k' for each of k terminal pairs
  - non-crossing paths will correspond to non-nested factors

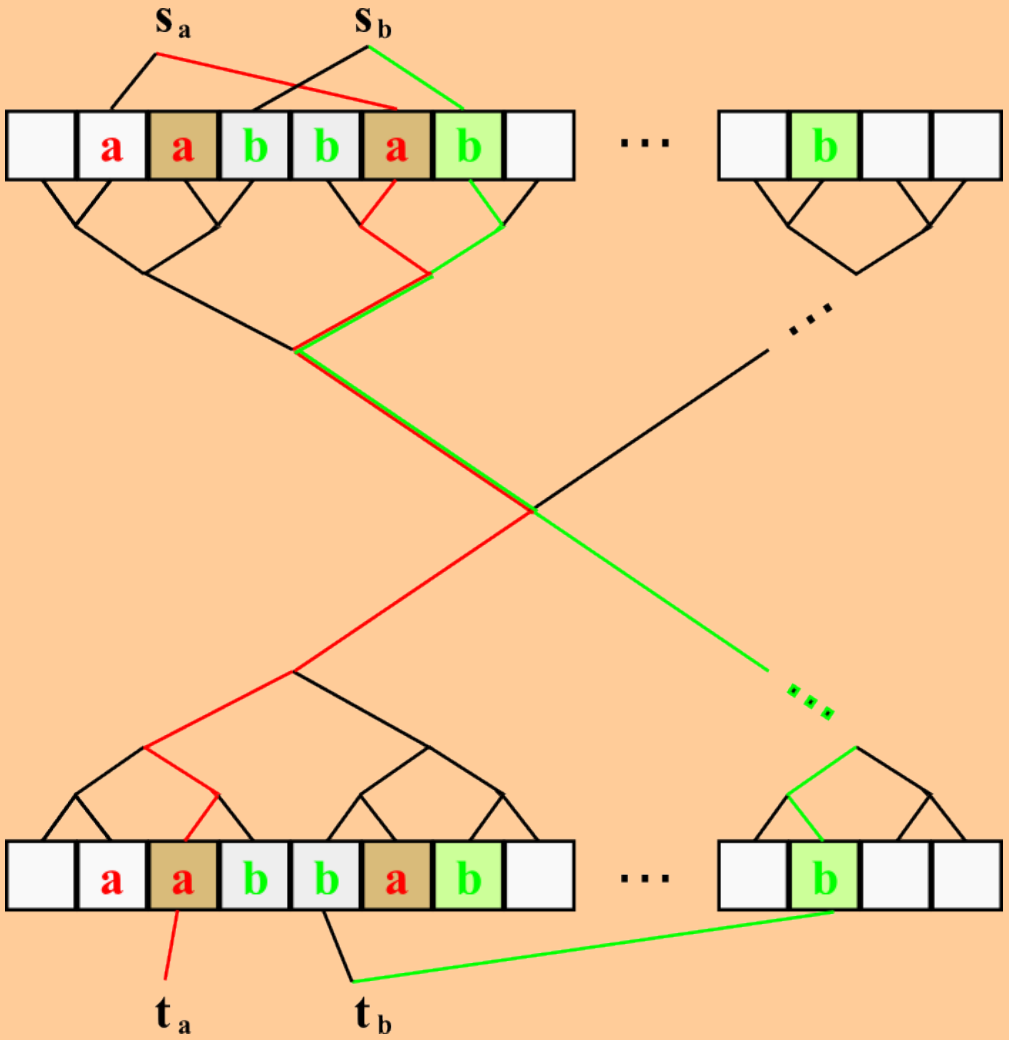
Disjoint Short Paths

- the high-level structure for Disjoint Short Paths



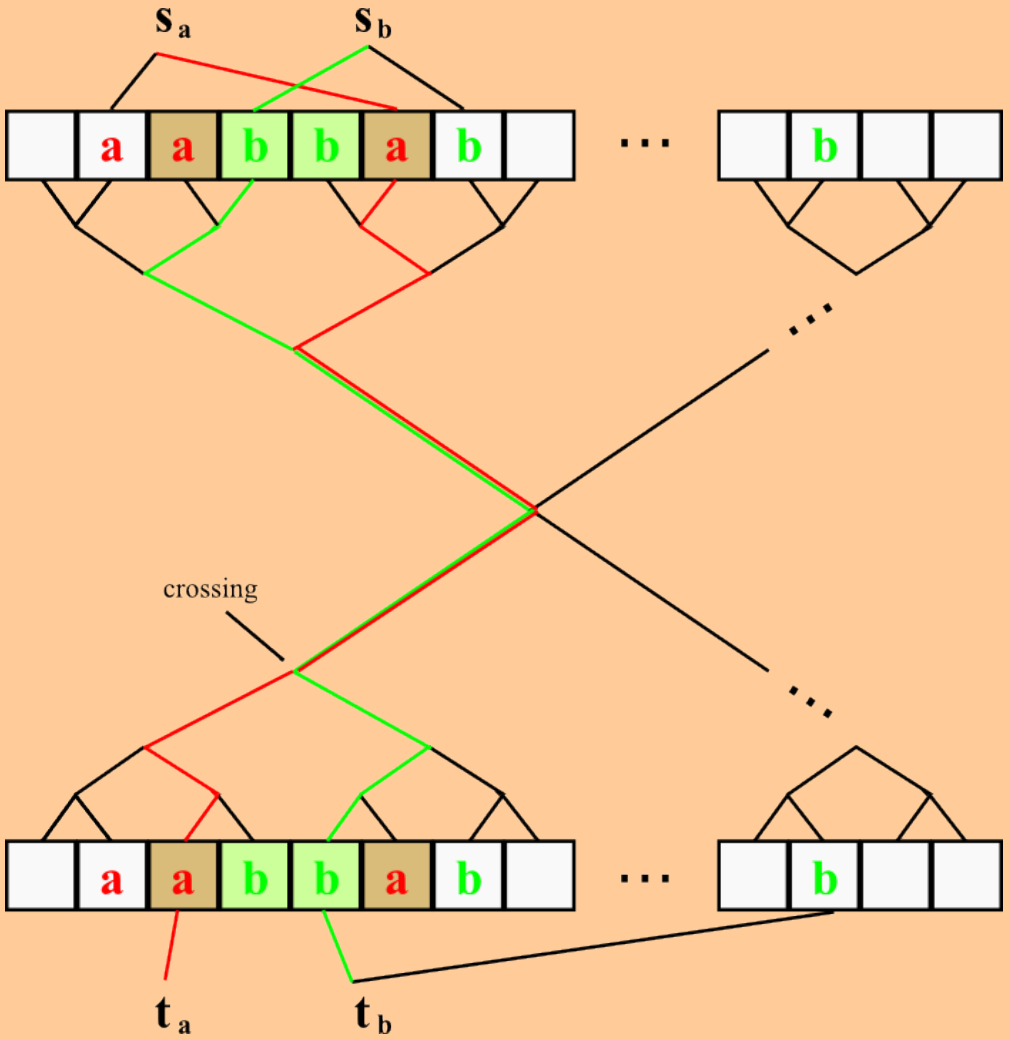
Disjoint Short Paths

- paths corresponding to non-overlapping factors can go alongside...



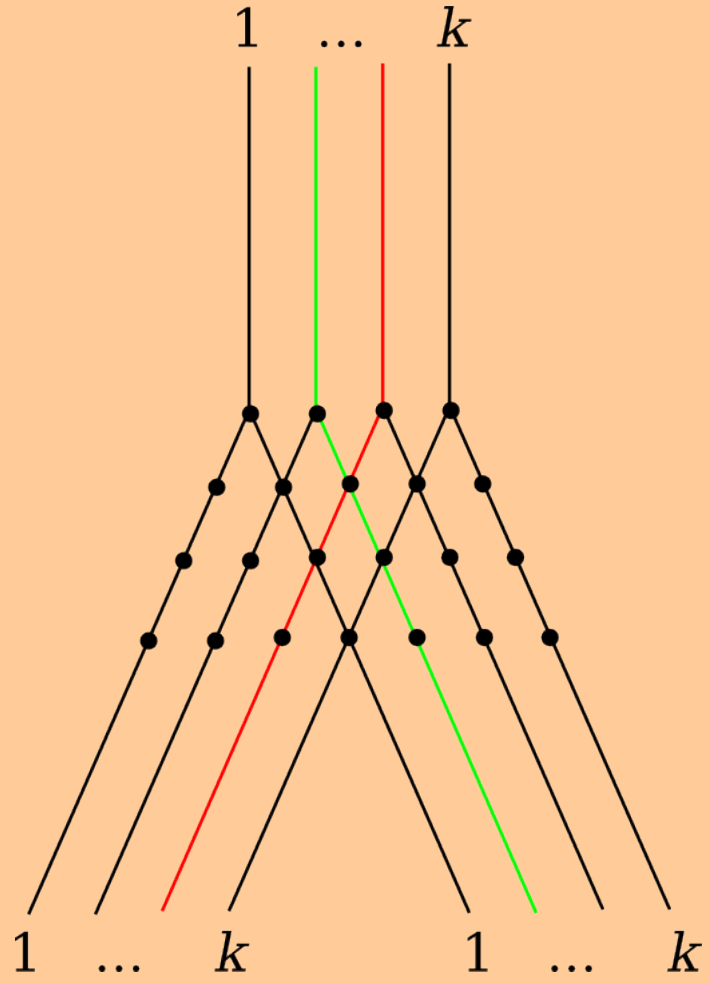
Disjoint Short Paths

- ...while paths corresponding to nested factors always have to cross



Disjoint Short Paths

- replacing edges by k-lane highways allows paths to go alongside



Open Problems

- k-Path, Directed k-Path
- (might be difficult) prove some separation results

Thank you! Questions?