Tools for Kernelizing Graph Cut Problems

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Graph cut problems

For many graph cut problems, the existence of polynomial kernels is/was unknown:

- Multiway Cut – separate terminals T by removing k vertices
- Directed Feedback Vertex Set – hit all directed cycles
- Multicut – fulfill all cut requests (s_i, t_i) using k vertices

Also related problems (graph cut problems in disguise):

- Graph Bipartization (OCT) – hit all odd cycles
- Almost 2-SAT, a.k.a. 2-CNF Deletion – remove k variables/clauses to make F satisfiable

Graph cut problems

Many found attackable by matroid theory:

This talk:

- Graph Bipartization (OCT)

Other problems:

- Almost 2-SAT, a.k.a. 2-CNF Deletion
- Multiway Cut – restricted cases
- Directed Feedback Vertex Set – unknown
- Multicut – unknown

Outline

Matroids 1:

Encoding Terminal Cuts

Matroids

Matroid theory is for the concept of dependence what group theory is for symmetry — (unknown)

A matroid $M = (U, \mathcal{I})$, $\mathcal{I} \subseteq 2^U$, is an independence system with independent sets $\mathcal{I}$ satisfying:

1. The empty set is independent
2. A subset of an independent set is independent
3. Augmentation property: If A, B are independent and $|B| > |A|$, then there is some $b \in B - A$ such that $A + b$ is independent

Rank $r(X)$: Size of largest independent subset of X

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Examples

Canonical examples:

1. Graphic matroids: Let $G = (V, E)$ be a graph.
   - $M = (E, \mathcal{I})$, $\mathcal{I}$ contains cycle-free edge sets
   - Rank: number of vertices minus number of components
2. Linear matroids $M = (U, \mathcal{I})$:
   - U is a collection of vectors in $\mathbb{F}^d$ for some field $\mathbb{F}$
   - Independence concept is linear independence
   - Rank: dimension

Linear matroids more conveniently represented by $d \times |U|$ matrix.
Many tools work only for linear matroids (our matroids are linear).

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Gammoids

Let $G = (V, E)$ be a graph. Say that $T \subseteq V$ is linked into S if there are $|T|$ vertex-disjoint paths from S to T (not only internally vertex-disjoint).

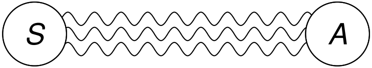
The gammoid defined by G and S is $M = (V, \mathcal{I})$ where:

- $\mathcal{I}$ contains all sets linked into S
- The rank $r(X)$ equals the size of an (S, X)-cut
- Augmentation property: see next slide

Also works for digraphs.

Gammoids: augmentation property

Let $G = (V, E)$, $S \subset V$, A and B linked into S, with $|B| > |A|$.
There exists a vertex $v \in B - A$ such that $A + v$ is linked into S.



1. Let C be the minimum (S, A)-cut closest to S. Claim: if $A + v$ is dependent then C cuts v from S.
   - $A + v$ dependent $\Rightarrow$ cut $(S, A + v)$ of size $< |A| + 1$
   - A independent $\Rightarrow$ cut has size $|A| = |C|$, dominated by C
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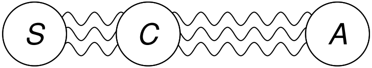

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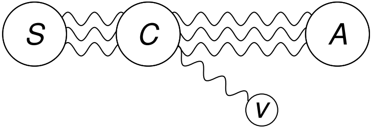

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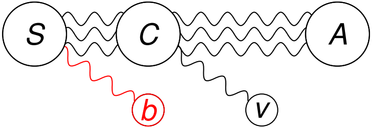

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Gammoids: small representation

1. Let $G = (V, E)$, $S, T \subset V$. We can represent the subsets of T which are linked into S in space $\mathcal{O}(|S| \cdot |T| \cdot |S| \log |T|)$.
2. Source-free form: Let $G = (V, E)$, $X \subseteq V$ a set of terminals. We can represent the flow from S to T in $G - R$ for arbitrary $S, T, R \subseteq X$ in space $\mathcal{O}(|X|^3)$.

Still works for digraphs. Randomized polynomial time with one-sided error (underestimates flow only).

Application:

Polynomial Kernel for OCT (Graph Bipartization)

OCT algorithm

Recall OCT iterative compression algorithm[1] ($G_i = G[v_1, \ldots, v_i]$)

1. Start with graph G_1, empty solution $X = \emptyset$
2. For $i = 2 \ldots n$:
   2.1 Have solution X of size k for G_{i-1}
   2.2 $X' = X + v_i$ is solution of size $k+1$ for G_i
   2.3 Use X' to find optimal solution for G_i

[1] Reed, Smith, Vetta 2004

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1. Start with graph G_1, empty solution $X = \emptyset$
2. For $i = 2 \ldots n$:
   2.* Compress $X + v_i$ to optimal solution X

Kernelization order:

1. Create approximate solution X (size k^c)
2. Feed X to compression step
3. Kernelize resulting graph cut problem

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OPT^c approximation

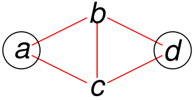
1. Run FPT algorithm with parameter $k = \log n$ ($3^k n^{O(1)} \to n^{O(1)}$)
2. Run $\mathcal{O}(\sqrt{\log n})$-approximation;[3] observe $\log n < k$

$\mathcal{O}(\text{OPT}^{1.5}) = \mathcal{O}(k^{1.5})$-sized solution.

[3] Agarwal, Charikar, Makarychev, Makarychev, STOC 2005

Algorithm sketch (compression step)

Let u–v be a normal edge ($u \neq v$), u–v an equality edge ($u = v$).

1. 

$X = \{a, d\}$. Partition $G - X$ as $U \cup V = \{b\} \cup \{c\}$

2.

Negate U: Toggle crossing edges

3.

Split X into positive, negative copies

4. Search for cuts that for $x \in X$ delete x and $\bar{x}$, or split $x \neq \bar{x}$

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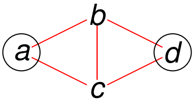
1. 

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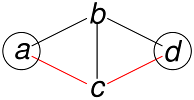
2. 

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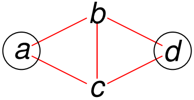
1. 

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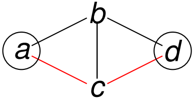
2. 

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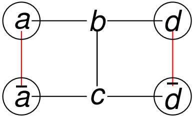
3. 

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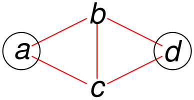
1. 

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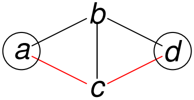
2. 

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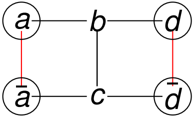
3. 

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OCT kernel, summary

1. Use known tools to get solution X with $|X| = \mathcal{O}(k^{3/2})$
2. Create cut problem on auxiliary graph, terminals X'
3. Encode terminal cuts over X' into $\mathcal{O}(|X|^3)$-size description
4. Yields a $\mathcal{O}(k^{4.5})$-sized instance description (compression, a.k.a. generalized kernel/bikernel)

(True (direct) kernel by NP-hardness.)

Matroids 2:

Irrelevant vertices

Direct kernels

Terminal Cuts Compression

Input: Graph $G = (V, E)$, sets $S, T \subset V$

Parameter: $|S| + |T|$

Task: Reduce G to a small graph G' while preserving sizes of (A, B)-cuts, $A \subseteq S$, $B \subseteq T$

Would give direct (combinatorial?) kernels for our problems.

Strategy: Irrelevant vertices

Terminal Cuts Compression

Reduce size of G while preserving (A, B)-cuts, $A \subseteq S$, $B \subseteq T$

- A vertex v is essential if for some A, B, every minimum (A, B)-cut uses v
- Otherwise irrelevant

Claim

There are at most k^4 essential vertices (and we can find them).

An irrelevant vertex may be removed (lifted) (then iterate).

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Case: Almost 2-SAT kernel

Digraph Pair Cut

Input: Digraph $D = (V, A)$, source vertex s, integer k, set of pairs $P \subseteq \binom{V}{2}$

Parameter: k

Problem: Remove k vertices such that for every pair $\{u, v\} \in P$, either u or v is not reachable from s

- Claim 1: Digraph Pair Cut has $\tilde{O}(k^4)$ kernel
- Claim 2 (omitted): This gives a polynomial kernel for Almost 2-SAT

Digraph Pair Cut

Algorithm:

1. Let $T = \emptyset$, $X = \emptyset$
2. While any pair is reachable and $|X| < k$:
   ($\{u, v\}$ reachable: both u and v reachable from s in $G - X$)
   2.1 Find reachable pair $\{u, v\}$
   2.2 Branch on ($T = T + u$) or ($T = T + v$)
   2.3 Let X be the min-(s, T)-cut closest to s

Claims

There are only k^2 non-irrelevant pairs $P' \subseteq P$ for step 2.1

We can encode the problem into terminal cuts, size $\tilde{O}(k^2|P'|)$.

Conclusions

- Parameterized graph cut problems (still) include many open problems for polynomial kernelization
- Matroid theory gives very powerful tools for these problems
  - Encode a problem compactly (as a matrix), in small space
  - Detect irrelevant vertices/objects to remove