

# Max- $r$ -Lin Above Average and its Applications

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## Reminder: Parameterizations above Tight Bounds

- ▶ If a natural parameter has a large lower bound then it doesn't work well as a parameter - the answer is trivially YES unless  $k$  is large, in which case  $f(k)$  will be impractical
- ▶ for example, in MAX-SAT, one can always satisfy at least  $m/2$  clauses, so “Does there exist an assignment satisfying at least  $k$  clauses?” isn't a good question.
- ▶ Instead, ask “is there an assignment satisfying at least  $m/2 + k$  clauses?” (where  $m$  is the total number of clauses)
- ▶ Here we are parameterizing above the known lower bound ( $m/2$ )

# MaxLin Problem

## MAX- $r$ -LIN2-AA

*Instance:* A system  $S$  of equations  $\sum_{i \in I_j} z_i = b_j$  over  $\mathbb{F}_2$ , where  $z_i, b_j \in \{0, 1\}$ ,  $j = 1, \dots, m$ ; equation  $j$  is assigned a positive integral weight  $w_j$ .

**Each equation contains at most  $r$  variables ( $|I_j| \leq r$ ).**

*Parameter:*  $k$ .

*Question:* Is the maximum possible weight of satisfied equations  $\geq W/2 + k$ ?

( $W$  denotes the total weight of all equations in the system)

# Tightness

- ▶ Consider any system consisting of pairs of equations with different left hand sides
- ▶ One may only satisfy one equation from each pair
- ▶ For example, the system:

$$x_1 = 0, x_2 = 0, \dots, x_n = 0, x_1 + x_2 = 0$$

$$x_1 = 1, x_2 = 1, \dots, x_n = 1, x_1 + x_2 = 1$$

## Previous results for MAX- $r$ -LIN2-AA

Theorem (Gutin, Kim, Szeider, Yeo (2009))

MAX- $r$ -LIN2-AA has a kernel with at most  $(2k - 1)^2 64^r$  variables.

Theorem (Kim, Williams (2011))

MAX- $r$ -LIN2-AA has a kernel with at most  $kr(r + 1)$  variables.

# Multilinear kernel for MAX- $r$ -LIN2-AA

## Theorem

MAX- $r$ -LIN2-AA has a kernel with at most  $(2k - 1)r$  variables.

## Proof - Reduction Rules

Apply known reduction rules to reduce the number of equations and variables:

### Reduction Rule (Linear Independence)

Let  $A$  be the matrix over  $\mathbb{F}_2$  corresponding to the set of equations in  $S$ , such that equation  $j$  is  $\sum_{i \in [n]} a_{ji} x_i = b_j$ . Let  $t = \text{rank} A$  and suppose columns  $a^{i_1}, \dots, a^{i_t}$  of  $A$  are linearly independent. Then delete all variables not in  $\{x_{i_1}, \dots, x_{i_t}\}$  from the equations of  $S$ .

### Reduction Rule (LHS Rule)

If we have, for a subset  $I$  of  $[n]$ , an equation  $\sum_{i \in I} x_i = b'_I$  with weight  $w'_I$ , and an equation  $\sum_{i \in I} x_i = b''_I$  with weight  $w''_I$ , then we replace this pair by one of these equations with weight  $w'_I + w''_I$  if  $b'_I = b''_I$  and, otherwise, by the equation whose weight is bigger, modifying its new weight to be the difference of the two old ones. If the resulting weight is 0, we delete the equation from the system.



## Algorithm $\mathcal{H}$

### ALGORITHM $\mathcal{H}$

While the system  $S$  is nonempty and the total weight of marked equations is less than  $2k$  do the following:

1. Choose an arbitrary equation  $\sum_{i \in I} x_i = b$  and mark an arbitrary variable  $x_l$  such that  $l \in I$ .
2. Mark this equation and delete it from the system.
3. Replace every equation  $\sum_{i \in I'} x_i = b'$  in the system containing  $x_l$  by  $\sum_{i \in I \Delta I'} x_i = b + b'$ , where  $I \Delta I'$  is the symmetric difference of  $I$  and  $I'$  (the weight of the equation is unchanged).
4. Apply Reduction Rule 2 to the system.

## Theorem

*Let  $S$  be an irreducible system and suppose that each equation contains at most  $r$  variables. Let  $n \geq (2k - 1)r + 1$  and let  $w_{\min}$  be the minimum weight of an equation of  $S$ . Then, in time  $m^{O(1)}$ , we can find an assignment  $x^0$  to variables of  $S$  such that it satisfies equations of total weight at least  $W/2 + k \cdot w_{\min}$ .*

# Sum-free Sets

- ▶ It would be good if we may mark an equation in the algorithm, and only few equations cancel out
- ▶ **Aim:** Find a set of equation that may be marked in turn, without any being cancelled out
- ▶ Let  $K \subseteq M$  be sets of vectors in  $\mathbb{F}_2^n$ .  $K$  is **M-sum-free** if no sum of two or more vectors in  $K$  is equal to a vector in  $M$

## Lemma

*Let  $M$  be the set of vectors formed from the equations in  $S$ . If there is a  $M$ -sum-free set  $K$  of size  $t$ , then we may run Algorithm  $\mathcal{H}$  for  $t$  iterations.*

# Proof of Theorem

## Proof.

- ▶ Consider a set  $M$  of vectors in  $\mathbb{F}_2^n$  corresponding to equations in  $S$ : for each equation in  $S$ , define a vector  $v = (v_1, \dots, v_n) \in M$ , where  $v_i = 1$  if  $i \in I$  and  $v_i = 0$ , otherwise.
- ▶  $M$  contains a basis for  $\mathbb{F}_2^n$ , and each vector contains at most  $r$  non-zero coordinates and  $n \geq (k-1)r + 1$
- ▶ Using a constructive lemma, we may find a Sum-Free Set  $K$  of size  $2k$ . To see this exists, consider  $K$  to be the minimal set of vectors whose sum is  $(1, 1, \dots, 1)$  (this exists, since  $M$  is a basis).
- ▶ Run Algorithm  $\mathcal{H}$  on  $K$ .
- ▶ Algorithm  $\mathcal{H}$  will run for  $2k$  iterations of the while loop as no equation from  $\{e_{j_1}, \dots, e_{j_{2k}}\}$  will be deleted before it has been marked.

# Kernel

- ▶ Each iteration of Algorithm  $\mathcal{H}$  gives a gain of 0.5 above average
- ▶ Hence, if  $n \geq (2k - 1)r + 1$ , the Theorem gives  $k$  above average
- ▶ Otherwise,  $n \leq (2k - 1)r + 1$ , the claimed kernel.

# Applications

MAX- $r$ -CSP parameterized above average (MAX- $r$ -CSP-AA)

*Instance:*

A set  $V$  of  $n$  boolean variables, and a set  $\mathcal{C}$  of  $m$  constraints, where each constraint  $C$  is a boolean function acting on at most  $r$  variables of  $V$ .

*Parameter:*  $k$ .

*Question:* Can we satisfy  $E + k$  constraints, where  $E$  is the expected number of constraints satisfied by a random assignment?

# MAX- $r$ -SAT-AA

We focus on MAX- $r$ -SAT-AA. The same methodology can be applied to MAX- $r$ -CSP-AA:

## MAX- $r$ -SAT-AA

*Instance:* A CNF formula  $F$  with  $n$  variables,  $m$  clauses, such that each clause has  $r$  variables.

*Parameter:*  $k$ .

*Question:* Can we satisfy  $\geq (1 - 1/2^r)m + k$  clauses?

- ▶ Given a random assignment, each clause is satisfied with probability  $(1 - 1/2^r)$
- ▶ The lower bound is the expected number of clauses satisfied by a random assignment.
- ▶  $(1 - 1/2^r)m$  is a tight lower bound, for example, the system of all  $2^r$  clauses on  $r$  variables.
- ▶ The condition “each clause has  $r$  variables” may be modified to “each clause has **at most**  $r$  variables”



## Pseudo-boolean functions

- ▶ Pseudo-boolean functions are both of independent interest, and a useful tool for moving between MAX- $r$ -LIN2-AA and MAX- $r$ -CSP-AA
- ▶ A **Pseudo-boolean function**  $f : \{-1, +1\}^n \rightarrow \mathbb{R}$
- ▶ Consider the Fourier Expansion of  $f$ :

$$f(x) = \sum_{S \subseteq [n]} c_S \prod_{i \in S} x_i$$

- ▶ Each term  $c_S \prod_{i \in S} x_i$  corresponds to an equation  $\sum_{i \in S} z_i = b_i$  of weight  $|c_S|$  for MAX- $r$ -LIN2-AA.
- ▶ If  $c_S$  is positive, the  $b_i = 0$ . Otherwise,  $b_i = 1$ .
- ▶  $z_i = 0$  if  $x_i = 1$ , and  $z_i = 1$  if  $x_i = -1$ . ( $x_i = (-1)^{z_i}$ )
- ▶  $f(x) = 2 \cdot (\text{Weight of satisfied equations} - W/2)$

## MAX- $r$ -SAT-AA as a pseudo-boolean function

$$f(x) = \sum_{C \in \mathcal{F}} (1 - \prod_{v_i \in C} (1 + \epsilon_i x_i))$$

- ▶ For each  $C$ ,  $\epsilon_i = 1$  if  $v_i \in C$ ,  $\epsilon_i = -1$  if  $\bar{v}_i \in C$
- ▶  $x_i = -1$  corresponds to TRUE,  $x_i = 1$  to FALSE
- ▶ Note  $f(x)$  is of degree  $r$ , so this defines a transformation to MAX- $r$ -LIN2-AA
- ▶ The transformation takes time  $O^*(2^r)$
- ▶  $(1 - \prod_{v_i \in C} (1 + \epsilon_i x_i))$  is 1 if  $C$  is satisfied,  $1 - 2^r$  if  $C$  is falsified.

## MAX- $r$ -SAT-AA as a pseudo-boolean function (contd)

$$f(x) = \sum_{C \in \mathcal{F}} (1 - \prod_{v_i \in C} (1 + \epsilon_i x_i))$$

- ▶  $f(x) = 2^r(\text{number of satisfied clauses} - (1 - 1/2^r)m)$ .
- ▶ Hence, the MAX- $r$ -SAT-AA instance is a YES-instance with parameter  $k$  iff MAX- $r$ -LIN2-AA is a YES-instance with parameter  $k' = 2^{r-1} \cdot k$ .
- ▶ But MAX- $r$ -LIN2-AA has a kernel with  $(2k' - 1)r$  variables. Hence MAX- $r$ -SAT-AA has a kernel with  $(2^r k - 1)r$  variables.

- ▶ The same approach can be applied to any general  $\text{MAX-}r\text{-CSP}$  problem
- ▶ In fact, if a class of  $\text{MAX-}r\text{-CSP}$  problems has certain symmetry, the running time/kernel may be better.

- ▶ Mahajan, Rama & Sikdar (2006) asked if MAX-LIN2-AA is FPT.

Theorem (Crowston, Fellows, Gutin, Jones, Rosamond, Thomasse, Yeo, 2011)

MAX-LIN-AA is fixed-parameter tractable, and has a kernel with  $O(k^2 \log k)$  variables.

# Recent results on MAX- $r$ -SAT-AA

Whilst MAX-LIN2-AA and MAX- $r$ -LIN2-AA have polynomial kernels, the same does not hold for MAX- $r$ -SAT-AA

Theorems (Crowston, Gutin, Jones, Raman, Saurabh (2011))

- ▶ MAX- $r$ -SAT-AA is *para-NP-complete* for  $r = \lceil \log n \rceil$ .
- ▶ Assuming the exponential time hypothesis, MAX- $r$ -SAT-AA is not fixed-parameter tractable for any  $r \geq \log \log n + \phi(n)$ , where  $\phi(n)$  is any unbounded strictly increasing function of  $n$ .
- ▶ MAX- $r$ -SAT-AA is fixed-parameter tractable for  $r \leq \log \log n - \log \log \log n - \phi(n)$ , for any unbounded strictly increasing function  $\phi(n)$ .

# The End

- ▶ Thank You
- ▶ Questions?