Complexity of Permutation Constraint Satisfaction Problems Parameterized Above Average

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Outline

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Fixed-parameter Tractability

- A parameterized problem Π: a set of pairs (I, k) where I is the main part and k (usually an integer) is the parameter; I is an instance.
- Π is fixed-parameter tractable (FPT) if membership of (I, k) in Π can be decided in time $O(f(k)|I|^c)$, where $|I|$ is the size of I, $c = O(1)$ and $f(k)$ is a computable function.
- The idea: for small values of k, $O(f(k)|I|^c)$ is **not too large**.
- Examples: $O(2^k m^2)$, $2^{O(k \log k)} m^3$.

Kernelization

- A kernelization of Π: a polynomial-time algorithm that maps an instance $(x,k) \in \Pi$ to an instance $(x',k') \in \Pi$ (the kernel) such that
  - (x,k) is YES iff (x',k') is YES
  - $k' \le f(k)$ and $|x'| \le g(k)$ for some functions f and g.
- The function $g(k)$ is called the size of the kernel.
- A parameterized problem is FPT if and only if it is decidable and admits a kernelization.
- Wanted: low degree polynomial-size kernels (for preprocessing).

Outline

Various Parameterizations

- Standard parameterizations: the parameter is the size of a set to optimize.
- Parameterizations above and below tight bounds. Formally suggested by Mahajan and Raman (1999).

Acyclic Subgraphs of Digraphs: Standard Parameterization

- Given a digraph $D = (V, A)$, find an acyclic subgraph $H = (V, B)$ of D with the maximum number of arcs.
- Standard parameterization: $k = |B|$. Namely, does D have an acyclic subgraph with at least k arcs?
- Standard parameterization is FPT as $|B| \geq |A|/2$: if $k \leq |A|/2$ the answer is YES otherwise $|V| \leq |A| + 1 \leq 2k$ and we use a brute-force algorithm of running time $|A|^{O(1)}(2k)!$ to check whether the answer is YES.
- k is supposed to be small (for $|A|^{O(1)}(2k)!$ to be tractable), but $k > |A|/2$ is large when $|A|$ is large.

Acyclic Subgraphs of Digraphs: Parameterization above the Average

- The expected number of arcs in an acyclic subgraph of a digraph $D = (V, A)$ is $|A|/2$.
- In particular, each digraph has an acyclic subgraph with with $\geq |A|/2$ arcs.
- The bound is tight: For symmetric digraphs, $k = 0$: a digraph D is symmetric if $xy \in A$ implies $yx \in A$.
- Parameterization Above Average (AA): Does $D = (V, A)$ have an acyclic subgraph with at least $|A|/2 + k$ arcs? [ACYCLIC AA]
- Mahajan, Raman and Sikdar (2009): Is ACYCLIC AA fixed-parameter tractable?

Other Problems PATLB-1

- Parameterization above average is a special case of pameterization above a tight bound.
- (Mahajan and Raman, 1999): A CNF formula F with m clauses. Is there a truth assignment satisfying at least $m/2 + k$ clauses of F? FPT, $O(k)$-size kernel. [It is not AA]
- (Crowston, Gutin, Jones, Yeo, 2010): A unit-conflict free CNF formula F with m clauses. Is there a truth assignment satisfying at least $pm + k$ clauses of F? Here $p = (\sqrt{5} - 1)/2$ and pm is an asymptotically tight lower bound (for some F we cannot satisfy $(p + \epsilon)m$ clauses). Kernel with $\lfloor 4k/(7 - 3\sqrt{5}) \rfloor$ vars.
- The standard parameterization should become that above/below (asymptotically) tight bounds !

Other Problems PATLB-2

- MAX-r-SAT: given a CNF formula with m clauses, each clause has r vars, find an assignment satisfying max number of clauses.
- Average $= (1-2^{-r})m$. MAX-r-SAT AA: Can we satisfy $\geq (1-2^{-r})m+k$ clauses?
- (Alon, Gutin, Kim, Szeider, Yeo 2010): MAX-r-SAT AA has a kernel of size $O(k^2)$ with $O(k^2)$ vars; for $r=2$ $O(k)$ vars.
- (Crowston, Gutin, Jones, Kim, Ruzsa, 2010): MAX-r-SAT AA has a kernel of size $O(k\log k)$ with $O(k\log k)$ vars.
- (Kim and Williams, 2010): MAX-r-SAT AA has a kernel with $\leq r(r+1)k$ vars.

Outline

Strictly Above/Below Expectation Method (SABEM): Symmetric Case

- Gutin, Kim, Szeider and Yeo (2009). Problem Π parameterized AA.
- Apply some reduction rules.
- Introduce a random variable X s.t. $\mathbb{E}(X) = 0$ and if $\text{Prob}(X \geq k) > 0$ then the answer to Π is YES.
- If X is symmetric ($\text{Prob}(X = a) = \text{Prob}(X = -a)$ for each a), then $\text{Prob}(\ X \geq \sqrt{\mathbb{E}(X^2)}\ ) > 0$.
- If $\sqrt{\mathbb{E}(X^2)} \geq k$ then YES. Otherwise, $\sqrt{\mathbb{E}(X^2)} < k$ and we can often solve the problem using a brute-force algorithm.

SABEM: Asymmetric Case

Lemma (Alon, Gutin, Krivelevich, 2004; Alon, Gutin, Kim, Szeider, Yeo, 2010)

Let X be a real random variable and suppose that its first, second and forth moments satisfy $\mathbb{E}(X) = 0$, $\mathbb{E}(X^2) = \sigma^2 > 0$ and $\mathbb{E}(X^4) \leq b(\mathbb{E}(X^2))^2$, respectively. Then $\text{Prob}(\ X > \frac{\sigma}{2\sqrt{b}}\ ) > 0$.

Lemma (Hypercontractive Inequality, Bonami, 1970)

If a random variable X is expressible as a polynomial of degree r in independent uniformly distributed variables with values -1 and 1, then $\mathbb{E}(X^4) \leq 9^r(\mathbb{E}(X^2))^2$.

Outline

Reduction Rule for Linear Ordering-2 Problem AA

- Linear Ordering-2 AA: each arc ij has positive integral weight w_{ij}, does $D = (V, A)$ have an acyclic subgraph of weight at least $W/2 + k$, where $W = \sum_{ij \in A} w_{ij}$?
- Reduction rule: Assume D has a directed 2-cycle iji;
  - if $w_{ij} = w_{ji}$ delete the cycle,
  - if $w_{ij} > w_{ji}$ delete the arc ji and replace w_{ij} by $w_{ij} - w_{ji}$,
  - if $w_{ji} > w_{ij}$ delete the arc ij and replace w_{ji} by $w_{ji} - w_{ij}$.
- Thus, we've reduced Linear Ordering-2 AA to the one on oriented graphs.

SABEM for Linear Ordering-2 AA-1

- Let $D = (V, A)$ be an oriented graph, let $n = |V|$.
- Consider a random bijection: $\alpha : V \rightarrow [n]$ and a random variable $X = \frac{1}{2}\sum_{ij\in A} x_{ij}$, where $x_{ij} = w_{ij}$ if $\alpha(i) < \alpha(j)$ and $x_{ij} = -w_{ij}$, otherwise.
- $X(\alpha) = \sum\{w_{ij} : \ ij \in A, \alpha(i) < \alpha(j)\} - W/2$. Thus, the answer is YES iff there is an $\alpha : V \rightarrow [n]$ such that $X(\alpha) \geq k$.
- Since $\mathbb{E}(x_{ij}) = 0$, we have $\mathbb{E}(X) = 0$.

SABEM for Linear Ordering-2 AA-2

Lemma

$\mathbb{E}(X^2) \geq W^{(2)}/12$, *where* $W^{(2)} = \sum_{ij \in A} w_{ij}^2$.

Since X is symmetric, we have Prob($X \geq \sqrt{W^{(2)}/12}$) > 0. Hence, if $\sqrt{W^{(2)}/12} \geq k$, there is an $\alpha : V \rightarrow \{1, \ldots, n\}$ such that $X(\alpha) \geq k$ and, thus, the answer is YES. Otherwise, $|A| \leq W^{(2)} < 12 \cdot k^2$. Thus, we have:

Theorem (Gutin, Kim, Szeider, Yeo, 2009)

LINEAR ORDERING-2 AA *is FPT and has an* $O(k^2)$*-size kernel.*

Outline

Betweenness

- Let $V = \{v_1, \ldots, v_n\}$ be a set of variables and let $\mathcal{C}$ be a set of m betweenness constraints of the form $(v_i, \{v_j, v_k\})$.
- Given a bijection $\alpha : V \to \{1, \ldots, n\}$, we say that a constraint $(v_i, \{v_j, v_k\})$ is satisfied if either $\alpha(v_j) < \alpha(v_i) < \alpha(v_k)$ or $\alpha(v_k) < \alpha(v_i) < \alpha(v_j)$.
- BETWEENNESS: find a bijection α satisfying the max number of constraints in $\mathcal{C}$.

Betweenness AA

- Tight Lower Bound: $m/3$, the expectation number of satisfied constraints is $m/3$.
- Betweenness AA: Is there α that satisfies $\geq m/3 + \kappa$ constraints? (κ is the parameter)
- Benny Chor's question in Niedermeier's book (2006): What is the parameterized complexity of Betweenness AA?

Difficulties for BETWEENNESS AA

- We could not find symmetric random var X for SABEM.
- Asymmetric random var X for SABEM is easy to obtain, but ...
- Difficult to estimate $\mathbb{E}(X^2)$, practically impossible to do $\mathbb{E}(X^4)$, and we cannot use Hypercontractive Inequality as X is not a polynomial of constant-bounded degree in n independent vars $\in \{-1, 1\}$.
- What to do?

Way Around Difficulties-1

- An instance $(V, \mathcal{C})$, where V is the set of variables and $\mathcal{C} = \{C_1, \ldots, C_m\}$ is the set of betweenness constraints.
- A function $\phi : V \to \{0, 1, 2, 3\}$.
- A ϕ-compatible bijection $\alpha : V \to [n]$: if $\phi(v_i) < \phi(v_j)$ then $\alpha(v_i) < \alpha(v_j)$.

Way Around Difficulties-2

- Let α be a random ϕ-compatible bijection and $\nu_p(\alpha) = 1$ if C_p is satisfied and 0, otherwise.
- Let the *weights* $w(C_p, \phi) = \mathbb{E}(\nu_p(\alpha)) - 1/3$ and $w(\mathcal{C}, \phi) = \sum_{p=1}^{m} w(C_p, \phi)$.

Lemma

If $w(\mathcal{C}, \phi) \geq \kappa$ then $(V, \mathcal{C})$ is a YES*-instance of* BETWEENNESS AA.

- Thus, to solve BETWEENNESS AA, it suffices to find ϕ for which $w(\mathcal{C}, \phi) \geq \kappa$.
- We may forget about bijections α !

Way Around Difficulties-4

We call a triple A, B, C of distinct betweenness constraints complete if $vars(A) = vars(B) = vars(C)$.
Reduction rule: if $\mathcal{C}$ contains a complete triple of constraints, delete these constraints from $\mathcal{C}$ and delete from V any variable that appears only in the triple.

Lemma

Let $(V, \mathcal{C}, \kappa)$ be an instance and let $(V', \mathcal{C}', \kappa)$ be obtained from $(V, \mathcal{C}, \kappa)$ by applying the reduction rule as long as possible. Then $(V, \mathcal{C}, \kappa)$ and $(V', \mathcal{C}', \kappa)$ are equivalent.

Way Around Difficulties-5

- Let $X = w(\mathcal{C}, \phi)$ and $X_p = w(C_p, \phi)$, $p \in [m]$.
- Let ϕ be a random function from V to $\{0, 1, 2, 3\}$. Then $X, X_1, \ldots, X_m$ are random vars.
- We have $\mathbb{E}[X] = 0$.
- X can be expressed as a polynomial of degree 6 in independent uniformly distributed random variables with values -1 and 1.

Way Around Difficulties-6

Lemma

For an irreducible instance $(V, \mathcal{C}, \kappa)$ *we have* $\mathbb{E}[X^2] \geq \frac{11}{768}m$.

Main Result for BETWEENNESS AA

Gutin, Kim, Mnich and Yeo, 2009:

Theorem

BETWEENNESS AA *has a kernel of size* $O(\kappa^2)$.

Proof: Let $(V, \mathcal{C})$ be an instance. In time $O(m^3)$ we can obtain an irreducible instance $(V', \mathcal{C}')$. Let $m' = |\mathcal{C}'|$ and let X be the random variable defined above. Then X is expressible as a polynomial of degree 6; hence by Hypercontractive Inequality $\mathbb{E}[X^4] \leq 9^6\mathbb{E}[X^2]^2$. Thus, $\mathbb{P}\left(X > \frac{1}{2 \cdot 9^3}\sqrt{\frac{11}{768}m'}\right) > 0$. By Lemma 5 if $\frac{1}{2 \cdot 9^3}\sqrt{\frac{11}{768}m'} \geq \kappa$ then $(V', \mathcal{C}')$ is a YES-instance. Otherwise, we have $m' = O(\kappa^2)$.

Outline

Ternary Permutation CSPs-1

- Let V be a set of n variables.
- Group $\mathcal{S}_3 = \{(123), (132), (213), (231), (312), (321)\}$.
- A constraint set over V is a multiset $\mathcal{C}$ of constraints, which are permutations of three distinct elements of V.
- For each subset $\Pi \subseteq \mathcal{S}_3$ and a bijection $\alpha: V \to [n]$, a constraint $(v_1, v_2, v_3) \in \mathcal{C}$ is Π-satisfied by α if there is a permutation $\pi \in \Pi$ such that $\alpha(v_{\pi(1)}) < \alpha(v_{\pi(2)}) < \alpha(v_{\pi(3)})$.
- For a subset $\Pi \subseteq \mathcal{S}_3$, the problem Π-CSP is to decide whether for a given pair $(V, \mathcal{C})$ of variables and constraints there is a bijection α of V that Π-satisfies all constraints in $\mathcal{C}$.

Ternary Permutation CSPs-2

- Guttmann and Maucher, 2006: There are 64 Π-CSP problems, but only 13 up to some symmetries. Of which 11 are nontrivial, i.e., $\Pi \notin \{\emptyset, \mathcal{S}_3\}$.
- Guttmann and Maucher, 2006: A complete dichotomy of the 11 Π-CSP problems with respect to their computational complexity (4 polynomial and 7 NP-complete problems).

Ternary Permutation CSPs-3

$\Pi \subseteq \mathcal{S}_3$	Name	Complexity
$\Pi_0 = \{(123)\}$	Linear Ordering-3	polynomial
$\Pi_1 = \{(123),(132)\}$		polynomial
$\Pi_2 = \{(123),(213),(231)\}$		polynomial
$\Pi_3 = \{(132),(231),(312),(321)\}$		polynomial
$\Pi_4 = \{(123),(231)\}$		NP-comp.
$\Pi_5 = \{(123),(321)\}$	Betweenness	NP-comp.
$\Pi_6 = \{(123),(132),(231)\}$		NP-comp.
$\Pi_7 = \{(123),(231),(312)\}$	Circular Ordering	NP-comp.
$\Pi_8 = \mathcal{S}_3 \setminus \{(123),(231)\}$		NP-comp.
$\Pi_9 = \mathcal{S}_3 \setminus \{(123),(321)\}$	Non-Betweenness	NP-comp.
$\Pi_{10} = \mathcal{S}_3 \setminus \{(123)\}$		NP-comp.

Ternary Permutation CSPs-4

- $\textsc{Max}$-Π-CSP problem: find the max number of constrains satisfied by a bijection.
- Gutin, van Iersel, Mnich, Yeo, 2010: All the 11 $\textsc{Max}$-Π-CSP problems are NP-hard.
- Average $= \frac{|\Pi|}{6}|\mathcal{C}|$.
- Π-CSP AA: Is there a bijection satisfying at least Average $+$ k constraints?

Ternary Permutation CSPs-3

- Gutin, van Iersel, Mnich, Yeo, 2010: All the 11 Π-$\mathrm{CSP\ AA}$ problems admit kernels with $O(k^2)$ vars.
- Proof Idea 1: All the 11 Π-$\mathrm{CSP\ AA}$ can be reduced to Π_0-$\mathrm{CSP\ AA}$, where $\Pi_0 = (1,2,3)$.
- Proof Idea 2: Π_0-$\mathrm{CSP\ AA} = \textsc{Betweenness}\ \mathrm{AA} \bigoplus \textsc{Linear Ordering-2}\ \mathrm{AA}$.
- Example: (u,v,w) is Π_0-satisfied by α iff $\alpha(u) < \alpha(v) < \alpha(w)$ iff $(v,\{u,w\})$ is Π_5-satisfied by α and both (u,v) and (v,w) are LinOrd-2-satisfied by α. Yet $(v,\{u,w\})$ and $((u,v)$ and $(v,w))$ are "independent".

Thank you!

- Questions?
- Comments?