# Where Myhill–Nerode Theorem Meets Parameterized Algorithmics

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- Explicit comb. extensions of this concept appeared e.g. in the works [Abrahamson and Fellows, 93], [PH, 03], or [Ganian and PH, 08].

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- Informally, the classes of $\approx_{\mathcal{P},k}$ capture all information about the property $\mathcal{P}$ that can "cross" our boundary of size $k$
(regardless of actual meaning of "boundary" and "join").

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- For simplicity, solution fragments $\varphi$ can be "embedded" in $\mathcal{U}_k$ and $\otimes$.
- Can, e.g., count the solutions in each class of $\approx_{\mathcal{P},k}$, or keep an opt. one.

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  – yes, still works quite nicely, cf. [Ganian, PH, Obdržálek, 09].

  – brings new application issues such as "quantification inside $\otimes$" (cf. sol. fragments), or a "second-level" congruence on top of $\approx_{\mathcal{P},k}$.

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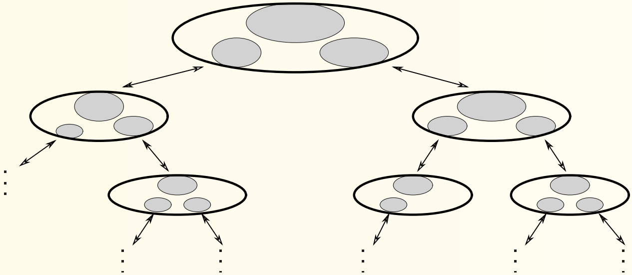
- This can be (visually) seen as...



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$\longrightarrow$ giving the *expression tree* (parse tree) for clique-width.

## Rank-decomposition

- [Oum and Seymour, 03] Bringing the branch-decomposition approach to measure "complexity" of vertex subsets $X \subseteq V(G)$ via *cut-rank*:

$$\varrho_G(X) = \text{rank of} \quad \begin{array}{cc} & V(G)-X \\ X & \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array} \text{ modulo } 2$$

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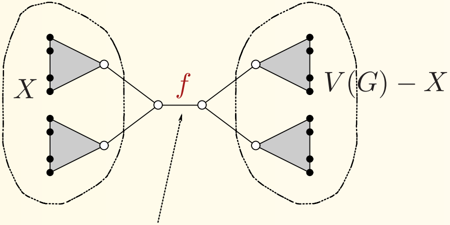


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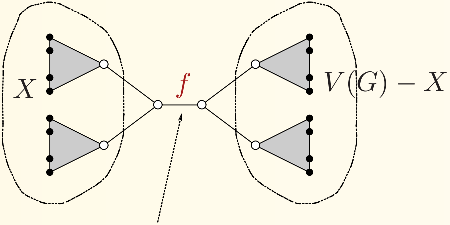


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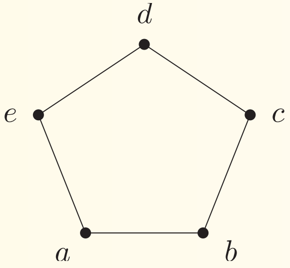
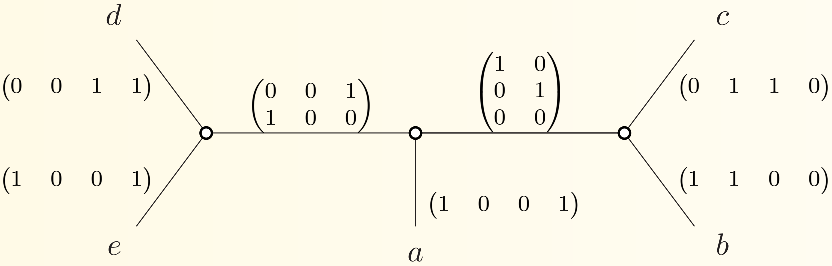
- **Rank-width** $= \min_{\text{rank-decs. of } G} \max \{\text{width}(f) : f \text{ tree edge}\}$

**An example.** Cycle $C_5$ and its *rank-decomposition* of width $2$:



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- [Oum and PH, 07] There is an *FPT algorithm* for computing an optimal width-$t$ rank-decomposition of a graph in time $O(f(t) \cdot n^3)$.
- And new results show that certain algorithms designed on rank-decompositions run faster than their analogues designed on clique-width expressions... (subst. $poly(t)$ in place of $cw$, instead of $2^t$)

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$$\text{edge } uv \;\leftrightarrow\; lab(u) \cdot \mathbf{g} \cdot lab(v) = 1.$$

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Unlike for tree- or clique- decompositions with obvious parse trees, what is the "boundary" and "join" operation for rank-width?

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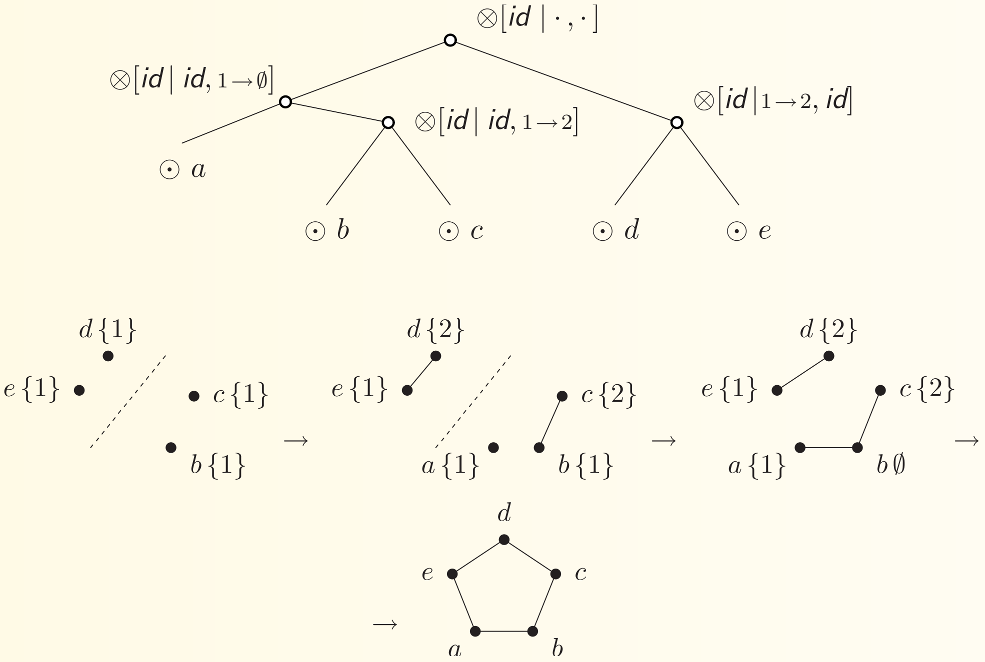
- Independently considered related notion of $R_t$-*join* decompositions by [Bui-Xuan, Telle, and Vatshelle, 08].

**A parse tree.** An example generating the cycle $C_5$ (of rank-width $2$):



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**Where is the problem?**

A resulting double-exponential worst-case dependency on a width estimate!

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**Our answer** – considering *rank-width*:

- No loss in the promissed width, and yet single-exponential in it.
- A clear and rigorous algorithm employing many of the above tricks.

**Theorem.** [Ganian, PH, Obdržálek, 10] #SAT solved in FPT time

$$\mathcal{O}(t^3 \cdot 2^{3t(t+1)/2} \cdot |\phi|)$$

where $t$ is the *signed rank-width* of the input instance (CNF formula) $\phi$.

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  Then

  $$G_1 \oplus G_2 \; = \; \left(G_1^+ \oplus G_2^+\right) \cup \left(G_1^- \oplus G_2^-\right)$$

  and the same decomposition is used.

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Easy to prove. . . , but does it help?

Subsets of labels from $2^{\{1,2,\dots,t\}}$ $\longrightarrow$ $\Omega\left(2^{2^t}\right)$ classes!

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**Conclusion.** Breaking the satisfying assignments of $\phi$ into $S(t)^4$ classes, and processing a node of the parse tree in $O^*\big(S(t)^6\big)$. $\square$

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