Does Treewidth Help in Modal Satisfiability?

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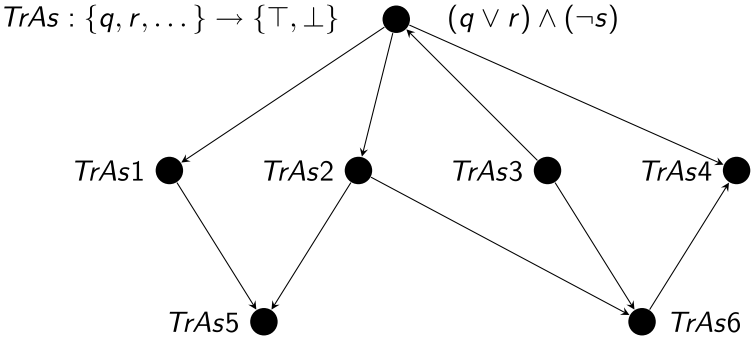
Introduction — Modal Logic

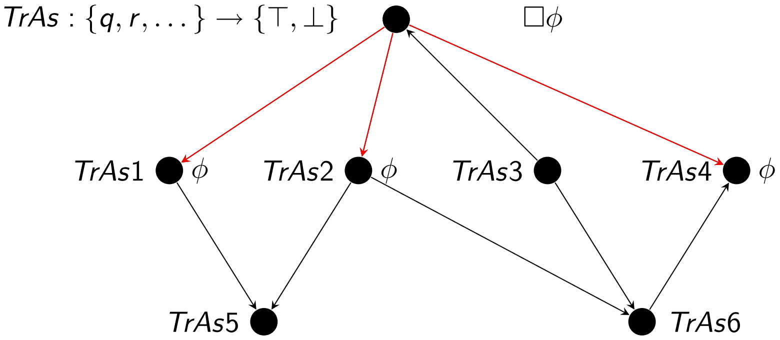
$TrAs : \{q, r, \dots\} \to \{\top, \bot\}$ ● $(q \vee r) \wedge (\neg s)$

Introduction — Modal Logic



Kripke structure

Introduction — Modal Logic



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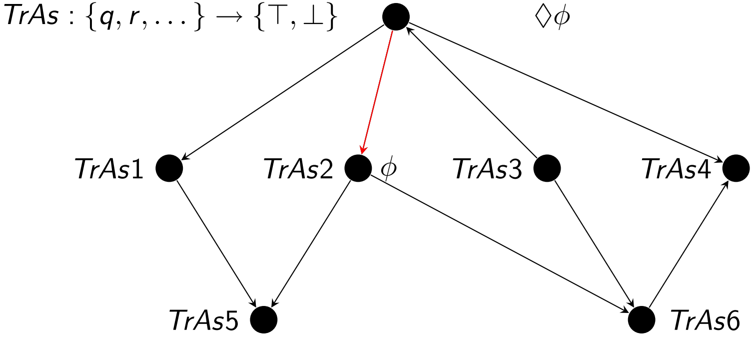

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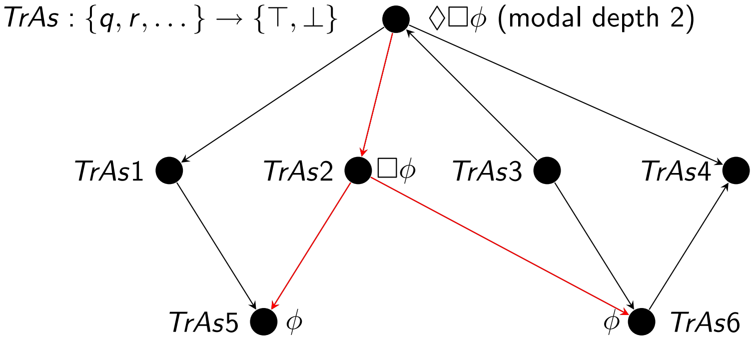

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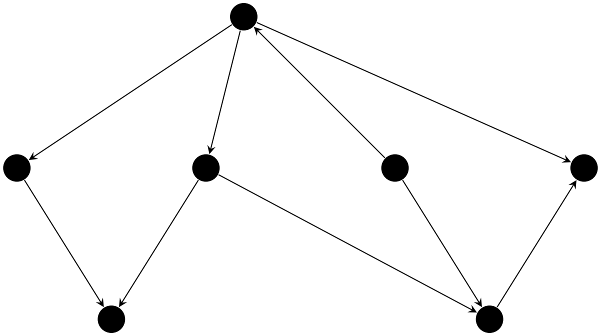

Underlying frame

Introduction — Problem Definition

- Modal logic is an extension of Propositional logic:

  $\phi ::= q$ (propositional variable) $\mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \phi_1 \vee \phi_2 \mid \Box\phi \mid \Diamond\phi$

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- [Ladner 1977]: Modal satisfiability, satisfiability in reflexive and transitive structures are PSPACE-complete. Satisfiability in structures with equivalence relation is NP-complete.

Related work

- [Halpern 1995]: Effect of bounding number of propositional variables and modal depth.
- [Dalmau, Kolaitis, Vardi 2002]: Treewidth, constraint satisfaction and finite variable logics. [Samer, Szeider 2010]: Constraint satisfaction with bounded treewidth.
- [Chen 2004]: Quantified constrained satisfaction and bounded treewidth. [Pan, Vardi 2006]: Non-elementary lower bound for QBF with bounded treewidth.
- [Nguyen 2005]: Complexity of syntactically restricted fragments of modal logic.
- [Halpern, Rêgo 2007]: Characterizing NP-PSPACE gap.
- [Hemaspaandra, Schnoor 2008]: Dichotomy based on restricting frames with first order horn formulas.
- [Kazakov,Pratt-Hartmann 2009]: Complexity of graded modal logics.

Related work (Contd. . .)

- [Adler, Weyer 2009]: Treewidth for model checking first order formulae.
- [Achilleos, Lampis, Mitsou 2010]: Many parameters — number of variables, modal depth, modal width, diamond dimension, box dimension.

Introduction — Parameterized Complexity

- For NP-complete decision problems, the best known deterministic algorithms usually have running time of the form $2^{\mathrm{poly}(n)}$, where n is the input size.
- If a problem instance has some underlying simplicity measured by a parameter k (e.g., number of propositional variables), we look for algorithms with running time of the form $f(k)\mathrm{poly}(n)$.
- Parameterized problems that can be solved by algorithms with running time of the form $f(k)\mathrm{poly}(n)$ are said to be Fixed Parameter Tractable (FPT).
- There is a hierarchy of hard parameterized problems $\mathrm{W}[1], \mathrm{W}[2], \ldots$. Problems hard for $\mathrm{W}[1]$ are unlikely to be FPT.

Our results

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- In the context of modal logic, we use treewidth to measure interconnectivity between different parts of a modal formula.
- With treewidth and modal depth as parameters, we study the parameterized complexity of modal satisfiability with various restrictions on the underlying frame.
- Main results: With treewidth and modal depth as parameters, satisfiability in general structures is FPT while it is $\mathrm{W}[1]$-hard for transitive structures.

Motivation

- Modal logic has many applications like reasoning about knowledge, programming, hardware verification etc.
- Despite being intractable in complexity theoretic sense, many tools have been built.
- Treewidth has been studied in the context of many other problems like constraint satisfaction, quantified constraint satisfaction etc. Helps to understand where and why treewidth works.
- Different conditions on the underlying frame affects complexity significantly. Parameterized complexity can strengthen this classification.

Treewidth and its relevance — Incidence Graph

$$
\begin{array}{ccccccc}
c_{\ell 1} & & c_{\ell 2} & & c_{\ell 3} & & c_{\ell 4} \\
(x_1 \vee \neg x_2) & \wedge & (x_3 \vee x_4) & \wedge & x_5 & \wedge & (\neg x_6 \vee \neg x_7)
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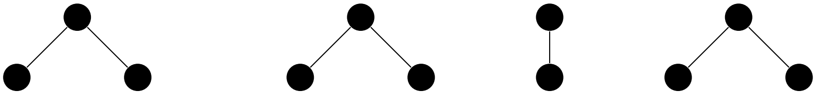

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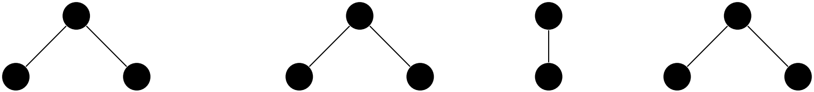

$c_{\ell 1}$ $c_{\ell 2}$ $c_{\ell 3}$ $c_{\ell 4}$

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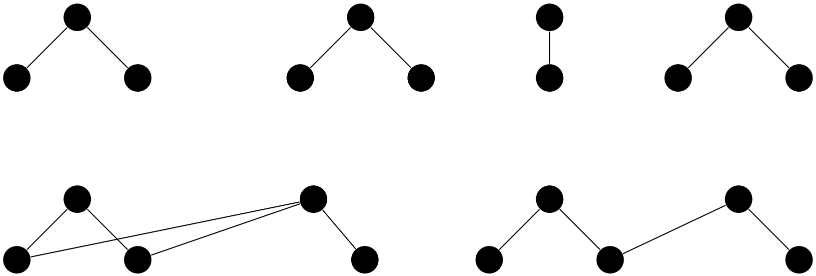

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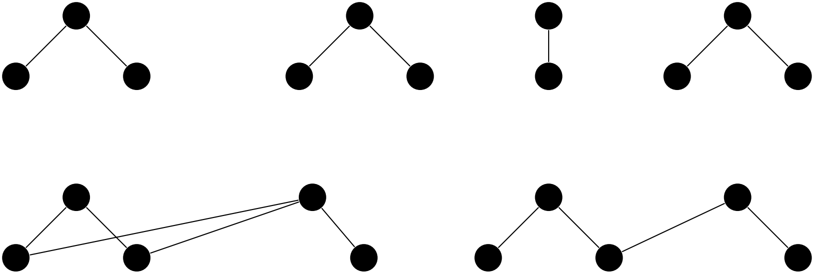

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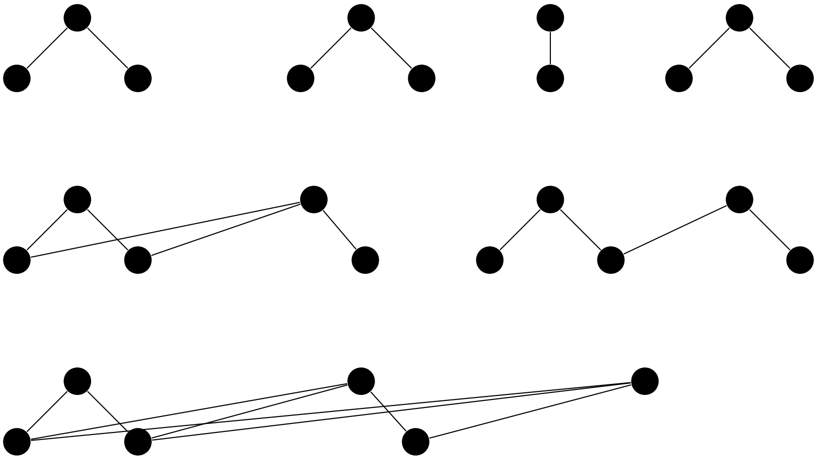

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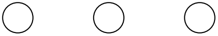

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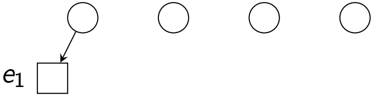

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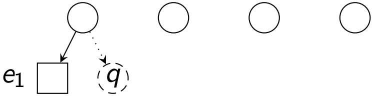

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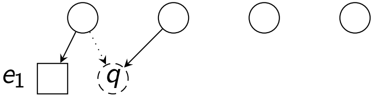

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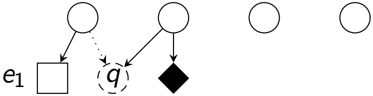

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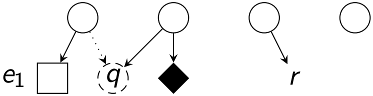

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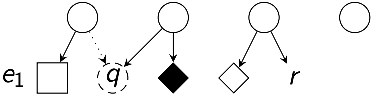

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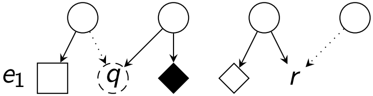

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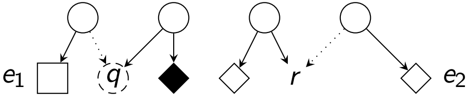

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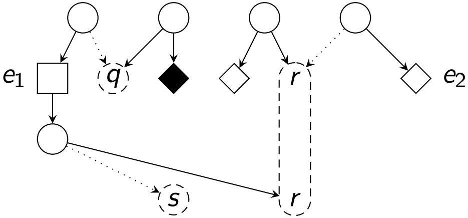

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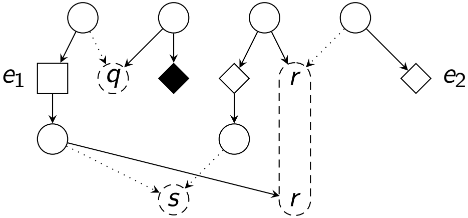

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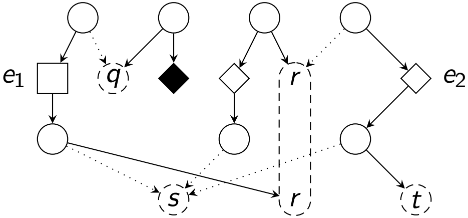

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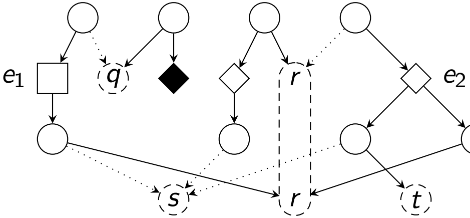

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With only treewidth as parameter, modal satisfiability is not FPT unless PTIME=NP (follows from [Achilleos, Lampis, Mitsou 2010]).	If modal depth is bounded, modal satisfiability is NP-complete [Halpern 1995].

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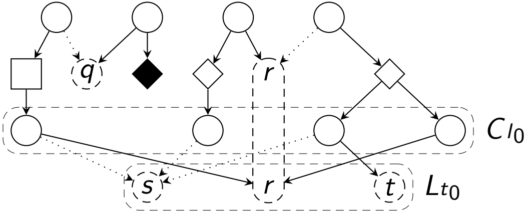
With treewidth and modal depth as parameters, satisfiability in transitive structures is W[1]-hard.	Even with modal depth 2, satisfiability in transitive structures is PSPACE-complete [Halpern 1995].
With only treewidth as parameter, satisfiability in transitive and Euclidean structures is FPT.	Adding Euclidean property results in a drop from PSPACE-hardness to NP-completeness in infinitely many modal logics [Halpern, Rêgo 2007].

Proof Sketch — FPT results



There is a subset of literals in L_{t0} such that, for every clause in C_{l_0}, there is a variable in the subset occurring positively or there is a variable not in the subset occurring negatively.

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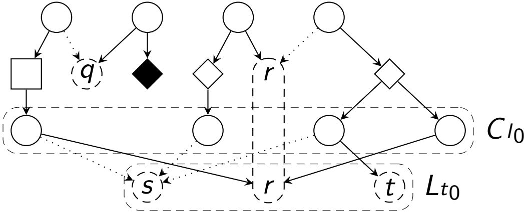

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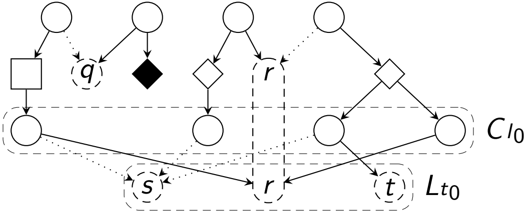

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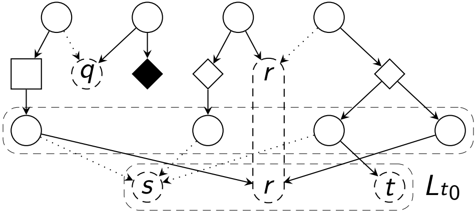

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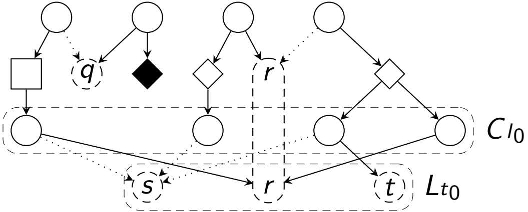

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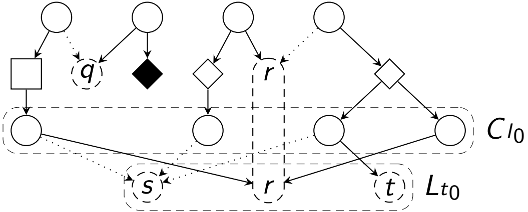

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Proof Sketch — FPT results (Contd. . .)

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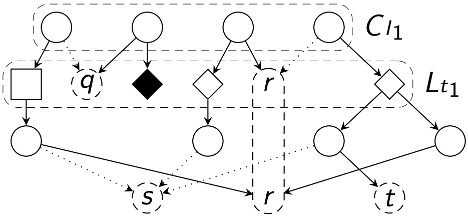
- Constant length MSO formula can be written for special cases like Euclidean property etc.

Proof Details — FPT results Skip



$$\xi[i](C_{\ell_i}) \stackrel{\triangle}{=} \exists T_{r_i} \subseteq (L_t \cap L_{v_i}) : \forall c_\ell \in C_{\ell_i} :$$
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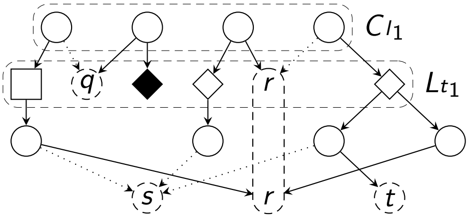

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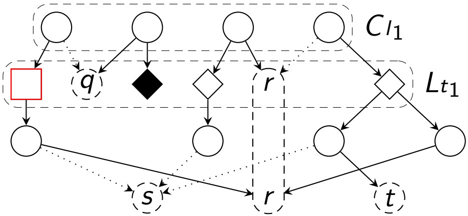
Proof Details — FPT results Skip



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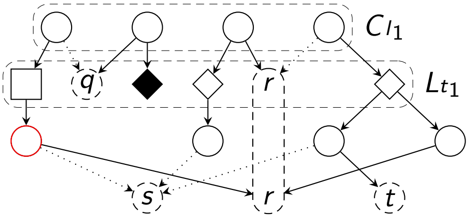

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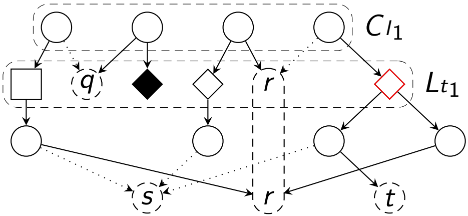

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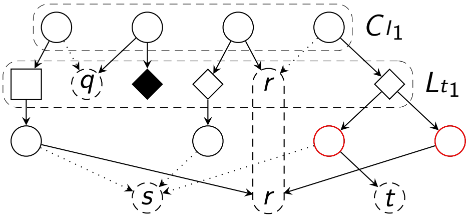

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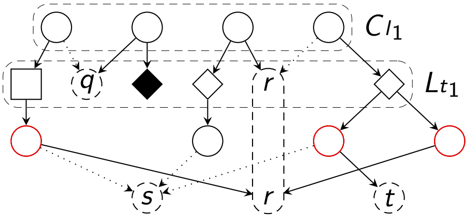

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Proof Sketch — $\mathrm{W}[1]$-hardness

- [Fellows, Fomin, Lokshtanov, Rosamund, Saurabh, Szeider, Thomassen 2007]: Counting on graphs is hard even when treewidth is constant.
- Observation: In presence of transitivity, short modal logic formulas can count easily.

Conclusion

- Satisfiability of a modal logic formula in CNF is a MSO property of its incidence graph.
- Courcelle's theorem gives a FPT algorithm for modal satisfiability when treewidth and modal depth are parameters.
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