Parameterized Graph Editing with Chosen Vertex Degrees

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Abstract. We study the parameterized complexity of the following problem: is it possible to make a given graph r-regular by applying at most k elementary editing operations; the operations are vertex deletion, edge deletion, and edge addition. We also consider more general annotated variants of this problem, where vertices and edges are assigned an integer cost and each vertex v has assigned its own desired degree $\delta(v) \in \{0, \ldots, r\}$. We show that both problems are fixed-parameter tractable when parameterized by (k, r), but $W[1]$-hard when parameterized by k alone. These results extend our earlier results on problems that are defined similarly but where edge addition is not available. We also show that if edge addition and/or deletion are the only available operations, then the problems are solvable in polynomial time. This completes the classification for all combinations of the three considered editing operations.

1 Introduction

Deciding whether a given graph has a regular subgraph is a well studied problem. Chvátal et al. [5] give one of the earliest results, showing that the CUBIC SUBGRAPH problem is NP-complete. Plesník [17] proves that it remains NP-complete even when restricted to a planar bipartite graph with maximum degree 4. In the same paper he also shows that the r-REGULAR SUBGRAPH problem for $r \geq 3$ is NP-complete even on bipartite graphs of degree at most $r + 1$. Cheah and Corneil [4] show that a similar result holds for general graphs of degree at most $r + 1$. A series of results for further constraints is given by Stewart [18–20]. Bodlaender et al. [2] give a polynomial-time algorithm for producing a Δ-regular supergraph of a graph with maximum degree Δ, using at most $\Delta + 2$ additional vertices. Moser and Thilikos [15] give a series of results for certain parameterized versions, showing that when parameterized by the size of the regular subgraph, the problem is $W[1]$-hard, but when parameterized by both the number k of vertices to remove to make the graph regular and the regularity r, the problem is fixed-parameter tractable. In previous work [14] we show that when parameterized by k alone, the problem is $W[1]$-hard (a question left open by Moser and Thilikos). We also introduce a generalized version of the problem where vertices and edges are weighted, and each vertex has a degree function that specifies the number of edges to be incident on the vertex, rather than simply a fixed number for all vertices. When parameterized by the number k of edges and vertices to remove to obtain the regular graph, the problem is $W[1]$-hard, but when parameterized by k and the bound r on the degree function, the problem is fixed-parameter tractable. Interestingly the latter result improves that of Moser and Thilikos even though it is generalized, and additionally allows edge deletion.

In this paper we extend the editing operations available to include edge addition besides vertex and edge deletion (see Section 2.1 for precise definitions), thus giving the following problems.

EDIT TO REGULAR GRAPH
Instance: A graph $G = (V, E)$, two nonnegative integers k and r.
Question: Is there an r-regular graph H obtainable from G by at most k edit operations?

WEIGHTED EDIT TO CHOSEN DEGREE GRAPH
Instance: A graph $G = (V, E)$, nonnegative integers k and r, a weight function $\rho : V \cup E \to \{1, \ldots, k+1\}$ and a degree function $\delta : V \to \{0, \ldots, r\}$.
Question: Is there a graph H obtainable from G by edit operations of total cost at most k such that $\sum_{e \in E(v)} \rho(e) = \delta(v)$ holds for each vertex in H?

Variants of the above two problems with only one or two of the three editing operations available are defined similarly.

We previously demonstrated [14] that these two problems are $W[1]$-hard when parameterized by k. In this paper we complete the classification and show that they are both fixed-parameter tractable with parameter (k, r). We also give a simpler proof that the weighted edit problem is $W[1]$-hard when parameterized by k alone, via the $W[1]$-hardness of the related subproblem EDGE REPLACEMENT SET (see Section 4). We also prove that EDGE REPLACEMENT SET is NP-complete, giving an indication that a polynomial-time kernelization of the form previously used for the deletion version of the problems is unlikely to exist for the edit version. Additionally, we show that WEIGHTED EDGE EDIT TO CHOSEN DEGREE GRAPH, and the unweighted counterpart EDGE EDIT TO REGULAR GRAPH, where the edit operations are edge addition and deletion, both have polynomial-time algorithms. The results are summarized in Table below.

2 Preliminaries

2.1 Graph Modification

Graph modification or graph editing problems are widespread throughout the literature appearing in various forms in such areas as bioinformatics [6], electronic commerce [9] and graph theory [1]. Three fundamental operations for graph editing are edge deletion, vertex deletion and edge addition. For any combination of these three operations Cai [3] demonstrates fixed-parameter tractability for graph properties with finitely many obstructions in the induced order.

In this paper we consider simple, undirected graphs (whether weighted or unweighted). The edge between two vertices u and v is denoted uv (or equivalently vu). The degree of a vertex u is denoted $d(u)$.

Problem	Operations	Parameter	
		k	(k, r)
Uniform	v	$W[1]$-hard[†]	FPT[‡†]
	e+v	$W[1]$-hard[†]	FPT[†]
	v+a	$W[1]$-hard[†]	FPT[*]
	e+v+a	$W[1]$-hard[*†]	FPT[*]
	a	P[¶*]	P[¶*]
	e	P[¶*]	P[¶*]
	e+a	P[*]	P[*]
Annotated	v	$W[1]$-hard[†]	FPT[†]
	e+v	$W[1]$-hard[†]	FPT[†]
	v+a	$W[1]$-hard[†]	FPT[*]
	e+v+a	$W[1]$-hard[*†]	FPT[*]
	a	P[¶*]	P[¶*]
	e	P[¶*]	P[¶*]
	e+a	P[*]	P[*]
ERS		$W[1]$-hard[*]	FPT[*]

Results shown in: * this paper, ‡ [15], † [14], ¶ follows from results on f-factors [13].
The editing operations are codified as 'e' - edge deletion, 'v' - vertex deletion and 'a' - edge addition. The 'uniform' version of the problem is where all vertices and edges have weight 1 and the desired graph is r-regular.
The final row indicates the complexity of EDGE REPLACEMENT SET.

The following operations alter a graph $G = (V, E)$ into a new graph $G' = (V', E')$. Deleting an edge uv simply removes that edge from the graph (i.e., $E' = E \setminus \{uv\}$, and $V' = V$). Deleting a vertex u removes that vertex and all incident edges (i.e., $V' = V \setminus \{u\}$, $E' = E \setminus \{ uv \mid v \in V \}$). Adding an edge uv of course inserts an edge between u and v (i.e., $E' = E \cup \{uv\}$, $V' = V$).

In this paper we also consider weighted versions of these operations, which are defined similarly. Given a weighted edge or vertex, the cost of deletion is simply that weight. Note particularly that the cost of deleting a vertex is the weight of the vertex alone, not the weight of the vertex plus the weights of the incident edges, even though they are also removed (this is consistent with the normal definition for unweighted graphs, where deleting a vertex counts as one step, regardless of any incident edges). Weighted edge addition works as defined, except where an edge already exists, which in the unweighted case would prevent addition. In the weighted case however, we simply increase the weight of the existing edge. Thus, in the presence of edge addition one needs to consider the weighted and the unweighted variants of a problem separately, as the former is not just a special case of the latter.

2.2 Basic Parameterized Complexity

Here we introduce some basic, relevant parameterized complexity theory. For a more in-depth coverage we refer to the books of Downey and Fellows [7], Flum and Grohe [11] and Niedermeier [16]. When considering the complexity of a problem in a classical, P vs. NP setting, the only measure available is n, the instance size (or some function thereof). Parameterized complexity adds a second measure, that of a parameter k, which is given as a special part of the input. If a problem has an algorithm that runs in time $O(f(k)p(n))$, where p is a polynomial and f is a computable function of k, then the problem is *fixed-parameter tractable*, or in the class FPT. Conversely, the demonstration of hardness for the class $W[t]$ for some $t \geq 1$ gives the intuition that the problem is unlikely to be fixed-parameter tractable. This is analogous to a problem being NP-hard in the classical set-up. For the sake of clarity any problem is understood to be a decision problem unless explicitly stated otherwise (and the parameterized complexity classes that are referenced are defined for decision problems).

Demonstration of $W[t]$-hardness is normally done via an FPT reduction, which is the parameterized complexity equivalent of a polynomial-time many-one reduction in classical complexity theory. Given two parameterized problems Π_1 and Π_2, an FPT reduction $\Pi_1 \leq_{FPT} \Pi_2$ is a mapping from Π_1 to Π_2 that maps an instance (I, k) of Π_1 to an instance (I', k') of Π_2 such that (i) $k' = h(k)$ for some computable function h, (ii) (I, k) is a YES-instance of Π_1 if and only if (I', k') is a YES-instance of Π_2 and (iii) the mapping can be computed in time $O(f(k)p(|I|))$, where f is some computable function of the parameter k alone and p is a polynomial.

Then if Π_2 is in FPT, so is Π_1, and if Π_1 is $W[t]$-hard, so is Π_2.

The classes $W[t]$, $t = 1, 2, \ldots$, are defined as the classes of problems that can be FPT-reduced to certain weighted satisfiability problems. The classes form the chain FPT $\subseteq W[1] \subseteq W[2] \subseteq \ldots$, where all inclusions are believed to be strict (Flum and Grohe [11] in particular give detailed coverage of this hierarchy) .

Reduction to a problem kernel, or *kernelization*, is one of the fundamental techniques for developing fixed-parameter tractable algorithms, and thus for demonstrating FPT membership. A problem is *kernelizable* if, given an instance (I, k) of the problem, where I is the input and k is the parameter, it is possible to produce in polynomial time an instance (I', k') where $|I'| \leq g(k')$ and $k' = h(k)$ for computable functions g and h, and (I, k) is a YES-instance if and only if (I', k') is a YES-instance. It can be shown that a problem is kernelizable in this sense if and only if it is fixed-parameter tractable. Kernelization is normally accomplished by the application of *reduction rules* to the instance. Further explanation of the theory can be found in Estivill-Castro et al.'s paper [10].

3 Easy Cases

Before moving to the general versions of the considered problems, let us examine restricted versions where *only* edge editing operations are allowed; we may not delete any vertices.

If only one of the operations is available, then the problems (weighted and uniform) can easily be seen to correspond the well-known polynomially solvable *f-factor problem* [13]. Although the f-factor problem does not explicitly include editing operations, edge deletions are dealt with implicitly as any f-factor of a graph has the same number of edges, we need merely then to compare the difference between this number and the total number of edges with the parameter. When the operation is edge addition, we simply use the complement of the input graph instead and modify the degree function appropriately. However it is not immediately apparent that these techniques may be directly applied to the case where we allow both edge addition and edge deletion. Hence we shall give a general construction for solving the WEIGHTED EDGE EDIT TO CHOSEN DEGREE GRAPH problem by application of Edmond's minimum weight perfect matching algorithm [13, 8].

Let G, k be an instance of WEIGHTED EDGE EDIT TO CHOSEN DEGREE GRAPH. By allowing edge weights of 0, we may assume that G is a complete graph. Now solving the problem is clearly equivalent to finding an edge weight function $\rho' : E(G) \to \{0, 1, 2, \dots\}$ of G such that for each $v \in V(G)$ we have $\sum_{vv' \in E(G)} \rho'(vv') = \delta(v)$ and the cost of ρ', $\sum_{vv' \in E(G)} |\rho(vv') - \rho'(vv')|$, is at most k.

We construct a graph H with edge-weight function η as follows: For each vertex v of G we introduce in H a set $V(v)$ of $\delta(v)$ vertices. For each edge $vv' \in E(G)$ we add the following vertices and edges to H'.

1. We add two sets $V_{\text{del}}(v, v')$ and $V_{\text{del}}(v', v)$ of vertices, each of size $\rho(vv')$.
2. We add two sets $V_{\text{add}}(v, v')$ and $V_{\text{add}}(v', v)$ of vertices, each of size $\min(\delta(v), \delta(v'))$.
3. We add all edges uw for $u \in V(v)$ and $w \in V_{\text{del}}(v, v') \cup V_{\text{add}}(v, v')$, and all edges uw for $u \in V(v')$ and $w \in V_{\text{del}}(v', v) \cup V_{\text{add}}(v', v)$.
4. We add edges that form a matching $M_{vv'}$ between the sets $V_{\text{del}}(v, v')$ and $V_{\text{del}}(v', v)$. We will refer to these edges as *deletion edges*.
5. We add edges that form a matching $M'_{vv'}$ between the sets $V_{\text{add}}(v, v')$ and $V_{\text{add}}(v', v)$ and subdivide the edges of $M'_{vv'}$ twice, that is, we replace $xy \in M'_{vv'}$ by a path x, x_y, y_x, y where x_y and y_x are new vertices. We will refer to the edges of the form $x_y y_x$ as *addition edges*.

This completes the construction of H. It remains to assign deletion and addition edges e the weight $\eta(e) = 1$, and all other edges the weight 0. It can be verified that (G, k) a yes-instance of WEIGHTED EDGE EDIT TO CHOSEN DEGREE GRAPH if and only if H perfect matching of weight at most k, but owing to space restrictions, we omit the proof. If we remove the addition (deletion) edges from H, then we also have a construction that can be used to solve the edge deletion (addition) problem. Naturally this construction allows solutions for the uniform versions of the problems as well, as a subcase.

4 A Thorn in the Paw

Previously [14] we demonstrated the following result:

Theorem 1. WEIGHTED DELETION TO CHOSEN DEGREE GRAPH *is fixed parameter tractable for parameter* (k, r).

This was shown by reduction to problem kernel, with a kernel of size $O(kr(k+r))$. It is interesting to note that the result holds not only for DELETION TO REGULAR GRAPH, but also for WEIGHTED VERTEX DELETION TO CHOSEN DEGREE GRAPH, and that the generalization gives a smaller kernel than by using a similar method without the annotation.

Naturally we would like to achieve a similar result for the edit versions of the problems. However the kernelization for deletion problems heavily relies on the fact that if we delete a vertex in a clean region (defined below) or an edge incident with a vertex in a clean region, then we must delete the entire clean region. Consequently, we can shrink large clean regions as their specific structure is not relevant. This reasoning fails for editing problems where edge addition is allowed.

We prove in Section 5 that the edit version is indeed fixed-parameter tractable for (k, r), but obtaining a kernel is difficult, at least when approached in a similar manner to the previous investigation. Note that demonstration of fixed-parameter tractability guarantees that *some* kernelization exists, what we show here is that it is unlikely to take a certain (useful) form.

Firstly it is useful to define the notion of a *clean region* (first introduced by Moser and Thilikos [15]). Given a graph $G = (V, E)$, a function $\delta : V \to \{0, \dots, r\}$, and a function $\rho : V \cup E \to \{1, 2, 3, \dots\}$, we say a vertex v is *clean* if $\sum_{e \in E(v)} \rho(e) = \delta(v)$, where $E(v)$ denotes the set of edges incident on v. Then a clean region is a maximal connected subgraph of clean vertices. In the case where the graph is unweighted, we implicitly assume that $\delta(v) = r$ for all $v \in V$ and $\rho(x) = 1$ for all $x \in V \cup E$.

In both EDIT TO REGULAR GRAPH and WEIGHTED EDIT TO CHOSEN DEGREE GRAPH it may be necessary to delete a set of vertices from clean regions so that the edges that become available may be used to complete the degree of a vertex of insufficient degree (indeed there are easily constructable instances where this is the only way to solve the instance). This gives the following sub-problem:

EDGE REPLACEMENT SET
Instance: A graph $G = (V, E)$, two positive integers k and t.
Question: Does there exist a set $X \subseteq V$ such that $|X| \leq k$ and there are exactly t edges between vertices in X and vertices in $V \setminus X$?

Unfortunately, EDGE REPLACEMENT SET is NP-complete, thus making the possibility of obtaining a kernel in polynomial time by somehow identifying all relevant sets in the clean regions unlikely. The proof is by reduction from the following:

REGULAR CLIQUE
Instance: An r-regular graph $G = (V, E)$, a positive integer k.
Question: Does G contain a clique on k vertices?

REGULAR CLIQUE is NP-complete, and $W[1]$-complete for parameter k, but fixed-parameter tractable for parameter (k, r). We refer to previous work [14] for a detailed proof of these statements.

The proof of the following theorem requires that the regularity r of the input graph in the REGULAR CLIQUE instance be sufficiently large. It is possible to construct a "fixing gadget" that allows the degree of each vertex to be increased effectively arbitrarily, without introducing any non-trivial cliques. We refer again to previous work [14], and particularly to the proof of Lemma 3.1 contained therein for proof of this claim.

Theorem 2. EDGE REPLACEMENT SET *is* NP-*complete and* $W[1]$-*hard for parameter* k.

Proof. We shall concentrate on the $W[1]$-hardness proof; the NP-hardness follows from the same result.

Let $(G = (V, E), k)$ be an instance of REGULAR CLIQUE where G is r-regular. We may assume that r is not bounded in terms of k, since REGULAR CLIQUE is fixed-parameter tractable for parameter (k, r). For a set $X \subseteq V$ let $d(X)$ denote the number of edges $uv \in E$ with $u \in X$ and $v \in V \setminus X$. If X forms a clique in G then $d(X) = k(r - k + 1)$. Therefore we put $t = k(r - k + 1)$ and consider (G, k, t) as an instance of EDGE REPLACEMENT SET.

Let $X \subseteq V$ with $|X| \leq k$ and $d(X) = t$. We show that X has exactly k elements and forms a clique in G. Assume for the sake of contradiction that $|X| < k$. It follows that $d(X) \leq |X|r$ and consequently $r < r(k - |X|) \leq k^2 - k$. This contradicts the assumption that r is not bounded in terms of k. Hence we conclude $|X| = k$. Each vertex $x \in X$ has at most $k - 1$ neighbors in X and at least $r - k + 1$ neighbors in $V \setminus X$. Therefore, if at least one $x \in X$ had fewer than $k - 1$ neighbors in X, then $d(X) > k(r - k + 1) = t$. Since $d(X) = t$, it follows that X is a clique in G. □

As the weighted edit problem contains EDGE REPLACEMENT SET as a subproblem we also have the following result:

Corollary 1. WEIGHTED EDIT TO CHOSEN DEGREE GRAPH *is* $W[1]$*-hard for parameter* k.

This can be observed by considering the following simple construction: an isolated vertex with $\rho(v) = k + 1$ and $\delta(v) = t \leq r$, along with a clique on y vertices inside a clean region, where each clique vertex has t/y 'outgoing' edges, low weight, and $k \geq y + t$. We cannot delete the isolated vertex, but the deleting the clique will give the requisite number of edges to 'fix' the isolated vertex.

Thus this proof demonstrates that a polynomial-time kernelization which relies upon identifying such candidate sets for deletion is unlikely to exist. Note also that the proof holds if we also demand in EDIT TO REGULAR GRAPH that the set X is connected.

5 Editing is Fixed Parameter Tractable for Parameter (k, r)

To demonstrate that EDIT TO REGULAR GRAPH is fixed-parameter tractable we take a logical approach and apply the following meta-theorem which is due to Frick and Grohe [12].

Theorem 3 ([12]). *Let C be a polynomial-time decidable class of structures of effectively bounded local tree-width. Then the model checking problem for first-order logic on the class C is fixed-parameter tractable parameterized by the length of the first-order formula.*

More particularly we use their corollary that the parameterized model checking problem for first-order logic is fixed-parameter tractable for graphs of *bounded degree*. Stewart [21] pointed out that this result also holds if the degree bound is not global but depends on the parameter. Furthermore he indicated how this can be used to show that REGULAR SUBGRAPH with parameter (k, r) is fixed-parameter tractable. In the following we extend this approach to EDIT TO REGULAR GRAPH and further to WEIGHTED EDIT TO CHOSEN DEGREE GRAPH.

First we introduce the following reduction rule, used previously [14, 15], that reduces an instance $(G, (k, r))$ of EDIT TO REGULAR GRAPH to another instance $(G', (k', r'))$ of EDIT TO REGULAR GRAPH with bounded degree:

Reduction Rule 1: If there exists a vertex v in G where $d(v) > k + r$, then $G' = G[V(G) \setminus \{v\}]$, $k' = k - 1$ and $r' = r$.

Therefore if we can formulate sentences ϕ_k, $k \geq 0$, of first-order logic such that ϕ_k is true for a graph G if and only if (G, k) is a YES-instance of EDIT TO REGULAR GRAPH, then we have established fixed-parameter tractability of EDIT TO REGULAR GRAPH, since by application of Reduction Rule 1 we have a graph of bounded degree. Note that the predicates Vx, Ey and Ixy mean that x is a vertex, y is an edge, and that y is incident on x, respectively. Furthermore we write $[n] = \{1, \ldots, n\}$.

The sentence is defined as

$$\phi_k = \bigvee_{k'+k''+k'''\leq k} \exists u_1, \ldots, u_{k'}, e_1, \ldots, e_{k''}, a_1, \ldots, a_{k'''}, b_1, \ldots, b_{k'''} (\phi'_k \wedge \forall v \; \phi''_k)$$

where ϕ'_k and ϕ''_k are defined below. ϕ'_k is the conjunction of the following clauses (1)...(4) that ensure that $u_1, \ldots, u_{k'}$ represent deleted vertices, $e_1, \ldots e_{k''}$ represent deleted edges, and a_i, b_i, $1 \leq i \leq k'''$ represent ends of added edges. Note that since added edges are not present in the given structure we need to express them in terms vertex pairs.

(1) $\bigwedge_{i\in[k']} Vu_i \wedge \bigwedge_{i\in[k'']} Ee_i$ "u_i is a vertex, e_i is an edge;"
(2) $\bigwedge_{i\in[k''']} Va_i \wedge Vb_i \wedge a_i \neq b_i \wedge \bigwedge_j (u_j \neq a_i \wedge u_j \neq b_i)$ "a_i and b_i are distinct vertices and not deleted;"
(3) $\bigwedge_{i\in[k''']} \forall y(\neg Ia_i y \vee \neg Ib_i y)$ "a_i and b_i are not adjacent;"
(4) $\bigwedge_{1\leq i<j\leq k''} (a_i \neq b_j \vee a_j \neq b_i) \wedge (a_i \neq a_j \vee b_i \neq b_j)$ "the pairs of vertices are mutually distinct."

The subformula ϕ''_k ensures that each vertex v has degree r after editing:

$$\phi''_k = (Vv \wedge \bigwedge_{i\in[k']} v \neq u_i) \rightarrow \bigvee_{\substack{r', r'' \in [r] \\ r' + r'' = r}} \exists x_1, \ldots, x_{r'}, y_1, \ldots, y_{r''} \; \phi'''_k$$

where ϕ'''_k is the conjunction of the following clauses:

(5) $\bigwedge_{i\in[r']} Ivx_i$ "v is incident with r' edges;"
(6) $\bigwedge_{1\leq i<j\leq r'} x_i \neq x_j$ "the edges are all different;"
(7) $\bigwedge_{i\in[r'],j\in[k'']} x_i \neq e_j$ "the edges have not been deleted;"
(8) $\bigwedge_{i\in[r'],j\in[k']} \neg Iu_j x_i$ "the ends of the edges have not been deleted;"
(9) $\forall x(Ivx \rightarrow \bigvee_{i\in[r']} x = x_i \vee \bigvee_{i\in[k'']} x = e_i \vee \bigvee_i Ixu_i)$ "v is not incident with any further edges except deleted edges;"
(10) $\bigwedge_{i\in[r'']} \bigvee_j (y_i = a_j \wedge v = b_j) \vee (y_i = b_j \wedge v = a_j)$ "v is incident with at least r'' added edges;"
(11) $\bigwedge_{j\in[r'']} (v = a_j \rightarrow \bigvee_i y_i = b_j) \wedge (v = b_j \rightarrow \bigvee_{j\in[r'']} y_i = a_j)$ "v is incident with at most r'' added edges."

By the above considerations, we have the following.

Theorem 4. EDIT TO REGULAR GRAPH *is fixed-parameter tractable for parameter* (k, r).

If we force k'' to be zero, then the same sentence suffices to prove that the variant with only edge addition and vertex deletion is also fixed-parameter tractable for parameter (k, r). This variant is $W[1]$-hard for parameter k by a previous result [14].

WEIGHTED EDIT TO CHOSEN DEGREE GRAPH can be classified by a similar approach, but first we must demonstrate that we can express the ρ and δ functions in first order logic. To this aim we introduce a series W_i, $1 \leq i \leq k+1$, of weight predicates such that $W_i x$ is true for a vertex x if and only if $\rho(x) = i$, and a series D_j, $0 \leq j \leq r$, of degree predicates such that $D_j x$ is true for a vertex x if and only if $\delta(x) = j$. We represent an edge of weight i by i parallel edges.

Hence we can formulate the following sentence to represent solutions of WEIGHTED EDIT TO CHOSEN DEGREE GRAPH.

$$\psi_k = \bigvee_{\substack{k', k'', k''', l_1, \ldots, l_{k'} \in [k] \\ l_1 + \cdots + l_{k'} + k'' + k''' \leq k}} \bigwedge_{i \in [k']} W_{l_i}(u_i) \wedge$$

$$\exists u_1, \ldots, u_{k'}, e_1, \ldots e_{k''}, a_1, \ldots, a_{k'''}, b_1, \ldots, b_{k'''} (\psi'_k \wedge \forall v\ \psi''_k)$$

$$\psi''_k = \bigvee_{j \in [r]} [(D_j v \wedge \bigwedge_{i \in [k']} v \neq u_i) \rightarrow \bigvee_{\substack{r', r'' \in [r] \\ r' + r'' = r}} \exists x_1, \ldots, x_{r'} \exists y_1, \ldots, y_{r''}\ \psi'''_k(j)]$$

The subformula ψ'_k is the conjunction of the above clauses (1) and (2) (we omit (3) and (4) as we use multiple edges to encode edge weights) and $\psi'''_k(j)$ is obtained from the above subformula ϕ'''_k by setting r to j. Hence, as above, we conclude:

Theorem 5. WEIGHTED EDIT TO CHOSEN DEGREE GRAPH *is fixed-parameter tractable for parameter* (k, r).

By similar reasoning as before, this sentence demonstrates the fixed-parameter tractability of the variant without edge deletion for parameter (k, r). Again this variant is $W[1]$-hard for parameter k by a previous result [14].

6 Conclusion

We demonstrated that when parameterized by (k, r), the editing problems are fixed parameter tractable, but when parameterized by k, the problems are $W[1]$-hard. The only exceptions are when the editing operations are limited to edge addition and/or deletion, in which case the problems are solvable in polynomial time, thus completing the classification for all combinations of the three editing operations.

A change in complexity is also apparent when moving from the deletion only problems to the edit problems.

It seems unlikely that a similar approach as used for the deletion problems can be used to develop a polynomial-time kernelization for the editing problems. For a feasible kernelization some new structural insight must be gained.

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