Introduction to Knowledge-based Systems
Exercise Sheet 2: Logic

Exercise 1: (2 pts.)
Which of the formulas below are valid and which are satisfiable? If a formula is only satisfiable, then give a model and provide an interpretation under which the formula evaluates to 0.

1. $(p \to (q \to p)) \to ((p \to q) \to p)$.
2. $\neg p \to (p \to q)$.
3. $\neg(p \wedge (p \vee q)) \leftrightarrow p$.
4. $((p \to q) \to p) \to p$.

Exercise 2: (1 pt.)
Show by means of the replacement theorem that

$$(\phi \wedge (\psi \vee \neg\psi)) \to \chi \text{ and } \phi \to \chi$$

are logically equivalent, for any formulas ϕ, ψ, and χ.

Exercise 3: (1 pt.)
Check whether the following two formulas are logically equivalent:

$$(p \to r) \wedge (\neg p \vee q) \text{ and } (p \to (r \vee q)).$$

Justify your answer by providing a semantical argument, i.e., show that the two formulas have the same models in case that they are equivalent or give a countermodel otherwise.

Exercise 4: (2 pts.)
Check directly from the definition of valid consequence whether the following statements hold:

1. $p,\ r \to q,\ \neg(p \wedge q) \models \neg r$;
2. $(p \to q) \to \neg q \models \neg p$.

Exercise 5: (2 pts.)
Consider the following arguments:

1. He said he would come (C) if the weather is fine (F). The weather is not fine. Therefore, he won't come.
2. If he is responsible for this rumour (R), he must be either stupid (S) or unprincipled (U) (or both). He is neither stupid nor unprincipled. Therefore, he is not responsible for the rumour.

Translate the argument into propositional logic and show by TC0 that the argument is either correct or else extract an interpretation from the tableau showing that the argument is not correct.

Exercise 6: (2 pts.)
Your boss tells you that any binary relation which is symmetric and transitive is also reflexive. Is he right?

Translate the argument into the symbolism of first-order logic and show by TC1 that the argument is either correct or else extract an interpretation from the tableau showing that the argument is not correct.

Exercise 7: (2 pts.)
Let $\phi = \forall x(car(x) \rightarrow \exists y\, owns(y, x))$ and $\psi = \exists y \forall x(car(x) \rightarrow owns(y, x))$.

1. Show using TC1 that $\psi \models \phi$ holds.
2. Show that $\phi \models \psi$ does not hold, i.e., construct an interpretation I such that $I \models \phi$ but $I \not\models \psi$.

Exercise 8: (1 pt.)
Check using TC1 whether the formula

$$\forall x\,(\phi(x) \rightarrow \psi(x)) \vee \forall x\,(\psi(x) \rightarrow \phi(x))$$

is valid, where $\phi(x)$ and $\psi(x)$ are atomic formulas. If the formula is not valid, then try to extract a countermodel from the tableau.

Exercise 9: (2 pts.)
Let $\mathcal{L}$ be a language of propositional logic where formulas are built only from Boolean variables using the primitive connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$ (thus, $\top$ and $\bot$ are not part of the language). Furthermore, let A be a formula of $\mathcal{L}$ containing no occurrence of $\neg$ and let B be any formula of $\mathcal{L}$.

Show the following propositions:

1. Let I be an interpretation assigning to all atomic formulas of A the truth value 1. Then, A is true under I.
2. If $\models B \leftrightarrow \neg A$, then B contains at least one occurrence of $\neg$.

Hint: Show Item 1 by induction on the logical complexity of A (i.e., on the number of occurrences of logical connectives in A). Show Item 2 using Item 1.