First-Order Logic for Forgetters

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Outline

- Introduction
- First-Order Logic (PL1)
  - Syntax of PL1
  - Semantics of PL1
  - Deduction in PL1
- Summary

Why PL1?

- PL0 atoms are either true or false, they have no internal structure
- Problems to express problems and deduce solutions in PL0:
  1. All rabbits have long ears
  2. Roger is a rabbit
  3. We would like to deduce that Roger has long ears
- Representing/solving this problem is not possible in PL0
- PL1 formalization:
  1. $\forall x \, (rabbit(x) \rightarrow has\_long\_ears(x))$
  2. $rabbit(Roger)$
  3. We deduce $has\_long\_ears(Roger)$ how?

Why PL1? Cont'd

- Atoms like $rabbit(Roger)$ have an internal structure
- Truth value of such atoms depend on the internal structure!
- One-place predicates can be used to classify objects:
  - $rabbit(Roger)$ is e.g., true (i.e., Roger is a rabbit) but
  - $rabbit(Hans)$ is is false (i.e., Hans is not a rabbit)
- Predicates can be *n*-ary like $likes(Ed, Red\_Wine)$ ($n = 2$)
- Such predicates express relations between objects
- Functions like $+(7, 5)$ can be arguments of predicates

Group Theory

- Axiomatization of group theory

$$\forall x \forall y \forall z\, x \circ (y \circ z) = (x \circ y) \circ z \qquad (1)$$

$$\forall x\, e \circ x = x \qquad (2)$$

$$\forall x\, i(x) \circ x = e \qquad (3)$$

- Some consequences of group theory:

$$\forall x\, x \circ e = x \qquad (4)$$

$$\forall x\, x \circ i(x) = e \qquad (5)$$

$$\forall x\, i(i(x)) = x \qquad (6)$$

- If (1), (2) and (3) are satisfied then (4), (5) and (6) hold, i.e., $(1) \wedge (2) \wedge (3) \rightarrow (4) \wedge (5) \wedge (6)$ is valid

Daily Life

- Formalize
  "If someone has knocked the door frame, then he has headache."
- A possible formula is

$$\forall x \left(knocked(x, door\_frame) \rightarrow headache(x) \right)$$

- Compare this formula with the formula obtained for
  "All rabbits have long ears."

Signatures

- Signature Σ: countably infinite set of (function or predicate) symbols together with their arity
- In PL0: Σ is the set of boolean variables (with arity 0)
- Elements from Σ are the building blocks for formulas

$\Sigma = (\mathit{Func}, \mathit{Pred})$

- *Func*: set of function symbols (+ arity)
  - With arity 0: constant symbols
  - With arity > 0: for building terms
- *Pred*: set of predicate symbols (+ arity)
  - For building atomic formulas

Terms

- Given a set *Var* of (object) variables and $\Sigma = (Func, Pred)$
- Variables are $x, y, z, x_1, x', \ldots$
- Inductive definition of the set of terms for given Σ and *Var*

  B1: Every $x \in Var$ is a term

  B2: Every constant symbol from *Func* in Σ is a term

  S1: If $t_1, \ldots, t_n$ are terms and f is a FS from *Func* in Σ with arity $n > 0$, then $f(t_1, \ldots, t_n)$ is a term
- Example: Given $Var = \{x\}$ and $Func = \{c/0, f/1\}$

$$Terms(\Sigma, Var) = \{x, c, f(x), f(c), f(f(x)), f(f(c)), \ldots\}$$

- Set of terms is infinite if there is a FS with arity > 0
- Ground terms: terms without variables, i.e., $Terms(\Sigma, \{\})$

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Formulas

- Given $\Sigma = (\mathit{Func}, \mathit{Pred})$ and *Var*
- Let p be a PS from Σ with arity $n \geq 0$ and $t_1, \ldots, t_n$ terms. Then $p(t_1, \ldots, t_n)$ is an atomic formula or atom
- Ground atoms: atoms without variables
- Inductive definition of the set of first-order formulas

  B1: Every atom is a formula
  B2: $\top$ (verum) and $\bot$ (falsum) are formulas
  S1: For $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$: same as for PL0
  S2: If ϕ is a formula and $x \in \mathit{Var}$, then so are $\forall x\ \phi$ and $\exists x\ \phi$

- $\forall$ is the universal quantifier, $\exists$ is the existential quantifier
- In S2, ϕ is called the scope of the quantifier

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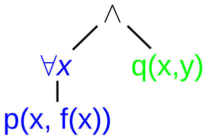
Formulas as Trees

- PL1 formulas can be depicted as formula trees (as for PL0)
- Example: $(\forall x\, p(x, f(x))) \wedge q(x, y)$



  - Var. occurrences can be free or bound
  - Occurrences x are bound ($\forall x$ above!)
  - Occurrence x is free (no $\forall x$, $\exists x$ above)

- Formulas without free vars are called closed or sentences

The Free Variables of a Formula

- Inductive definition of the set of free variables in a term

  B1: $\mathit{free}(x) = \{x\}$ for a variable x
  B2: $\mathit{free}(a) = \{\}$ for a constant a
  S1: $\mathit{free}(f(t_1, \ldots, t_n)) = \bigcup_{i=1}^{n} \mathit{free}(t_i)$ for a term $f(t_1, \ldots, t_n)$

- Inductive definition of the set of free variables in a formula

  B1: $\mathit{free}(p(t_1, \ldots, t_n)) = \bigcup_{i=1}^{n} \mathit{free}(t_i)$ for an atom $p(t_1, \ldots, t_n)$
  S1: $\mathit{free}(\neg\phi) = \mathit{free}(\phi)$
  S2: $\mathit{free}(\phi \circ \psi) = \mathit{free}(\phi) \cup \mathit{free}(\psi)$ for $\circ \in \{\vee, \wedge, \rightarrow, \leftrightarrow\}$
  S3: $\mathit{free}(Qx\, \phi) = \mathit{free}(\phi) \setminus \{x\}$ for $Q \in \{\forall, \exists\}$

Summary of Syntax of PL1

- Terms
  - (Object) variables, constants, functions
  - Set of terms, set of ground terms
- Literals
  - Atoms and ground atoms
  - Membership predicates and relations
- Formulas, formula trees, free and bound variables

The Semantics of PL1

- Semantics of PL1 more difficult than for PL0 because of
  - the term structure,
  - the quantifiers, and
  - the free variables which can occur in formulas
- First-order (interpretation) structure wrt Σ: consists of
  - Domain $\mathcal{U}$ = nonempty set of symbols
  - Interpretation function $I(\cdot)$
- For CS (0-ary FS) $c \in Func$: $I(c) \in \mathcal{U}$
- For n-ary FS $f \in Func$ (n>0): $I(f) : \mathcal{U}^n \mapsto \mathcal{U}$
- For n-ary PS $p \in Pred$: $I(p) \subseteq \mathcal{U}^n$ (or $I(p) : \mathcal{U}^n \mapsto \{0, 1\}$)
  If $n = 0$: $I(p) = \{\}$ is 0 (false); $I(p) = \{()\}$ is 1 (true)

How to Interpret the Different Kinds of Symbols

symbol	arity	interpretation
constant symbol	0	element of $\mathcal{U}$
function symbol	$n > 0$	n-ary function over $\mathcal{U}$
predicate symbol	0	truth value
predicate symbol	1	subset of $\mathcal{U}$
predicate symbol	> 1	relation over $\mathcal{U}$

How to Handle Free Variables?

- Free variables in a formula cause problems
  What is the meaning of a free x?
- Two solution possible:
  - Close a formula by $\forall$ (universal closure), or
  - interpret the formula modulo a variable assignment

$$\alpha\colon \mathit{Var} \mapsto \mathcal{U}$$

- We use variable assignments in the following

The Evaluation of a Term

- The evaluation of a term t under an interpretation I and a variable assignment α (modulo the signature Σ): $I_{\Sigma,\alpha}(t)$
- We often omit Σ for better readability!
- $I_\alpha(t)$ is defined inductively as follows:

B1: $I_\alpha(x) = \alpha(x)$ for $x \in \mathit{Var}$
B2: $I_\alpha(c) = I(c)$ for a constant symbol c ($I(c) \in \mathcal{U}$!)
S1: $I_\alpha(f(t_1, \ldots, t_n)) = I(f)(I_\alpha(t_1), \ldots, I_\alpha(t_n))$ for $f/n \in \mathit{Func}$ and $t_1, \ldots, t_n$ are terms

The Evaluation of a Formula

- The evaluation of a formula under an interpretation I and a variable assignment α (modulo the signature Σ) is defined inductively as follows:
  - B1: $I_\alpha(p(t_1, \ldots, t_n)) = 1$ iff $(I_\alpha(t_1), \ldots, I_\alpha(t_n)) \in I(p)$ where $p/n \in Pred$ and $t_1, \ldots, t_n$ are terms
  - S1: Negations, conjunctions, disjunctions, etc. as in PL0
  - S2: $I_\alpha(\forall x\ \phi) = 1$ iff $I_{\alpha \cup \{x \leftarrow c\}}(\phi) = 1$ for each $c \in \mathcal{U}$
  - S3: $I_\alpha(\exists x\ \phi) = 1$ iff $I_{\alpha \cup \{x \leftarrow c\}}(\phi) = 1$ for at least one $c \in \mathcal{U}$
- Evaluation of a PL1 formula is undecidable in general
- Notions like tautology, valid, (un)satisfiable, model, etc. remain essentially unchanged

Example for an Evaluation

- Let ϕ: $\forall x\,(p(x) \to p(f(f(x))))$
- Let $\mathcal{U} = \mathit{Nat}$
  - $f/1 \in \mathit{Func}$ with the intended meaning "successor of"
  - $p/1 \in \mathit{Pred}$ with the intended meaning "is odd number"
- ϕ's intended reading: for every odd nbr x, $x + 2$ is also odd
- Let $I(f)\colon \mathcal{U} \mapsto \mathcal{U}$ with $f(u) = u + 1$
- Moreover, $I(p) = \{1, 3, 5, \ldots\} \subset \mathcal{U}$
- Since ϕ is closed, $\alpha = \{\}$ at the beginning

Example for an Evaluation Cont'd

- $I_{\{\}}(\phi) = 1$ iff, for each $c \in \mathcal{U}$,
  $I_{\{x \leftarrow c\}}(p(x) \rightarrow p(f(f(x)))) = I_{\{\}}(p(c) \rightarrow p(f(f(c)))) = 1$
- Case distinction for c:

  1: c is odd (i.e., $c \in I(p)$):

  - $p(c) \rightarrow p(f(f(c)))$ is true iff $c \notin I(p)$ or $f(f(c)) \in I(p)$
  - Since $I(f(f(c))) = I(c) + 2$, $c \in I(p)$ implies $f(f(c)) \in I(p)$
  - Since $c \in I(p)$, $f(f(c)) \in I(p)$ and the implication is true

  2: c is even (i.e., $c \notin I(p)$):

  - Then $p(c) \rightarrow p(f(f(c)))$ is true because $c \notin I(p)$

- Hence, ϕ is true under the chosen interpretation

Equivalent Notations Again

ϕ is true under I and α (modulo Σ)	iff $I_{\Sigma,\alpha}$ satisfies ϕ
	iff $I_{\Sigma,\alpha}(\phi) = 1$
	iff $I_{\Sigma,\alpha} \models \phi$
	iff $I_{\Sigma,\alpha}$ is a model of ϕ

ϕ is false under I and α (modulo Σ)	iff $I_{\Sigma,\alpha}$ does not satisfy ϕ
	iff $I_{\Sigma,\alpha}(\phi) = 0$
	iff $I_{\Sigma,\alpha} \not\models \phi$

Recall the Notations

- *Mod*(ψ) is the set of all models of ψ
- ϕ is satisfiable if there is some I_α that satisfies ϕ
- ϕ is falsifiable if there is some I_α that does not satisfy ϕ
- ϕ is valid if every I_α is a model of ϕ
  - This means: for all I and for all α!
- ϕ is unsatisfiable if ϕ is not satisfiable
- Formulas ϕ and ψ are equivalent, denoted by $\phi \equiv \psi$, iff they have exactly the same models, i.e., $Mod(\phi) = Mod(\psi)$
  In other words, for all I_α, we have $I_\alpha \models \phi$ iff $I_\alpha \models \psi$
- Note: $p(x) \not\equiv p(y)$ why?

Some Useful Equivalences

Commutativity	$\phi \circ \psi$	$\equiv$	$\psi \circ \phi$	for $\circ \in \{\vee, \wedge, \leftrightarrow\}$
Idempotence	$\phi \circ \phi$	$\equiv$	ϕ	for $\circ \in \{\vee, \wedge\}$
Tautology	$\phi \vee \top$	$\equiv$	$\top$	
Unsatisfiability	$\phi \wedge \bot$	$\equiv$	$\bot$	
Neutrality	$\phi \wedge \top$	$\equiv$	ϕ	
	$\phi \vee \bot$	$\equiv$	ϕ	
Negation	$\phi \vee \neg\phi$	$\equiv$	$\top$	
	$\phi \wedge \neg\phi$	$\equiv$	$\bot$	
Double Negation	$\neg\neg\phi$	$\equiv$	ϕ	
Implication	$\phi \rightarrow \psi$	$\equiv$	$\neg\phi \vee \psi$	
De Morgan	$\neg(\phi \vee \psi)$	$\equiv$	$\neg\phi \wedge \neg\psi$	
	$\neg(\phi \wedge \psi)$	$\equiv$	$\neg\phi \vee \neg\psi$	

Some Useful Equivalences Cont'd

Absorption	$\phi \vee (\phi \wedge \psi)$	$\equiv$	ϕ
	$\phi \wedge (\phi \vee \psi)$	$\equiv$	ϕ
Distributivity	$\phi \wedge (\psi \vee \chi)$	$\equiv$	$(\phi \wedge \psi) \vee (\phi \wedge \chi)$
	$\phi \vee (\psi \wedge \chi)$	$\equiv$	$(\phi \vee \psi) \wedge (\phi \vee \chi)$
Associativity	$\phi \vee (\psi \vee \chi)$	$\equiv$	$(\phi \vee \psi) \vee \chi$
	$\phi \wedge (\psi \wedge \chi)$	$\equiv$	$(\phi \wedge \psi) \wedge \chi$
$\forall$-Shifting $(*)$	$(\forall x\ \phi) \wedge \psi$	$\equiv$	$\forall x\ (\phi \wedge \psi)$
	$(\forall x\ \phi) \vee \psi$	$\equiv$	$\forall x\ (\phi \vee \psi)$
$\exists$-Shifting $(*)$	$(\exists x\ \phi) \wedge \psi$	$\equiv$	$\exists x\ (\phi \wedge \psi)$
	$(\exists x\ \phi) \vee \psi$	$\equiv$	$\exists x\ (\phi \vee \psi)$

$(*)$: x not free in ψ

Some Useful Equivalences Cont'd

$\forall$-Distribution	$(\forall x\ \phi) \wedge (\forall x\ \psi)$	$\equiv$	$\forall x\ (\phi \wedge \psi)$
$\exists$-Distribution	$(\exists x\ \phi) \vee (\exists x\ \psi)$	$\equiv$	$\exists x\ (\phi \vee \psi)$
$\forall$ De Morgan	$\neg\forall x\ \phi$	$\equiv$	$\exists x\ \neg\phi$
$\exists$ De Morgan	$\neg\exists x\ \phi$	$\equiv$	$\forall x\ \neg\phi$
Renaming $(*)$	$\forall x\ \phi$	$\equiv$	$\forall y\ \phi'$
	$\exists x\ \phi$	$\equiv$	$\exists y\ \phi'$
Duality	$\forall x\ \phi$	$\equiv$	$\neg\exists x\ \neg\phi$
	$\exists x\ \phi$	$\equiv$	$\neg\forall x\ \neg\phi$
Exchange	$\forall x \forall y\ \phi$	$\equiv$	$\forall y \forall x\ \phi$
	$\exists x \exists y\ \phi$	$\equiv$	$\exists y \exists x\ \phi$
Attention	$\forall x \exists y\ \phi$	$\not\equiv$	$\exists y \forall x\ \phi$

$(*)$: all free occurrences of x in ϕ are replaced by y (resulting in ϕ')

Recall: Connections Between the Different Notations

- Distinguish between
  - tautologies: all interpretations I_α are models
  - satisfiable formulas: some interpretations I_α are models
  - contradictions: no interpretation I_α is a model
- Important: For closed formulas, the properties satisfiability, logical equivalence, entailment, etc. do not depend on variable assignments
- A formula ϕ is valid iff $\neg\phi$ is unsatisfiable
- A formula ϕ is satisfiable iff $\neg\phi$ is not valid
- Two formulas ϕ and ψ are equivalent iff $\phi \leftrightarrow \psi$ is valid
- A formula ϕ is valid iff ϕ is equivalent to $\top$
- A formula ϕ is unsatisfiable iff ϕ is equivalent to $\bot$

Entailment (or Logical Implication)

- So far, $\models$ relates an interpretation and a formula
- Allow also a set of formulas on the left side
- Important: a set of formulas coincides with the conjunction of its elements, i.e., $\{\phi_1, \ldots, \phi_n\}$ is $\bigwedge_{i=1}^{n} \phi_i$
- Important: an empty conjunction is 1 in all interpretations i.e., it is equivalent to $\top$
- W entails ϕ, $W \models \phi$, iff $Mod(W) \subseteq Mod(\phi)$
- $W \models \phi$ iff $I_\alpha \models \phi$ for all models I_α of W
- Important for KBSs: Does KB W entails query ϕ

Entailment: Example 1

Show that $\models \phi$ holds where ϕ: $\forall x\ (p(x) \vee \neg p(x))$

- The formula is closed and therefore $\alpha = \{\}$
- Choose $\mathcal{U}$ and I arbitrarily
- $I(\phi) = 1$ iff $I_{\{x \leftarrow c\}}(p(x) \vee \neg p(x)) = 1$ for all $c \in \mathcal{U}$
- $I_{\{x \leftarrow c\}}(p(x) \vee \neg p(x)) = 1$ for all $c \in \mathcal{U}$, because:
  - If $c \in I(p)$, then $p(c)$ is true
  - If $c \notin I(p)$, then $p(c)$ is false and $\neg p(c)$ is true
  - In both cases, the disjunction is true
- Consequently, $\models \phi$ holds

Entailment: Example 2

Show: $\phi \models \psi$ with ϕ: $\exists x\,(p(x) \wedge (p(x) \rightarrow q(x)))$ and ψ: $\exists y\, q(y)$

- We show that each model of ϕ is also a model of ψ
- Take an arbitrary domain $\mathcal{U}$ and let I be a model of ϕ
- Then there is $c \in \mathcal{U}$, s.t. $I_{\{x \leftarrow c\}}(p(x) \wedge (p(x) \rightarrow q(x))) = 1$
- Moreover, $c \in I(p)$ and $c \in I(q)$ why?
- Evaluate ψ under the model of ϕ
- $I(\exists y\, q(y)) = 1$ iff $I_{\{y \leftarrow d\}}(q(y)) = 1$ for some $d \in \mathcal{U}$
- Let $d = c$ and observe that I is then also a model of ψ

Properties of Entailment

- $W \models \psi$ implies $W \cup \{\phi\} \models \psi$ — Monotonicity for PL0
- $W \cup \{\phi\} \models \psi$ iff $W \models \phi \rightarrow \psi$ — Deduction Thm
- $W \cup \{\phi\} \models \neg\psi$ iff $W \cup \{\psi\} \models \neg\phi$ — Contraposition Thm
- $W \cup \{\phi\}$ is unsatisfiable iff $W \models \neg\phi$ — Contradiction Thm

Reduction to Satisfiability (like in PL0)

Reduce validity, entailment, equivalence to satisfiability

1. Validity
   - $\neg\phi$ is unsatisfiable iff ϕ is valid
2. Entailment
   - ϕ entails ψ ($\phi \models \psi$) iff $\phi \rightarrow \psi$ is valid (apply Deduction Thm)
   - Hence, $\phi \models \psi$ iff $\phi \wedge \neg\psi$ (i.e., $\neg(\phi \rightarrow \psi)$) is unsatisfiable
3. Equivalence
   - ϕ is equivalent to ψ ($\phi \equiv \psi$) iff $\phi \leftrightarrow \psi$ is valid
   - Hence, $\phi \equiv \psi$ iff $\phi \models \psi$ and $\psi \models \phi$ hold
   - Consequently, $\phi \equiv \psi$ iff $\phi \wedge \neg\psi$ and $\psi \wedge \neg\phi$ are unsatisfiable

Sound and complete procedure for satisfiability is sufficient!

Table of Synonym Notions

All four statements in each line amount the same

entailment(s)	validity	satisfiability	equivalence
$\phi \models \psi$	$\phi \rightarrow \psi$ valid	$\phi \wedge \neg\psi$ unsat	$(\phi \rightarrow \psi) \equiv \top$
$\top \models \psi$	ψ valid	$\neg\psi$ unsat	$\psi \equiv \top$
$\top \not\models \neg\psi$	$\neg\psi$ not valid	ψ sat	$\neg\psi \not\equiv \top$
$\phi \models \psi$ and $\psi \models \phi$	$\phi \leftrightarrow \psi$ valid	$\phi \leftrightarrow \neg\psi$ unsat	$\phi \equiv \psi$

The Tableau Calculus for PL1 (TC1)

- TC1 is a semi-decision procedure
  - Construction always terminates for unsatisfiable formulas
  - Result is then a closed tableau (all braches have clashes)
  - Termination for satisfiable formulas not guaranteed
- For satisfiable formula ϕ with a terminating construction: TC1 constructs a model of ϕ
- For simplicity: Input formulas are again in NNF
- NNF characterized by two conditions (like in PL0):
  1. Negation signs occur only in front of atoms
  2. The only connectives are $\wedge$ and $\vee$
- NNF of ϕ (denoted by *nnf*(ϕ)) and ϕ are equivalent!
- Translation procedures are available

Equivalence Replacement Again

Lemma (Equivalent Replacement Lemma)

Let I be an interpretation, α *a variable assignment, and* $I_\alpha \models \psi_1 \leftrightarrow \psi_2$*. Then* $I_\alpha \models \phi[\psi_1] \leftrightarrow \phi[\psi_2]$*.*

Theorem (Equivalent Replacement Theorem)

Let $\psi_1 \equiv \psi_2$*. Then* $\phi[\psi_1] \equiv \phi[\psi_2]$*.*

Basics of the NNF Translation for PL1

- Replace $\leftrightarrow$ by $\rightarrow$ using $(\phi \leftrightarrow \psi) \equiv ((\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi))$
- Replace $\rightarrow$ using $(\phi \rightarrow \psi) \equiv (\neg\phi \vee \psi)$
- Replace the left side of the following equivalences by the right side (order does not matter!)

$$
\begin{array}{rclrcl}
\neg\forall x\ \phi & \equiv & \exists x\ \neg\phi & \neg\exists x\ \phi & \equiv & \forall x\ \neg\phi \\
\neg(\phi \vee \psi) & \equiv & \neg\phi \wedge \neg\psi & \neg(\phi \wedge \psi) & \equiv & \neg\phi \vee \neg\psi \\
\phi \vee \top & \equiv & \top & \top \vee \phi & \equiv & \top \\
\phi \wedge \bot & \equiv & \bot & \bot \wedge \phi & \equiv & \bot \\
\phi \wedge \top & \equiv & \phi & \top \wedge \phi & \equiv & \phi \\
\phi \vee \bot & \equiv & \phi & \bot \vee \phi & \equiv & \phi \\
\neg\neg\phi & \equiv & \phi & & &
\end{array}
$$

- Translation process is terminating with the NNF

The Completion (Inference) Rules of TC1

- As for PL0, there is a rule for conjunctions and disjunctions
- Quantifier rules will be presented on the next slide

$$\frac{\phi \wedge \psi}{\begin{array}{c}\phi\\ \psi\end{array}}$$

If a model satisfies the conjunction, then it satisfies each of the conjuncts

$$\frac{\phi \vee \psi}{\phi \mid \psi}$$

If a model satisfies a disjunction, then it also satisfies one of the disjuncts. It is a non-deterministic rule, and it generates two alternative branches of the tableaux.

The Completion (Inference) Rules of TC1 Cont'd

$$\frac{\forall x\ \phi}{\phi\{x \leftarrow t\}}$$

If a model satisfies a universally quantified formula, it also satisfies the formula (without the quantifier), where the former quantified (and now free) variable is substituted with some (ground) term. The prescription is to use terms which occur in the tableau.

$$\frac{\exists x\ \phi}{\phi\{x \leftarrow a\}}$$

If a model satisfies an existentially quantified formula $\exists x\ \phi$, then it also satisfies the formula $\phi\{x \leftarrow a\}$, where the former quantified (and now free) variable is substituted with a fresh new Skolem constant.

When are the Completion Rules Applicable?

$$\frac{\phi \land \psi}{\begin{array}{c}\phi\\ \psi\end{array}}$$

This rule can be applied if ϕ and ψ are not both is on the current branch

$$\frac{\phi \lor \psi}{\phi \mid \psi}$$

This rule can be applied if neither ϕ nor ψ on the current branch

$$\frac{\exists x\, \phi}{\phi\{x \leftarrow a\}}$$

This rule can be applied if $\phi\{x \leftarrow b\}$ (for a Skolem constant b) is not on the current branch

- Applicability conditions prevent redundant rule applications
- For the $\forall$-rule, no restriction can be given in general!

Remarks on Quantifier Rules and TC1 in General

- Quantifier rules are conceptually simple, but sufficient for our purpose later
- For first-order theorem proving, advanced quantifier rules are widely used which use Skolem functions in general
- Additionally, usually free variable tableaux are used which use unification in order to determine the term t
- The use of sophisticated quantifier rules and unification result in better/faster implementation because some problems wrt permutability of inferences are avoided

Is ϕ: $(\exists x\,(p(x) \wedge (p(x) \rightarrow q(x)))) \rightarrow \exists z\, q(z)$ valid?

- Compute $nnf(\neg\phi)$ and check satisfiability
- If $nnf(\neg\phi)$ is unsatisfiable, then ϕ is valid why?

formula	use
$\neg((\exists x\,(p(x) \wedge (p(x) \rightarrow q(x)))) \rightarrow \exists z\, q(z))$	$\phi \rightarrow \psi \equiv \neg\phi \vee \psi$
$\neg(\neg(\exists x\,(p(x) \wedge (\neg p(x) \vee q(x)))) \vee \exists z\, q(z))$	$\neg(\phi \vee \psi) \equiv \neg\phi \wedge \neg\psi$
$\neg\neg(\exists x\,(p(x) \wedge (\neg p(x) \vee q(x)))) \wedge \neg\exists z\, q(z))$	$\neg\neg\phi \equiv \phi$
$(\exists x\,(p(x) \wedge (\neg p(x) \vee q(x)))) \wedge \neg\exists z\, q(z))$	$\neg\exists x\, \phi \equiv \forall x\, \neg\phi$
$(\exists x\,(p(x) \wedge (\neg p(x) \vee q(x)))) \wedge \forall z\, \neg q(z)$	

$nnf(\neg\phi)$: $(\exists x\,(p(x) \wedge (\neg p(x) \vee q(x)))) \wedge \forall z\, \neg q(z)$

Is $(\exists x\, (p(x) \wedge (\neg p(x) \vee q(x)))) \wedge \forall z\, \neg q(z)$ unsat?

$$(\exists x\, (p(x) \wedge (\neg p(x) \vee q(x)))) \wedge \forall z\, \neg q(z)$$

$$\exists x\, (p(x) \wedge (\neg p(x) \vee q(x)))$$

$$\forall z\, \neg q(z)$$

$$p(a) \wedge (\neg p(a) \vee q(a))$$

$$p(a)$$

$$\neg p(a) \vee q(a)$$

$$\neg p(a) \qquad q(a)$$

$$* \qquad \neg q(a)$$

$$*$$

Is $(\exists y\,(p(y) \land \neg q(y))) \land (\forall z\,(p(z) \lor q(z)))$ satisfiable?

$$(\exists y\,(p(y) \land \neg q(y))) \land (\forall z\,(p(z) \lor q(z)))$$

$$\exists y\,(p(y) \land \neg q(y))$$

$$\forall z\,(p(z) \lor q(z))$$

$$p(a) \land \neg q(a)$$

$$p(a)$$

$$\neg q(a)$$

$$p(a) \lor q(a)$$

$p(a)$ completed | $q(a)$ $*$

- Formula is satisfiable
- Left branch b is completed
- Why is b completed?
- Take b: make all literals true
- $I(p) = \{a\}$, i.e., $I(p(a)) = 1$
- $I(q) = \{\}$, i.e., $I(\neg q(a)) = 1$
- $\mathcal{U} = \{a\}$

Summary

We recapitulated important definitions and notations like

- the set of (well-formed) formulas (for PL0 and PL1)
- the set of terms for PL1
- the concept of an interpretation (for PL0 and PL1),
- models and related notions like (un)sat, valid, entailment, etc.
- negation normal form in PL1,
- TC1 and its use