VU Nichtmonotones Schließen

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§1 Introduction

1.1 Phenomenology of Nonmonotonic Reasoning
Slogan

“Nothing is certain, but death and taxes.”

(Benjamin Franklin, 1789)

Practically all inferences in daily life are uncertain.
Slogan (ctd.)

Examples:

- We all believe that the sun will rise tomorrow ... but we cannot prove it!

- I don’t have an elder brother, because otherwise I would know it. * However, it is nevertheless possible, unbeknownst to me, that I have an elder brother . . .

- The remaining lecture will not be taught in Japanese.

Conclusions of this kind are ubiquitous! ... But they are amazingly difficult to formalise.

In this course we deal with formal methods to describe this kind of reasoning, generally referred to as common-sense reasoning.
Basic Ideas

➢ Traditional logics satisfy the *monotonicity condition*: If $A$ is derivable from a theory $T$, then $A$ is also derivable from $T'$, for *any* $T' \supseteq T$.

➢ This principle is not adequate in the context of common-sense reasoning.

➢ Common-sense reasoning deals with *incomplete information*.

   ➢ We are constantly faced with situations for which not all relevant information is known.

   ➢ Yet, we are not paralysed by this obstacle and are able to draw “plausible” inferences.

   ➢ These inferences, however, are not ironclad—they can be invalidated by new, more accurate information.

   ➢ They are *defeasible*, i.e., they may be retracted in the light of new information.
Example: Defeasible Conclusions

Assume it is Saturday afternoon and James Bond must urgently meet M, his superior (and head of MI6).

The following facts are known:
- M cannot be reached by phone.
- At Saturday afternoons, M usually visits his club.

Given no information to the contrary, it is thus reasonable to assume that M is at his club and James Bond should therefore seek him there.

On his way to the club (in his Aston Martin DB 5), Bond receives a phone call from Miss Moneypenny. She informs him:
- M had a car accident on Friday.

Therefore, it is no longer reasonable to go to the club, rather to go to the hospital and meet M there.
General Definition

Definition 1 (Minsky, 1975)

1. By *nonmonotonic reasoning* we understand the drawing of conclusions which may be invalidated in the light of new information.

2. A logical system is *nonmonotonic* iff its provability relation violates the property of monotonicity. □
Monotonic vs. Nonmonotonic Logics

Principal difference:

- Classical logic formalises *truth* and *valid conclusions*.
- Nonmonotonic logics formalise *rationality* and *plausible conclusions*.

Note:

- Rationality is central for common-sense reasoning:
  * although conclusions can be invalidated by new information, *they are not chosen at random*,
  * rather our knowledge and experience is employed for accepting conclusions, demanding that they are *rational*.

- At least one rational justification is required to accept a conclusion in common-sense reasoning.
Rational Conclusions

- Rationality is a complex concept and a precise definition is nigh impossible.

- In fact, the different nonmonotonic reasoning formalisms introduced in the literature can be viewed as approaches for capturing different aspects of rationality.
Rational Conclusions (ctd.)

Rationality has at least two specific properties distinguishing it from the concept of truth, namely:

Rationality is

- **agent-dependent**
  - different agents may have different opinions on what is rational in a certain situation;

- **purpose-dependent**
  - the acceptance of a proposition as a rational conclusion depends on the purpose it is used for.

- Example:
  * Given only cursory evidence, I may well assume that a person is honest and lend him 20 Euros.
  * But I would be more cautious if I want to consider that person as a business partner, and would conduct a thorough investigation.
Beliefs

To emphasise the subjective nature of common-sense conclusions, they are referred to as beliefs.

Two fundamental questions:

1. What actually are common-sense conclusions (beliefs)?
2. What techniques do we use for reaching them?

Concerning the first question:

Definition 2 (Perlis, 1987)

A proposition $A$ is a belief of an agent $G$, i.e., $G$ considers $A$ as a rational conclusion, if $G$ is prepared to use $A$ as if it were true. □
Beliefs (ctd.)

Example (after Winograd (1980)):

- Assume I plan a trip by car.
- To begin with, I must decide where my car actually is.
- Given no better evidence, the following is plausible:
  (A) The car is there, where I parked it last.
- According to Def. 2, (A) is considered to be a belief if I act as if it were true.
- That is,
  – if I ignore all circumstances in which (A) could be false, and
  – I base my actions on the assumption that (A) is true,
then I believe in (A), even if I cannot be sure that (A) is actually true.
Beliefs (ctd.)

Example (ctd.):

➤ On the other hand:
  – Even if I am almost certain that (A) holds,
  – but at the same time I improve my chances by checking whether a bus will pass by, in case the car is missing,

then (A) is *not regarded as a belief*, rather *a very likely contingency*. 
Beliefs (ctd.)

Now to the question *how* beliefs are reached.

Consider the earlier example involving the famous British agent that

(C) M is in his club

since:

(1) It is Saturday afternoon.
(2) Usually, at Saturday afternoons, M visits his club.

Obviously, in virtue of (2), (1) gives evidence supporting (C).

It might seem that the belief (C) is inferred using the following rule:

(R) If (1), conclude (C).

But this cannot be the case!
Beliefs (Forts.)

One cannot accept (R) and at the same time be prepared to reject (C) in the light of new information.

The inference pattern must be more complex!

The rule should be blocked whenever its use would be intuitively unacceptable.

I.e., I am prepared to conclude (C) unless there is some evidence making this inference irrational.

Instead of (R), the following ignorance-dependent rule should be used:

(R′) From (1), in the absence of evidence to the contrary, infer (C).
Nonmonotonic Inference Rules

To summarise:

**Definition 3**

By a *nonmonotonic inference pattern*, or a *nonmonotonic inference rule*, we understand a rule of the following form:

Given A, in the absence of evidence B, infer C. □

**Remark:**

- In practical applications, B is often identified with contradictory evidence, i.e., considering rules of the following form:

  Given A, in the absence of evidence to the contrary, infer C.
Def. 3 actually leaves the following fundamental questions open:

- How are nonmonotonic rules to be represented?
- Under which criteria is a piece of information to be considered as given (or known)?

The answers to these questions are non-trivial!

The formalisms we will deal with in this course are approaches to answer these questions!
1.2 Nonmonotonic Reasoning and Reasoning about Actions
Two Problems

- Two problems were fundamental in the development of nonmonotonic reasoning formalisms:
  - frame problem;
  - qualification problem.

- Both problems are relevant for *reasoning about actions*.
Frame Problem

The frame problem deals with the following scenario:

➢ Suppose we perform an action.

➢ Which properties of the world change and which persist?

Example: A cube is painted \( \Rightarrow \) its colour changes, all other properties stay the same.

(1) \( \forall t \forall c (\text{paint}(t, c) \rightarrow \text{color}(t + 1, c)) \)

(2) \( \text{paint}(0, \text{red}), \quad \text{weight}(0, 500) \)

We want to derive: \( \text{weight}(1, 500), \text{color}(2, \text{red}) \).

Problem: This is not derivable from (1) and (2)!
Frame Problem (ctd.)

We need additional *frame axioms*:

(3) \( \forall t \forall c[(\text{color}(t, c) \land \text{do\_nothing}(t)) \rightarrow \text{color}(t + 1, c)] \);

\( \text{do\_nothing}(t) \ldots \) no action at time point \( t \).

(4) \( \forall t \forall k(\text{weight}(t, k) \rightarrow \text{weight}(t + 1, k)) \).

Together with fact \( \text{do\_nothing}(1) \).
Frame Problem (ctd.)

➤ Problem:

– In general, a large number of frame axioms is required $\Rightarrow$ inefficient!

➤ Idea:

– Replace frame axioms by a simple nonmonotonic rule:

   “All aspects of the world remain invariant, except those which explicitly change”. 
Qualification Problem

The qualification problem describes the following scenario:

» Which preconditions ("qualifications") must hold that an action succeeds?

Example: Suppose a robot needs to move a block A on top of another block B.

» Requirements:
  – B should have a clear top;
  – A must not be too heavy;
  – the robot’s arm must not be broken;
  – etc.

⇒ In general, a large number of qualifications is necessary.
Qualification Problem (ctd.)

Even worse:

- In real-world domains, *the satisfiability of the qualifications is often unknown*.
- For instance, suppose I plan a trip to the airport.
  
  * I may know sufficiently enough to be convinced that my trip succeeds.
  
  * **BUT:** I can always imagine situations making my trip impossible
    
    - my car is broken,
    - a meteor destroys the road to the airport,
    - I will be abducted by aliens, etc.

Idea: use a nonmonotonic rule

"*In the absence of evidence to the contrary, assume that an action succeeds.*"
Ramification Problem

Related problem: the *ramification problem*.

➢ Describes the implicit consequences from performed actions.

**Example:** A robot moves a dusty block.

⇒ The movement of the dust particles should not be specified explicitly, rather it should be derivable from the fact that the cube has been moved.
1.3 Typology of Nonmonotonic Reasoning
Types of Nonmonotonic Inferences

- Different types of nonmonotonic inferences can be identified.
- The classification is according to the criteria under which nonmonotonic conclusions are accepted.
- Following Moore (1983), one should distinguish between
  - genuine incomplete information;
  - incomplete representation of principally complete information.
Types of Nonmonotonic Inferences (ctd.)

Example:

(1) In the absence of evidence to the contrary, assume that birds fly.

(2) Unless your name is on a list of winners, assume that you are a loser (Gabbay & Sergot, 1986).

Discussion: Rule (1) involves genuinely incomplete information.

➢ If all we know about Tweety is that it is a bird, we may conclude (at least tentatively) that it flies.

➢ This conclusion is plausible but not ironclad:
  — if we later learn that Tweety is actually a penguin, then we have to retract our prior conclusion since penguins don’t fly.
Types of Nonmonotonic Inferences (ctd.)

Rule (2) describes a different scenario.

➤ Here, it is assumed that the list of winners is complete, so that it contains information about losers implicitly.

➤ That is, the rule refers not to incomplete information but to incomplete representation of virtually complete knowledge.

➤ The rule is logically valid:

- if it is true that a list of winners is complete, and it is true that your name is not on the list, then it must be true that you are a loser.
- This conclusion cannot be invalidated by providing better evidence.
- At the moment of announcing the list, our knowledge is complete and cannot be extended by a new piece of information.
Types of Nonmonotonic Inferences (ctd.)

Rule (2) is nevertheless nonmonotonic since it is context-dependent:

- It refers to the absence of some evidence and thus depends on the context within it operates.
- Embedding the rule in a larger context may render previous conclusions underivable.
- Example:

  (3) The list of winners consists of James and Felix.

Applying Rule (1), we obtain:

(*) Miss Moneypenny is a loser.

Suppose (3) is replaced by

(4) The list of winners consists of James, Felix, and Miss Moneypenny.

\[ \implies (*) \text{ is no longer derivable.} \]
Types of Nonmonotonic Inferences (ctd.)

In summarising, we can distinguish between two kinds of nonmonotonic inferences:

- **Default reasoning**
  - Refers to the drawing of rational conclusions from less than conclusive information, *in the absence of evidence making these inferences implausible*.
  - Conclusions are defeasible.

- **Autoepistemic reasoning**
  - Refers to the drawing of conclusions using incomplete representation of (theoretically) complete information.
    - Principle: if a conclusion would be false, I would know it (by assuming that my knowledge is complete).
  - Conclusions are *not* defeasible, yet nonmonotonic since context-dependent.
Default Reasoning: Further Classification

➤ Prototypical reasoning

- Principle:
  * $A$ describes a typical situation. Therefore, it is plausible that $A$ holds.

- Conclusions are based on *implicit* statistical knowledge
  * explicit probability values are not known and cannot be determined in general
  * we simply know that they are “sufficiently” high.

- Example: The majority of birds fly.
Default Reasoning: Further Classification (ctd.)

➢ No-risk reasoning

- Principle:
  * If \( \neg A \) would be accepted, but turning out to be false, the consequences would be disastrous.
  * Therefore, if one has to choose between \( A \) and \( \neg A \), select \( A \).

- Example: In the absence of evidence to the contrary, assume that the accused is innocent.
Default Reasoning: Further Classification (ctd.)

➤ **Best-guess reasoning**

- **Principle:**
  * No supporting evidence can be found, but a decision must be made.
  * So, taking the currently available knowledge into account, choose A as the best conclusion.

- **Example:**
  * It is Sunday and I want to go to a Sushi restaurant for lunch.
  * There are two Sushi restaurants in my neighbourhood; one of them is closed on Sundays but I don’t know which one.

  ➤ The best guess is to assume that the one nearest to my home is open.
Default Reasoning: Further Classification (ctd.)

Probabilistic default reasoning

- Here, explicit numerical probability values are embedded in the inference process.

- Principle:
  * If the probability for $A$ is sufficiently high, infer $A$.

In this lecture, we only deal with qualitative, logic-based methods for modelling nonmonotonic reasoning.

* Probabilistic methods are a separate field ("probabilistic reasoning", "uncertain reasoning")
* Based on classical logic and probability theory (e.g., belief networks, Dempster-Shafer theory, $\varepsilon$-semantics due to Adams and Pearl, etc.).
Autoepistemic Reasoning: Further Classification

**Subjective autoepistemic reasoning**

- **Principle:**
  * In view of my own (subjective) knowledge, I would know if $A$ is false.
  * Since I do not know that $A$ is false, $A$ must be true.

- **Example:**
  * I don’t have an elder brother, otherwise I would know it (Moore, 1983).
Autoepistemic Reasoning: Further Classification (ctd.)

- **Explicit autoepistemic reasoning**
  - **Principle:**
    * According to an explicit convention, I would know whether $A$ is false.
    * Since I do not know that $A$ is false, $A$ must be true.
  - **Example:**
    * If you are not on a list of winners, assume that you are a loser (in view of the convention that lists of winners are always complete).
1.4 Formal Approaches to Nonmonotonic Reasoning
Sandewall’s Formalism

- First attempt to formalise nonmonotonic inferences (1972).

- Extends classical first-order logic by introducing a unary operator ‘UNLESS’.
  
  - Intuitive meaning:
    
    * for a theory $T$ and a proposition $A$:
      
      $UNLESS(A)$ is derivable from $T$ (in Sandewall’s formalism)
      
      $\iff$ $A$ is not derivable from $T$ by means of first-order logic.
  
  - Symbolically:
    
    $T \sim UNLESS(A) \iff T \not\models A.$
Sandewall’s Formalism (ctd.)

- Sandewall’s formalism possesses undesired properties:
  - There are cases where both \( A \) as well as \( UNLESS(A) \) is derivable.
    * Example: \( T = \{ C, (C \land UNLESS(B)) \rightarrow A \} \).
      \[ \implies T \vdash A \text{ and } T \vdash UNLESS(A). \]
  - The formalism is too strong to deal with conflicting rules in an adequate manner:
    * Example: Assume \( T \) consists of
      \( Quaker(Nixon) \land Republican(Nixon); \)
      \[ \forall x \left( (Quaker(x) \land UNLESS(\neg Pacifist(x))) \rightarrow Pacifist(x) \right) \); \)
      \[ \forall x \left( (Republican(x) \land UNLESS(Pacifist(x))) \rightarrow \neg Pacifist(x) \right) ; \)
      \[ \implies T \text{ is inconsistent: It holds that } T \vdash UNLESS(Pacifist(Nixon)) \text{ and } T \vdash UNLESS(\neg Pacifist(Nixon)), \text{ so } T \vdash Pacifist(Nixon) \text{ and } T \vdash \neg Pacifist(Nixon) \text{ hold as well.} \]

- This lead to the development of more refined methods!
Modal Nonmonotonic Logics

➢ Refers to a family of formalisms employing the language of modal logic for representing the notions of consistency and belief.

   - Employ a unary operator ‘M’ for representing the notion of consistency directly in the object language:
     * MA: A can be consistently assumed.

   - Dual operator L = ¬M¬:
     * LA: A is nonmonotonically derivable (A is a belief).

   - Example: Birds usually fly:

     \[ \forall x ((\text{Bird}(x) \land M\text{Flies}(x)) \rightarrow \text{Flies}(x)) \]

     Interpretation: For all objects x,
     * if x is a bird and it is consistent to assume that x flies, then infer that x flies.
Modal Nonmonotonic Logics (ctd.)

- Important formalism: autoepistemic logic (Moore, 1983)
  - models autoepistemic reasoning
    - describes inference patterns of an ideally rational agent reasoning about his own knowledge
  - is a reconstruction of earlier (problematic) modal nonmonotonic logics due to McDermott & Doyle (1980) and McDermott (1982).
Default Logic

- Formalises default reasoning.
- Proposed by Reiter in 1980.
  - Uses special inference rules, called defaults, for allowing nonmonotonic inferences.
  - Example: “Birds usually fly” is represented by

\[
\frac{\text{Bird}(x) : \text{Flies}(x)}{\text{Flies}(x)}
\]

Interpretation: For all objects \( x \),

* if \( x \) is a bird and it is consistent to assume that \( x \) flies (i.e., \( \neg \text{Flies}(x) \) is not derivable),
* then infer that \( x \) flies.
Circumscription was proposed by McCarthy in 1980 and is a form of minimal-model reasoning:

- Circumscription of a predicate $P$ is tantamount to minimising the extension of $P$ (i.e., the set of objects satisfying $P$).

- Principle:
  * Only those objects satisfy a property $E$ for which it can be shown that they satisfy $E$.

- This is realised by means of a schema of second-order logic.

- Example: “Birds usually fly” is represented here by means of
  \[ \forall x((\text{Bird}(x) \land \neg \text{Abnormal}(x)) \rightarrow \text{Flies}(x)), \]
  and by minimising the interpretation of abnormal birds (i.e., minimising the extension of $\text{Abnormal}(x)$).
Circumscription (ctd.)

➢ Model theoretically, circumscription is reasoning under *minimal models*:
  
  – For checking whether $A$ follows from $T$, not all models of $T$ are taken into consideration, *but only those which are minimal in a certain sense*.
  
  – To this end, a partial pre-order, $\leq$, over the models is defined
    ➢ a model is minimal iff it is a minimal element with respect to $\leq$.
  
  – A similar model-theoretic method has been introduced by Shoham (1988).
Closed-World Assumption

The Closed-world assumption (CWA) was postulated by Reiter in 1978 in the context of database theory.

Idea:

- Only positive information is stored in a database.
- All facts which are not derivable from the database are assumed to be false.
- Example: train schedule
  - all train connections which are not explicitly given in the train schedule are assumed to be non-existent.
1.5 Literature (Selection of Standard Texts)


§2 Prerequisites from Classical Logic
Language

Two important forms of classical logic:

– Classical propositional logic
– Classical first-order logic (or *predicate logic*)
Language of Propositional Logic

Formulas of propositional logic are built from a set of atomic formulas using

- the primitive logical connectives \( \neg \) and \( \to \)
- and the logical constant \( \top \) (verum)

in the usual manner.

The remaining operators \( \lor, \land, \equiv, \) and \( \bot \) (falsum) are defined from the primitive ones as usual, i.e., by setting

- \( (A \lor B) = (\neg A \to B) \),
- \( (A \land B) = \neg(\neg A \lor \neg B) \),
- \( (A \equiv B) = ((A \to B) \land (B \to A)) \),
- \( \bot = \neg \top \).
Language of First-Order Logic

- Formulas of first-order logic are built from a set of atomic formulas using
  - the operators from propositional logic
  - and the universal quantification symbol, $\forall$,
in the usual manner.

- Here, we assume a first-order language with function symbols.
  - We denote (object) variables by $x, y, z, \ldots$,
  - and (object) constants by $a, b, c, \ldots$ or by proper names.
  - Terms are built from the variables and the function symbols.

- The existential quantification is defined by setting

$$\exists x A = \neg \forall x \neg A.$$
Some Concepts and Notation

Let $Q \in \{\forall, \exists\}$.

- For any formula $QxA$, $A$ is the scope of $Qx$.
- Each occurrence of a variable $x$ in $A$ as well as the occurrence of $x$ directly preceded by $Q$ is in $QxA$ bound.
- Each occurrence of a variable $x$ in $A$ is free iff it is not bound.
- A formula in which all variable occurrences are bound is closed, otherwise open.

A theory is

- a set of formulas, in case of propositional logic, and
- a set of closed formulas, in case of first-order logic.
Some Concepts and Notation (ctd.)

➢ A term containing no variables is called ground.

➢ Let $x_1, \ldots, x_n$ be the free variables in a formula $A$.
  
  – The universal closure of $A$ is $\forall A = \forall x_1 \ldots \forall x_n A$.

➢ Let $A(x_1, \ldots, x_n)$ be a formula containing the free variables $x_1, \ldots, x_n$ and let $\alpha_1, \ldots, \alpha_n$ be terms.
  
  – $A[x_1/\alpha_1, \ldots, x_n/\alpha_n]$ denotes the formula resulting from $A(x_1, \ldots, x_n)$ by uniformly replacing all free occurrences of $x_i$ by $\alpha_i$, for $1 \leq i \leq n$.
  
  – E.g., for $A(x, y) = (p(x) \lor \exists x q(x, y)) \rightarrow r(x, y)$, we have that

\[
A[x/a, y/f(b)] = (p(a) \lor \exists x q(x, f(b))) \rightarrow r(a, f(b)).
\]

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Semantics vs. Proof Theory

- Each “proper” logic is assigned with two basic ingredients:
  - semantics;
  - proof theory.

- The semantics associates a meaning to the language elements
  - i.e., it defines the truth values to formulas.

- The proof theory takes care for defining the notion of a formal proof or a formal derivation
  - it is defined in terms of a calculus, comprising axioms and inference rules.
Semantics

The semantics is defined in terms of interpretations and the semantic consequence operator $\models$:

- In propositional logic, an interpretation is a function assigning each atom a truth value.
- In first-order logic, an interpretation is a bit more complicated, involving
  - a domain for the values of variables and objects,
  - assignments of the predicate letters and function symbols.
Semantics (ctd.)

- Interpretations induce truth values for composite formulas in an inductive fashion.

- A *model* of a formula $A$ is an interpretation making $A$ true. Similarly, a model of a theory $T$ is an interpretation which is a model of all elements of $T$.

- The relation $T \models A$ holds if all models of $T$ are models of $A$.
  - In particular:
    * for $T = \emptyset$, $T \models A$ means that $A$ is *valid*, i.e., $A$ is true under every interpretation.
    * We write $\models A$ instead of $\emptyset \models A$. \
Proof Theory

The proof theory involves the following concepts:

- An **axiom system**, consisting of
  - a set of formulas, called **axioms**,
  - a set of relations between formulas, called **inference rules**.
  * If $R$ is an inference rule and if $\langle X_1, \ldots, X_n, Y \rangle \in R$, then $Y$ is a **direct consequence** of $X_1, \ldots, X_n$ under $R$.

- A **proof** is a finite sequence $A_1, \ldots, A_n$ of formulas s.t. for each $A_i$ ($1 \leq i \leq n$)
  - $A_i$ is an axiom, or
  - $A_i$ is a direct consequence of earlier elements in the sequence under some inference rule.

- A formula $A$ is **provable**, symbolically $\vdash A$, iff there is a proof $A_1, \ldots, A_n$ s.t. $A_n = A$ holds.
Proof Theory (ctd.)

- A *derivation* from a theory $T$ is a finite sequence $A_1, \ldots, A_n$ of formulas s.t. for each $A_i$ $(1 \leq i \leq n)$
  - $A_i \in T$, or
  - $A_i$ is an axiom, or
  - $A_i$ is a direct consequence of earlier elements in the sequence under some inference rule.

- A (closed) formula $A$ is *derivable* from a theory $T$, symbolically $T \vdash A$, iff there is a derivation $A_1, \ldots, A_n$ of $T$ s.t. $A_n = A$ holds.
Central properties:

- Soundness of an axiom system:
  \[ T \vdash A \implies T \models A. \]

- Completeness of an axiom system:
  \[ T \models A \implies T \vdash A. \]

Concrete sound and complete axiom systems for propositional logic and first-order logic can be readily found in the literature, like

- Hilbert-type systems,
- sequent-type calculi,
- tableau calculi,
- resolution calculi,
- natural deduction systems, etc.

We present a sequent-type calculus for propositional logic later on!
Deductive Closure

The deductive closure of a theory $T$ is given by

- $Th(T) = \{ A \mid T \vdash A \}$, in case of propositional logic, and
- $Th(T) = \{ A \mid T \vdash A \text{ and } A \text{ is closed} \}$, in case of first-order logic.

Properties:

- $T \subseteq Th(T)$; ("inflationaryness")
- $Th(T) = Th(Th(T))$ ("idempotency");
- $T \subseteq T'$ implies $Th(T) \subseteq Th(T')$ ("monotonicity").
A formula $A$ is in \textit{prenex normal form} if $A$ is of the form

$$Q_1x_1 \ldots Q_kx_k \left( \bigwedge_{i=1}^n \left( \bigvee_{j=1}^{m(i)} B_{ij} \right) \right),$$

where

- $Q_l \in \{\forall, \exists\}$, $l = 1, \ldots, k$,
- $x_1, \ldots, x_k$ are pairwise distinct variables of $( \bigwedge_{i=1}^n ( \bigvee_{j=1}^{m(i)} B_{ij}) )$,
- $B_{ij}$ is an atom or the negation of an atom.

For each formula $A$ there is a formula $B$ in prenex normal form s.t. $\models (A \equiv B)$. 

\textbf{Prenex Normal Form}
Skolemisation

The skolemisation of a closed formula \( A \) in prenex normal form is constructed as follows:

- Consider a subformula of \( A \) of the form \( \exists y \ B(y, x_1, \ldots, x_n) \)
  * \( x_1, \ldots, x_n \) are those free variables of \( \exists y \ B(y, x_1, \ldots, x_n) \) which are bound by \( \forall \)-quantifiers in \( A \).

- Replace \( \exists y \ B(y, x_1, \ldots, x_n) \) by \( B(f(x_1, \ldots, x_n), x_1, \ldots, x_n) \)
  * \( f \) is a new \( n \)-ary function symbol (“Skolem function”).

- Repeat this for any subformula of \( A \) of the form \( \exists y \ B(y) \).

A skolemised form of a closed formula \( A \) is a closed formula resulting from a prenex form of \( A \) by skolemisation.
§3 Default Logic

3.1 Preliminaries
Default Rules

In default logic, nonmonotonic conclusions are represented in terms of special *inference rules*.

Let $A(\tilde{x}), B_1(\tilde{x}), \ldots, B_n(\tilde{x}), C(\tilde{x})$ be first-order formulas, whose free variables are among $\tilde{x} = x_1, \ldots, x_k$.

A *default*, $d$, is an expression of form

$$
\frac{A(\tilde{x}) : B_1(\tilde{x}), \ldots, B_n(\tilde{x})}{C(\tilde{x})}
$$

- Intuitive meaning:
  * for all objects $\tilde{x} = x_1, \ldots, x_k$, if $A(\tilde{x})$ is known and $B_1(\tilde{x}), \ldots, B_n(\tilde{x})$ can be consistently assumed, then infer $C(\tilde{x})$.

- Notation:
  * $A(\tilde{x})$ ist the *prerequisite*,
  * $\{B_1(\tilde{x}), \ldots, B_n(\tilde{x})\}$ is the *justification*, and
  * $C(\tilde{x})$ is the *consequent* of $d$. 
Examples

➤ Prototypical reasoning:

\[
\text{Bird}(x) : \text{Flies}(x) \\
\hline \\
\text{Flies}(x)
\]

- Typically, birds fly.

➤ No-risk reasoning:

\[
\text{Accused}(x) : \text{Innocent}(x) \\
\hline \\
\text{Innocent}(x)
\]

- The accused is innocent unless proven otherwise.
Examples (ctd.)

➤ Best-guess reasoning:

\[
\text{Solution\_found\_so\_far}(x) : \text{Best\_Solution}(x) \quad \frac{\text{Best\_Solution}(x)}{}
\]

- In the absence of evidence to the contrary, assume that the best solution found so far is the best one (McCarthy, 1984).

➤ Subjective autoepistemic reasoning:

\[
\top : \neg\text{Have\_elder\_brother} \quad \frac{\neg\text{Have\_elder\_brother}}{}
\]

- In the absence of evidence to the contrary, assume that you have no elder brother (Moore, 1983).
Notation

Let \( d \) be a default of form

\[
\frac{A : B_1, \ldots, B_n}{C}
\]

- If \( A = \top \), then \( d \) is called \textit{prerequisite-free}.
  In this case, \( d \) is also written as

\[
\frac{B_1, \ldots, B_n}{C}
\]

- If \( n = 0 \), then \( d \) is \textit{justification-free}. 
In default logic, knowledge is represented in terms of default theories.

A default theory is an ordered pair $T = \langle W, D \rangle$, where

- $W$ is a set of closed first-order formulas;
- $D$ is a set of defaults.

Elements of $W$ are the premisses of $T$, representing certain yet in general incomplete information about the world.

$D$ represents plausible although not necessarily true conclusions (i.e., conclusions which hold typically—“rules of thumb”).
Problem:
- What totality of knowledge is induced by a default theory?

Answer:
- notion of an extension.

Basic idea of obtaining extensions:
- Apply the defaults in $D$ to the premisses in $W$ to extend certain knowledge by plausible conclusions;
- apply the defaults as long as possible, until no new knowledge is generated
  ➡️ the result is an extension of $T$. 

Default Theories (ctd.)
Recall: intuitive meaning of

\[
A : B_1, \ldots, B_n \\
\frac{}{C}
\]

- If \( A \) is known and \( B_1, \ldots, B_n \) can be consistently assumed, then infer \( C \).

Problem:

- What means “\( A \) is known”?
- When can \( B_1, \ldots, B_n \) be “consistently assumed”?
- Can default conclusions be obtained from applications of the premisses in \( W \) alone?
Example

Consider the following default theory $T = \langle W, D \rangle$:

$W = \{\text{Friend}(Tom, Bob), \text{Friend}(Bob, Sally), \text{Friend}(Sally, Tina)\}$

$D = \left\{ \frac{\text{Friend}(x, y) \land \text{Friend}(y, z) : \text{Friend}(x, z)}{\text{Friend}(x, z)} \right\}$

On the basis of the premisses (by using the defaults), $\text{Friend}(Tom, Sally), \text{Friend}(Bob, Tina)$ can be inferred.

But it cannot be further derived that $\text{Friend}(Tom, Tina)$ holds, since $\text{Friend}(Tom, Sally)$ and $\text{Friend}(Bob, Tina)$ can only be derived by default, i.e., they do not belong to the premisses.

The default inference mechanism must be more involved!
Extensions

- An extension $E$ of a default theory $T = \langle W, D \rangle$ should satisfy the following conditions:
  1. $E$ should be a set of first-order formulas.
  2. $E$ should contain the premisses, i.e., $W \subseteq E$.
  3. $E$ should be deductively closed, i.e., $Th(E) = E$.
  4. $E$ should contain “a maximal number of formulas which are derivable by defaults”.
  5. $E$ should contain no “unfounded” formulas.

Conditions 4 and 5 are informal and need to be recast into formal terms!
Example: Tweety

Consider the following default theory $T = \langle W, D \rangle$:

$$W = \{Bird(Tweety)\};$$

$$D = \left\{ d = \frac{\text{Bird}(x) : \text{Flies}(x)}{\text{Flies}(x)} \right\}.$$ 

An extension $E$ of $T$ should contain $\text{Bird}(Tweety)$ and $\text{Flies}(Tweety)$, and all formulas which can be classically derived from them.

$\implies$ It should hold that $E = \text{Th}\{\text{Bird}(Tweety), \text{Flies}(Tweety)\}$.
Example: Tweety (ctd.)

Consider now $T' = \langle W', D \rangle$, where

$$W' = W \cup \{ \text{Penguin(Tweety)}, \forall x (\text{Penguin}(x) \rightarrow \neg \text{Flies}(x)) \}.$$

Extension $E'$ of $T'$ should contain $W'$, but not $\text{Flies}(\text{Tweety})$, since the application of $d$ is "blocked":

- with $\text{Penguin}(\text{Tweety}) \in E'$ and $\forall x (\text{Penguin}(x) \rightarrow \neg \text{Flies}(x)) \in E'$, it follows that $\neg \text{Flies}(\text{Tweety}) \in E'$ (since $E'$ is closed under classical logic).

$\text{Flies}(\text{Tweety})$ can no longer be consistently assumed!

It should hold that $E' = \text{Th}(W')$. 
Example: Nixon Diamond

Consider the following default theory \( T = \langle W, D \rangle \):

\[
W = \{ \text{Republican}(Nixon), \text{Quaker}(Nixon) \}\};
\]

\[
D = \left\{ d_1 = \frac{\text{Republican}(x) : \neg \text{Pacifist}(x)}{\neg \text{Pacifist}(x)}, \right. \\
\left. d_2 = \frac{\text{Quaker}(x) : \text{Pacifist}(x)}{\text{Pacifist}(x)} \right\}.
\]

Defaults \( d_1 \) and \( d_2 \) are mutually conflicting! (The application of \( d_1 \) blocks \( d_2 \) and vice versa.)

There are two alternatives for extensions:

\[
E_1 = \text{Th}(\{ \text{Republican}(Nixon), \text{Quaker}(Nixon), \neg \text{Pacifist}(Nixon) \}));
\]

\[
E_2 = \text{Th}(\{ \text{Republican}(Nixon), \text{Quaker}(Nixon), \text{Pacifist}(Nixon) \})).
\]

Formal definition of an extension is given later on!
3.2 Open Defaults
Open Defaults

- A default \( d = (A : B_1, \ldots, B_n/C) \) is open iff one of the formulas \( A, B_1, \ldots, B_n, C \) is open; otherwise \( d \) is closed.

- A set \( D \) of defaults is open iff at least one default \( d \in D \) is open; otherwise \( D \) is closed.

- A default theory \( T = \langle W, D \rangle \) is open iff \( D \) is open; otherwise \( T \) is closed.
Open Defaults (ctd.)

Let \( d \) be a default of form

\[
\frac{A(\tilde{x}) : B_1(\tilde{x}), \ldots, B_n(\tilde{x})}{C(\tilde{x})},
\]

where \( \tilde{x} = x_1, \ldots, x_k \) comprises the free variables in \( A(\tilde{x}), B_1(\tilde{x}), \ldots, B_n(\tilde{x}), C(\tilde{x}) \).

Let furthermore \( \alpha_1, \ldots, \alpha_k \) be ground terms.

Then, the closed default

\[
\frac{A[x_1/\alpha_1, \ldots, x_k/\alpha_k] : B_1[x_1/\alpha_1, \ldots, x_k/\alpha_k], \ldots, B_n[x_1/\alpha_1, \ldots, x_k/\alpha_k]}{C[x_1/\alpha_1, \ldots, x_k/\alpha_k]}
\]

is called an instance of \( d \).
Open Defaults (ctd.)

- Open defaults represent *general inference schemata* which can be applied to arbitrary objects.
  - Open defaults can be identified with the set of all its instances.
  - We will replace open defaults by their closed instances.

- It therefore suffices to consider only *closed default theories* in the following.

- To this end, we define the *closure* of an open default theory.
Closure of a Default Theory: Motivation

- An instance of a default is obtained by replacing the free variables by ground terms.
  - Ground terms can intuitively be viewed as representing individuals.

- Problem:
  - which totality of individuals is determined by a default theory?
Examples

Let $T = \langle W, D \rangle$ be the following default theory:

$$W = \{\text{Musician}(\text{Lemmy}), \text{Musician}(\text{Phil}), \text{Philosopher}(\text{Immanuel})\};$$

$$D = \left\{ d = \frac{\text{Musician}(x) : \text{Deaf}(x)}{\text{Deaf}(x)} \right\}.$$

Lemmy, Phil, and Immanuel are the only individuals mentioned by $T$.

The closure of $T$ should therefore be comprised of $W$, together with the following instances of $d$:

$$\frac{\text{Musician}(\text{Lemmy}) : \text{Deaf}(\text{Lemmy})}{\text{Deaf}(\text{Lemmy})};$$

$$\frac{\text{Musician}(\text{Phil}) : \text{Deaf}(\text{Phil})}{\text{Deaf}(\text{Phil})};$$

$$\frac{\text{Musician}(\text{Immanuel}) : \text{Deaf}(\text{Immanuel})}{\text{Deaf}(\text{Immanuel})}.$$
Suppose $T = \langle W, D \rangle$ is the following default theory:

\[
W = \{ Kryptonian(Kal-El), \forall x (Kryptonian(x) \rightarrow Kryptonian(father(x))) \} ; \\
D = \left\{ d = \frac{Kryptonian(x) : Superpowers(x)}{Superpowers(x)} \right\} .
\]

Clearly, here, the set of individuals induced by $T$ should be given by \{Kal-El, father(Kal-El), father(father(Kal-El)), \ldots \}.

The closure of $T$ should be given by $W$, together with the following infinitely many instances of $d$:

\[
\begin{align*}
Kryptonian(Kal-El) & : Superpowers(Kal-El) ; \\
\frac{Kryptonian(father(Kal-El)) : Superpowers(father(Kal-El))}{Superpowers(father(Kal-El))} ; \\
\frac{Kryptonian(father(father(Kal-El)))) : Superpowers(father(father(Kal-El))))}{Superpowers(father(father(father(Kal-El)))))} ; \ldots
\end{align*}
\]
Examples (ctd.)

Let $T = \langle W, D \rangle$ now be the following default theory:

$$W = \{ \exists x \ Kryptonian(x) \};$$

$$D = \left\{ d = \frac{Kryptonian(x) \ : \ Superpowers(x)}{Superpowers(x)} \right\}.$$

$T$ makes here only *implicit* reference to an object, namely one which is determined by the existential quantifier.

We intuitively expect that this object possesses superpowers, i.e.,

$$\exists x \ (Kryptonian(x) \land Superpowers(x))$$

should be contained in an extension of $T$. 
To achieve this, we introduce a new constant, $a$, and replace the premiss $\exists x \text{Kryptonian}(x)$ by $\text{Kryptonian}(a)$.

The closure of $T$ is then given as follows:

\[
\begin{align*}
\text{Kryptonian}(a), \\
\text{Kryptonian}(a) : \text{Superpowers}(a) \\
\hline
\text{Superpowers}(a)
\end{align*}
\]

This default theory has one extension, namely

\[ E = \text{Th} (\{ \text{Kryptonian}(a), \text{Superpowers}(a) \} ) \]

and it holds that $\exists x (\text{Kryptonian}(x) \land \text{Superpowers}(x)) \in E$. 
Examples (ctd.)

Remark:

- In the last example, we have the problem that $E$ is defined over another language than $T$. In particular:
  - $E$ contains formulas which have no direct connection to $T$ (namely formulas containing $a$).

One identifies extensions of $T$ as those sets, which are extensions of the closure of $T$, but restricted to the original language of $T$.
Examples (ctd.)

As final example, we consider $T = \langle W, D \rangle$ given as follows:

$$W = \{ \text{Superhero}(\text{Batman}) \};$$

$$D = \left\{ d_1 = \frac{\exists x \text{ Archfiend}(x, \text{Batman})}{\exists x \text{ Archfiend}(x, \text{Batman})}, \\
    d_2 = \frac{\text{Archfiend}(x, y) : \text{Fights}(x, y)}{\text{Fights}(x, y)} \right\}.$$  

$T$ explicitly mentions the person Batman and refers implicitly to an archfiend of Batman.

We expect that $\exists x (\text{Archfiend}(x, \text{Batman}) \land \text{Fights}(x, \text{Batman}))$ should be contained in an extension of $T$.

To this end, we introduce a new constant, $b$, and replace $d_1$ by its skolemised form

$$: \exists x \text{ Archfiend}(x, \text{Batman}) \frac{\text{Archfiend}(b, \text{Batman})}{.}$$
Examples (ctd.)

As closure of $T$, we obtain:

\[
Superhero(Batman),
\]

\[
: \exists x \Archfiend(x, Batman) \\
\Archfiend(b, Batman)
\]

\[
\Archfiend(Batman, Batman) : Fights(Batman, Batman)
\]

\[
Fights(Batman, Batman)
\]

\[
\Archfiend(Batman, b) : Fights(Batman, b)
\]

\[
Fights(Batman, b)
\]

\[
\Archfiend(b, Batman) : Fights(b, Batman)
\]

\[
Fights(b, Batman)
\]

\[
\Archfiend(b, b) : Fights(b, b)
\]

\[
Fights(b, b)
\]

Note that the extension of the closure of $T$ contains \(\Archfiend(b, Batman) \land Fights(b, Batman)\), and thus \(\exists x (\Archfiend(x, Batman) \land Fights(x, Batman))\) as desired.
Closure of a Default Theory: Definition

A skolemised form of a default \( d = (A : B_1, \ldots, B_n/C) \) results from \( d \) as follows:

1. Let \( C_1 \) be a skolemised form of \( \forall C \) such that no Skolem function already occurs in \( A, B_1, \ldots, B_n \).
2. Eliminate all quantifiers in \( C_1 \). The result is \( C_2 \).
3. Replace \( C \) in \( d \) by \( C_2 \). This is the desired skolemised form of \( d \).

A skolemised form of a default theory \( T = \langle W, D \rangle \) results from \( T \) by replacing each premiss in \( W \) and each default in \( D \) by their skolemised forms, respectively.

- All introduced Skolem functions must be pairwise distinct and may not already occur in \( T \).
Closure of a Default Theory: Definition (ctd.)

Let $T = \langle W, D \rangle$ be a default theory.

- The set $\text{TERMS}(T)$ consists of all ground terms which can be constructed from the function symbols occurring in $T$.

- The closure, $\overline{T}$, of $T$ is defined as follows:
  
  * If $T$ is closed, then $\overline{T} = T$.
  * Otherwise: Let $T_1$ be a skolemised form of $T$.
    
    · Replace all open defaults in $T_1$ by their instances over $\text{TERMS}(T_1)$.
    · The result is $\overline{T}$.

From now on, we only deal with closed default theories.
3.3 Extensions
Properties of Extensions: Reminder

An extension $E$ of a default theory $T = \langle W, D \rangle$ should satisfy the following conditions:

- $W \subseteq E$.
- $Th(E) = E$.
- $E$ should contain a maximal number of formulas which can be derived by means of defaults.
- $E$ should contain no unfounded formulas.
Definition of Extensions

We define extensions of a closed default theory $T = \langle W, D \rangle$ by means of a two-step procedure:

**Step 1:** We define a closure operator, $\Gamma_T$, mapping sets of closed formulas to sets of closed formulas.

- **Intuitive meaning:**
  - For constructing $\Gamma_T(E)$, $E$ serves as a “context” for testing the consistency conditions of the defaults in $D$.
  - $\Gamma_T(E)$ itself collects all formulas derivable from $W$ by means of
    * classical logic
    * and those defaults in $D$ satisfying the consistency condition relative to $E$.

**Step 2:** $E$ is an extension of $T$ iff $\Gamma_T(E) = E$, i.e., if $E$ is a fixed point of $\Gamma_T$.

- **Intuitively,** extensions of a default theory are those contexts which *reproduce themselves* under the closure operator $\Gamma_T$. 
Extensions of an open default theory $T$ are defined in terms of extensions of the closure $\bar{T}$ of $T$.

Formally:

- Let $T = \langle W, D \rangle$ be an open default theory.
- $E$ is an extension of $T$ iff $E = F \cap \mathcal{L}_T$, where
  * $F$ is an extension of the closure $\bar{T}$ of $T$, and
  * $\mathcal{L}_T$ is the language generated by $T$, i.e.,
    - $\mathcal{L}_T$ is the set of all formulas which can be constructed from the function and predicate symbols occurring in $T$. 
The Operator $\Gamma_T$

We now formally define $\Gamma_T$, for each closed default theory $T$.

Let $S$ be a set of closed formulas, and let $D$ be a set of closed defaults.

- The \textit{classical reduct of} $D$ \textit{relative to} $S$ is the following set of classical inference rules:

$$D_S = \left\{ \frac{A}{C} \mid \frac{A : B_1, \ldots, B_n}{C} \in D, \neg B_1 \notin S, \ldots, \neg B_n \notin S \right\}.$$

- Let $\vdash^{D_S}$ be the derivability relation resulting from the classical derivability relation $\vdash$ by adding the inference rules in $D_S$.

* For every theory $W$, we define

$$Th^{D_S}(W) = \{ A \mid W \vdash^{D_S} A, \text{ A is closed} \}.$$

Then, for $T = \langle W, D \rangle$, $\Gamma_T(S)$ is defined as follows:

$$\Gamma_T(S) = Th^{D_S}(W).$$
The Operator $\Gamma_T$ (ctd.)

Remarks:

- The set $Th^{Ds}(\cdot)$ can be expressed in terms of $Th(\cdot)$:

$$Th^{Ds}(W) = Th(W \cup \bigcup_{i \geq 0} E_i),$$

where

$$E_0 = \{C \mid A/C \in D_S \text{ and } W \vdash A\};$$

$$E_i = \{C \mid A/C \in D_S \text{ and } W \cup E_{i-1} \vdash A\}.$$  

- Consequently:

  - Each extension $E$ of a closed default theory $T = \langle W, D \rangle$ is necessarily of the form

    $$E = Th(W \cup G),$$

    where $G \subseteq \{C \mid (A : B_1, \ldots, B_n/C) \in D\}$. 

  - This property allows to select suitable candidates of sets for being an extension of a default theory!
Example: Nixon Diamond

We consider the following propositional form of the Nixon diamond:

\[ W = \{q, r\}; \]
\[ D = \{(q : p/p), (r : \neg p/\neg p)\}. \]

Intuitively:

- Nixon is both a Quaker and a Republican.
- A Quaker is normally a pacifist.
- A Republican is normally not a pacifist.
Example: Nixon Diamond (ctd.)

- $D = \{(q : p/p), (r : \neg p/\neg p)\}$ has two defaults $\implies$ four candidates:
  
  $$E_1 = Th(\{q, r\}); \quad E_3 = Th(\{q, r, \neg p\});$$
  $$E_2 = Th(\{q, r, p\}); \quad E_4 = Th(\{q, r, p, \neg p\}).$$

- Construction of $\Gamma_T(E_i) (= Th^{DE_i}(W))$:
  
  $$\begin{align*}
  D_{E_1} &= \{q/p, r/\neg p\}; & \Gamma_T(E_1) &= Th(\{q, r, p, \neg p\}) &= E_4; \\
  D_{E_2} &= \{q/p\}; & \Gamma_T(E_2) &= Th(\{q, r, p\}) &= E_2; \\
  D_{E_3} &= \{r/\neg p\}; & \Gamma_T(E_3) &= Th(\{q, r, \neg p\}) &= E_3; \\
  D_{E_4} &= \emptyset; & \Gamma_T(E_4) &= Th(\{q, r\}) &= E_1.
  \end{align*}$$

$\implies E_2$ and $E_3$ are extensions of $T$ (and they are the only extensions of $T$).
Assume we extend $T$ to $T' = \langle W', D \rangle$, where

$$W' = W \cup \{ s, s \rightarrow \neg p \} = \{ q, r, s, s \rightarrow \neg p \}$$

($s$: supporter of the military industry).

We have again four candidates:

$$F_1 = Th(W'); \quad F_3 = Th(W' \cup \{ \neg p \});$$
$$F_2 = Th(W' \cup \{ p \}); \quad F_4 = Th(W' \cup \{ p, \neg p \}).$$

Note: it holds that $F_1 = F_3$ and $F_2 = F_4$.

Construction of $\Gamma_T(F_i) (= Th^{D_{F_i}}(W))$:

$$D_{F_1} = \{ r/\neg p \}; \quad \Gamma_T(F_1) = Th(W') = F_1;$$
$$D_{F_2} = \emptyset; \quad \Gamma_T(F_2) = Th(W') = F_1.$$

$\implies F_1$ is the only extension of $T'$. 

Example: Nixon Diamond (ctd.)

Remark:

- For $T = \langle W, D \rangle$ and $T' = \langle W', D \rangle$ it holds that $W \subset W'$, but the beliefs have *not* increased monotonically (as $p \in E_2$ but $p \notin F_1$, i.e., $E_2 \not\subseteq F_1$).
Example: Non-Existence of Extensions

Consider \( T = \langle W, D \rangle \), where:

\[
W = \emptyset; \\
D = \{(p/\neg p)\}.
\]

We have two candidates:

\[
E_1 = Th(\emptyset); \quad E_2 = Th(\{\neg p\}).
\]

Construction of \( \Gamma_T(E_i) \) \((= Th_{D_{E_i}}(W))\):

\[
D_{E_1} = \{\top/\neg p\}; \quad \Gamma_T(E_1) = Th(\{\neg p\}) = E_2; \\
D_{E_2} = \emptyset; \quad \Gamma_T(E_2) = Th(\emptyset) = E_1.
\]

\( \implies \) \( T \) has no extension (as \( E_1 \neq E_2 \))!
The Operator $\Gamma_T$ Revisited

**Theorem 3.1**

Let $T = \langle W, D \rangle$ be a closed default theory. Furthermore, let $I_T(S)$ be the family of all sets $K$ satisfying the following conditions:

1. $K = \text{Th}(K)$;
2. $W \subseteq K$;
3. If $(A : B_1, \ldots, B_n/C) \in D$, $A \in K$, $\neg B_1 \notin S, \ldots, \neg B_n \notin S$, then $C \in K$.

Then,

$$\Gamma_T(S) = \bigcap_{K \in I_T(S)} K.$$ 

Furthermore, $\Gamma_T(S)$ is the *smallest* set satisfying Conditions (1)-(3).
3.4 Properties of Extensions
Closure Property

Let $E$ be a set of closed formulas and $D$ a set of closed defaults.

- $E$ is closed relative to $D$ iff every $(A : B_1, \ldots, B_n/C) \in D$ satisfies:
  - if $A \in E$ and $\neg B_1 \notin E, \ldots, \neg B_n \notin E$, then $C \in E$.

**Theorem 3.2**

Each extension of a closed default theory $T = \langle W, D \rangle$ is closed relative to $D$.

**Corollary 3.1**

If a set $E$ is not closed under $D$, then $E$ cannot be an extension of $T = \langle W, D \rangle$. 
Semi-Recursive Characterisation of Extensions

Theorem 3.3
Let $E$ be a set of closed formulas and let $T = \langle W, D \rangle$ be a closed default theory.

Define a sequence $(E_i)_{i \geq 0}$ of sets of formulas as follows:

$$E_0 = W;$$
$$E_{i+1} = Th(E_i) \cup \{C \mid (A : B_1, \ldots, B_n/C) \in D, E_i \vdash A$$
and $\neg B_1, \ldots, \neg B_n \notin E\}.$$

Then, $E$ is an extension of $T$ iff

$$E = \bigcup_{i \geq 0} E_i.$$
Semi-Recursive Characterisation of Extensions (ctd.)

**Theorem 3.4**

Let $T = \langle W, D \rangle$ be a closed default theory without justification-free defaults. Furthermore, let $E$ be a set of closed formulas such that $E = \bigcup_{i \geq 0} E_i$, where $(E_i)_{i \geq 0}$ is defined as in Theorem 3.3.

Then: $E$ is consistent $\iff$ $W$ is consistent.
Proof of Theorem 3.4

Proof: \((\Rightarrow)\) Trivial, since \(W \subseteq E = \bigcup_{i \geq 0} E_i\).

\((\Leftarrow)\) Suppose \(E\) is inconsistent.

Consider \(H_i = \{C \mid (A : B_1, \ldots, B_n/C) \in D, \ E_i \vdash A, \neg B_1, \ldots, \neg B_n \notin E\}\).

\(\Rightarrow\) \(E_{i+1} = \text{Th}(E_i) \cup H_i\), for all \(i \geq 0\).

Since it is easy to see that \(E = \text{Th}(E)\), and \(E\) is assumed to be inconsistent, it holds that \(F \in E\), for every closed formula \(F\).

Since \(D\) contains no justification-free defaults, it follows that \(H_i = \emptyset\), for all \(i \geq 0\).

\(\Rightarrow\) \(\bigcup_{i \geq 0} E_i = \text{Th}(E_0) = \text{Th}(W)\).

\(\Rightarrow\) \(E = \text{Th}(W)\), since \(E = \bigcup_{i \geq 0} E_i\).

\(\Rightarrow\) \(W\) is inconsistent since \(E\) is inconsistent. \(\square\)
Semi-Recursive Characterisation of Extensions (ctd.)

Corollary 3.2

Let $T = \langle W, D \rangle$ be a closed default theory without justification-free defaults.

$T$ has an inconsistent extension $\iff$ $W$ is inconsistent.

Proof: ($\Leftarrow$) Let $W$ be inconsistent.

Then, it is easy to verify that $Th(W)$ is an extension of $T$.

Moreover, $Th(W)$ is clearly inconsistent.

($\Rightarrow$) Let $E$ be an inconsistent extension of $T$.

In view of Theorem 3.3, we have that $E = \bigcup_{i \geq 0} E_i$ (where $E_i$ is defined as in Theorem 3.3).

From Theorem 3.4 it follows that $W$ is inconsistent.  \qed
Semi-Recursive Characterisation of Extensions (ctd.)

Corollary 3.3

Let $T = \langle W, D \rangle$ be a closed default theory without justification-free defaults. If $T$ has an inconsistent extension, then it is the only extension of $T$.

Proof: Let $E$ be an inconsistent extension of $T$. According to Corollary 3.2, $W$ is inconsistent.

Let $F$ be an arbitrary extension of $T$. Since $W \subseteq F$, we obtain that $F$ is also inconsistent. Moreover, it holds that $\text{Th}(E) = E$ and $\text{Th}(F) = F$.

$\implies$ It follows that $E = F$, since both $\text{Th}(E)$ and $\text{Th}(F)$ coincide with the set of all formulas. $\Box$
Semi-Recursive Characterisation of Extensions (ctd.)

Theorem 3.5 ("Maximality of Extensions")
Let $T = \langle W, D \rangle$ be a closed default theory and let $E, F$ be extensions of $T$.
If $E \subseteq F$, then $E = F$.

Proof: Consider sequences $(E_i)_{i \geq 0}$ and $(F_i)_{i \geq 0}$ s.t. $E = \bigcup_{i \geq 0} E_i$ and $F = \bigcup_{i \geq 0} F_i$ as in Theorem 3.3, and assume that $E \subseteq F$.
By induction, it is easy to show that $F_i \subseteq E_i$, for all $i \geq 0$.

$$\Rightarrow \bigcup_{i \geq 0} F_i \subseteq \bigcup_{i \geq 0} E_i,$$ i.e. $F \subseteq E$.
Since $E \subseteq F$ by hypothesis, $E = F$ follows. $\square$
Generating Defaults

Let $E$ be a set of closed formulas.

- A default $(A : B_1, \ldots, B_n/C)$ is active in $E$ iff $E \vdash A$ and $\neg B_1 \notin E, \ldots, \neg B_n \notin E$.
- The set of generating defaults of $E$ with respect to $T = \langle W, D \rangle$ is defined as 
  \[ GD(E, T) = \{ d \in D \mid d \text{ is active in } E \}. \]

For a set $D$ of defaults, define 

\[ CONS(D) = \{ C \mid (A : B_1, \ldots, B_n/C) \in D \}. \]

Prior results imply:

- Let $T = \langle W, D \rangle$ be a closed default theory and let $E$ be an extension of $T$. Then: 
  \[ E = Th(W \cup CONS(GD(E, T))). \]
Properties of Active/Non-Active Defaults

Theorem 3.6

Let \( d = (A : B_1, \ldots, B_n/C) \) be a default and \( E \) a set of closed formulas.

1. If \( d \) is not active in \( E \), then:
   - \( E \) is an extension of \( \langle W, D \rangle \iff E \) is an extension of \( \langle W, D \cup \{d\} \rangle \).

2. If \( d \) is active in \( E \), then:
   - \( E \) is an extension of \( \langle W, D \cup \{d\} \rangle \implies E \) is an extension of \( \langle W \cup \{C\}, D \rangle \).

3. If \( W \vdash A \) and \( \neg B_1 \notin E, \ldots, \neg B_n \notin E \), then:
   - \( E \) is an extension of \( \langle W \cup \{C\}, D \rangle \implies E \) is an extension of \( \langle W, D \cup \{d\} \rangle \).
3.5 Normal Default Theories
Definition

- A default $d$ is normal iff it is of the form
  \[ A : B \]
  \[ \frac{B}{B}. \]

- A default theory $T = \langle W, D \rangle$ is normal iff all defaults in $D$ are normal.
Properties of Normal Default Theories

Theorem 3.7
Each closed default theory which is normal has an extension.

Proof: Let $T = \langle W, D \rangle$ be a closed normal default theory.

If $W$ is inconsistent, then $T$ has an extension, namely the inconsistent set $Th(W)$ (cf. Corollary 3.2).

Assume now that $W$ is consistent. We define a sequence $(E_i)_{i \geq 0}$ as follows:

\begin{align*}
E_0 &= W; \\
E_{i+1} &= Th(E_i) \cup T_i,
\end{align*}

where $T_i$ is a maximal set of closed formulas satisfying the following conditions:

1. $E_i \cup T_i$ is consistent;

2. if $B \in T_i$, then there is a default $(A : B/B) \in D$ s.t. $E_i \vdash A$. 
Proof of Theorem 3.7

Let $E = \bigcup_{i \geq 0} E_i$.

Claim: $E$ is an extension of $T$.

Proof of claim: Let $H_i = \{ B \mid (A : B/B) \in D, E_i \vdash A, \neg B \notin E \}$.

We show that $T_i = H_i$, for all $i$. From this, the claim follows.

1. We first show that $T_i \subseteq H_i$.
   - Let $B \in T_i$. Then, by definition of $T_i$, there is some default $(A : B/B) \in D$ s.t. $E_i \vdash A$.
   - Suppose $\neg B \in E$. Since $E = \bigcup_{i \geq 0} E_i$, there is some $k \geq 0$ s.t. $\neg B \in E_k$.
   - Since $B \in T_i$ and $T_i \subseteq E_{i+1}$, we have that $B \in E_{i+1}$.
   - Let $m = \max(i + 1, k)$. Since $E_l \subseteq E_{l+1}$, for all $l \geq 0$, it follows that $B \in E_m$ and $\neg B \in E_m$. $\Rightarrow$ $E_m$ is inconsistent. Contradiction.
   - Therefore, $\neg B \notin E$ must hold. We obtain $B \in H_i$.
   $\Rightarrow$ This proves that $T_i \subseteq H_i$. 
Proof of Theorem 3.7 (ctd.)

2. Suppose that $T_i \subset H_i$ holds. Then, there is some $B \in H_i$ s.t. $B \notin T_i$.

- By the maximality of $T_i$, we have that $E_i \cup T_i \cup \{B\}$ is inconsistent. Since $E_i \cup T_i \subseteq E_{i+1} \subseteq E$ holds, $E \cup \{B\}$ is also inconsistent.

- It follows that $E \vdash \neg B$. Furthermore, it is easy to see that $E = Th(E)$. Hence, $\neg B \in E$.

- But $B \in H_i$, and so $\neg B \notin E$ must hold, by definition of $H_i$. Contradiction. $\square$
Properties of Normal Default Theories (ctd.)

Theorem 3.8 ("Orthogonality of Extensions")
If a closed normal default theory \( T = \langle W, D \rangle \) has two distinct extensions \( E, F \), then \( E \cup F \) is inconsistent.

Intuitive meaning:

- An agent cannot consistently accept both extensions simultaneously.

Theorem 3.9 ("Semi-Monotonicity")
Let \( D, D' \) be sets of closed normal defaults s.t. \( D \subseteq D' \), and let \( E \) be an extension of \( T = \langle W, D \rangle \).

Then, \( T' = \langle W, D' \rangle \) has an extension \( E' \) s.t. \( E \subseteq E' \).
Semi-Normal Defaults

- A default is *semi-normal* iff it is of the form
  \[ A : B \land C \]
  \[
  \frac{C}{C}.
  \]

- A default theory \( T = \langle W, D \rangle \) is semi-normal iff all defaults in \( D \) are semi-normal.
Semi-Normal Defaults (ctd.)

Semi-normal defaults possess in general not the properties of normal default theories.

- In particular:
  - for semi-normal default theories, it is not guaranteed that they always have extensions.

- Example:
  - Consider \( T = \langle W, D \rangle \), where
    \[
    W = \emptyset, \quad D = \left\{ \frac{p \land \neg q}{\neg q}, \frac{q \land \neg r}{\neg r}, \frac{r \land \neg p}{\neg p} \right\}.
    \]
  
  \( T \) has no extension!
3.6 Proof Theory for Default Logic
Basic Reasoning Tasks

Due to the multiplicity of extensions, two canonical inference relations can be defined in the context of default logic:

- **Brave reasoning:**

  * a formula $A$ is a *brave consequence* of a default theory $T$ iff there is an extension of $T$ containing $A$.

- **Skeptical reasoning:**

  * a formula $A$ is a *skeptical consequence* of a default theory $T$ iff $A$ is contained in all extensions of $T$.

In what follows, we describe a *sequent calculus* for brave default reasoning (in the propositional case) due to Bonatti (1993).
A Sequent Calculus for Default Logic

- The sequent calculus consists of three parts:
  - a sequent calculus, $\text{LK}$, for classical propositional logic;
  - a complementary sequent calculus, $\text{LK}^C$, formalising invalid propositions; and
  - special default inference rules.

- Remarks:
  - The complementary calculus is needed for formalising the consistency condition of default rules.
  - $\text{LK}^C$ is due to Bonatti (1993), although a similar system was independently developed by Goranko (1994).
The Sequent Calculus LK

- Sequent-type calculi are axiom systems optimised for proof search.
  → Important for automated deduction!

- The sequent method was introduced by Gerhard Gentzen in his Ph.D. thesis in 1934.
  - Actually, the system LK was introduced by him (but our version of LK is slightly different from Gentzen’s original one).

- Formal objects of LK are sequents, which are ordered pairs of form \( \Gamma \Rightarrow \Sigma \), where \( \Gamma, \Sigma \) are finite sets of propositional formulas.
  - Notation:
    * “\( \Gamma, A \Rightarrow \Sigma \)” for “\( \Gamma \cup \{A\} \Rightarrow \Sigma \)”;
    * “\( \Gamma, \Delta \Rightarrow \Sigma \)” for “\( \Gamma \cup \Delta \Rightarrow \Sigma \)”;
    * “\( \Rightarrow \Sigma \)” for “\( \emptyset \Rightarrow \Sigma \)”;
    * etc.
The Sequent Calculus LK (ctd.)

Semantical meaning:

- a sequent \( S = A_1, \ldots, A_n \Rightarrow B_1, \ldots, B_m \) has the same semantical meaning as the formula

\[
\iota(S) = (\bigwedge_{i=1}^{n} A_i) \rightarrow (\bigvee_{i=1}^{m} B_i)
\]

- empty conjunctions are identified with \( \top \), and empty disjunctions with \( \bot \).

- In particular, the following is defined:
  - \( S \) is valid if and only if \( \iota(S) \) is valid.
The Sequent Calculus LK (ctd.)

Axioms of LK:
- \( \Rightarrow \top \);
- \( A \Rightarrow A \), for each formula \( A \).

Inference rules of LK:

\[
\frac{\Gamma \Rightarrow \Sigma, A}{\Gamma, A \Rightarrow \Sigma} \quad \text{wl}
\]
\[
\frac{\Gamma \Rightarrow \Sigma, \neg A}{\Gamma, \neg A \Rightarrow \Sigma} \quad \text{wl}
\]
\[
\frac{\Gamma \Rightarrow \Sigma}{\Gamma, A \Rightarrow \Sigma} \quad \text{wr}
\]
\[
\frac{\Gamma, \Delta, (A \rightarrow B) \Rightarrow \Sigma, \Pi}{\Gamma \Rightarrow \Sigma, \Pi} \quad \text{l}
\]
\[
\frac{\Gamma \Rightarrow \Sigma, (A \rightarrow B)}{\Gamma, A \Rightarrow \Sigma} \quad \text{r}
\]
\[
\frac{\Gamma \Rightarrow \Sigma, \neg A}{\Gamma \Rightarrow \Sigma, -A} \quad \text{r}
\]
\[
\frac{\Sigma \Rightarrow \Gamma, A}{\Sigma \Rightarrow \Gamma} \quad \text{wr}
\]
From the inference rules for the primitive connectives $\rightarrow$ and $\neg$, we obtain derived rules for the defined connectives $\land$, $\lor$, and $\equiv$, like:

$$
\frac{\Gamma, A \Rightarrow \Sigma}{\Gamma, (A \land B) \Rightarrow \Sigma} \quad \land l_1
$$

$$
\frac{\Gamma, B \Rightarrow \Sigma}{\Gamma, (A \land B) \Rightarrow \Sigma} \quad \land l_2
$$

$$
\frac{\Gamma \Rightarrow \Sigma, A \quad \Lambda \Rightarrow \Pi, B}{\Gamma, \Lambda \Rightarrow \Sigma, \Pi, (A \land B)} \quad \land r
$$

$$
\frac{\Gamma, A \Rightarrow \Sigma \quad \Pi, B \Rightarrow \Sigma}{\Gamma, \Pi, (A \lor B) \Rightarrow \Sigma} \quad \lor l
$$

$$
\frac{\Gamma \Rightarrow \Sigma, A}{\Gamma \Rightarrow \Sigma, (A \lor B)} \quad \lor r_1
$$

$$
\frac{\Gamma \Rightarrow \Sigma, B}{\Gamma \Rightarrow \Sigma, (A \lor B)} \quad \lor r_2
$$
The Sequent Calculus LK (ctd.)

- Proofs in LK are written in the form of trees.
  - The root of such a tree is the sequent to be proven;
  - top-most sequents are axioms;
  - transitions between adjacent nodes are sanctioned by means of the inference rules.

Adequacy of LK:
- $\Gamma \Rightarrow \Sigma$ is provable in LK $\iff \Gamma \Rightarrow \Sigma$ is valid.
The Complementary Sequent Calculus \( \LKc \)

- \( \LKc \) formalises sequents which are *not* provable in \( \LK \).

- Formal objects of \( \LKc \) are *anti-sequents*:
  
  - An anti-sequent is an ordered pair of form \( \Gamma \not\Rightarrow \Sigma \), where \( \Gamma, \Sigma \) are finite sets of propositional formulas.
  
  - An anti-sequent \( \Gamma \not\Rightarrow \Sigma \) is *true* iff the sequent \( \Gamma \Rightarrow \Sigma \) is not valid.
The Complementary Sequent Calculus LK$^C$ (ctd.)

 màn Axioms of LK$^C$:

- anti-sequents of form $\Phi \not\vdash \Psi$, where $\Phi, \Psi$ are finite sets of atomic formulas s.t.
  * $\Phi \cap \Psi = \emptyset$, and
  * $\{\top\} \cap \Psi = \emptyset$.

 màn Inference rules of LK$^C$:

\[
\begin{align*}
\frac{\Gamma \not\vdash \Theta, A}{\Gamma, (A \rightarrow B) \not\vdash \Theta} & \rightarrow l_1^c \\
\frac{\Gamma, B \not\vdash \Theta}{\Gamma, (A \rightarrow B) \not\vdash \Theta} & \rightarrow l_2^c \\
\frac{\Gamma, A \not\vdash \Theta, B}{\Gamma \not\vdash \Theta, (A \rightarrow B)} & \rightarrow r^c \\
\frac{\Gamma \not\vdash \Theta, A}{\Gamma, \neg A \not\vdash \Theta} & \neg l^c \\
\frac{\Gamma, A \not\vdash \Theta}{\Gamma, \not\vdash \Theta, \neg A} & \neg r^c
\end{align*}
\]
The Complementary Sequent Calculus LK^c (ctd.)

For $\land$ and $\lor$ we obtain the following derived rules:

$$
\frac{\Gamma, A, B \not\in \Theta}{\Gamma, (A \land B) \not\in \Theta} \quad \land_l^c
$$

$$
\frac{\Gamma \not\in \Theta, A}{\Gamma \not\in \Theta, (A \land B)} \quad \land_r^c_1
$$

$$
\frac{\Gamma \not\in \Theta, B}{\Gamma \not\in \Theta, (A \land B)} \quad \land_r^c_2
$$

$$
\frac{\Gamma, A \not\in \Theta}{\Gamma, (A \lor B) \not\in \Theta} \quad \lor_l^c_1
$$

$$
\frac{\Gamma, B \not\in \Theta}{\Gamma, (A \lor B) \not\in \Theta} \quad \lor_l^c_2
$$

$$
\frac{\Gamma \not\in \Theta, A, B}{\Gamma \not\in \Theta, (A \lor B)} \quad \lor_r^c
$$
The Complementary Sequent Calculus $\text{LK}^C$ (ctd.)

- Central property:
  - $\Gamma \not\models \Theta$ is provable in $\text{LK}^C$ $\iff$ $\Gamma \not\models \Theta$ is true.

- Immediate consequence:
  - $\Gamma \not\models \Theta$ is provable in $\text{LK}^C$ $\iff$ $\Gamma \Rightarrow \Theta$ is not provable in $\text{LK}$. 
Default Sequents

- A default sequent is an ordered quadruple of form $\Gamma; \Delta \leadsto \Sigma; \Theta$, where
  - $\Gamma, \Sigma, \Theta$ are finite sets of propositional formulas, and
  - $\Delta$ is a finite set of propositional defaults (i.e., defaults containing only propositional formulas).

- A default sequent $\Gamma; \Delta \leadsto \Sigma; \Theta$ is true iff there is an extension $E$ of the default theory $\langle \Gamma, \Delta \rangle$ s.t.
  - $\Sigma \subseteq E$ and $\Theta \cap E = \emptyset$ ($E$ is called witness of $\Gamma; \Delta \leadsto \Sigma; \Theta$).
Postulates of the Default Sequent Calculus

The calculus for default logic consists of \( \text{LK}, \text{LK}^c \), and the following axioms and inference rules:

- **Axioms:**
  - \( \Gamma; \emptyset \vdash \emptyset; \emptyset \), for each finite set \( \Gamma \) of propositional formulas.

- **Inference rules:**
  
  \[
  \frac{\Gamma \Rightarrow A}{\Gamma; \emptyset \vdash A; \emptyset} \quad l_1 \quad \frac{\Gamma \not\Rightarrow A}{\Gamma; \emptyset \vdash \emptyset; A} \quad l_2
  \]

  \[
  \frac{\Gamma; \emptyset \vdash \Sigma_1; \Theta_1 \quad \Gamma; \emptyset \vdash \Sigma_2; \Theta_2}{\Gamma; \emptyset \vdash \Sigma_1, \Sigma_2; \Theta_1, \Theta_2} \quad cu
  \]

  \[
  \frac{\Gamma; \Delta \vdash \Sigma; \Theta, A}{\Gamma; \Delta, (A : B_1, \ldots, B_n/C) \vdash \Sigma; \Theta} \quad d_1 \\
  \frac{\Gamma; \Delta \vdash \Sigma, \neg B; \Theta}{\Gamma; \Delta, (A : \ldots, B, \ldots/C) \vdash \Sigma; \Theta} \quad d_2
  \]

  \[
  \frac{\Gamma, C; \Delta \vdash \Sigma; \Theta, \neg B_1, \ldots, \neg B_n}{\Gamma; \Delta, (A : B_1, \ldots, B_n/C) \vdash \Sigma; \Theta} \quad d_3
  \]

  \[
  \frac{\Gamma; \emptyset \vdash A; \emptyset}{\Gamma \vdash A} \quad d_2
  \]
Adequacy of the Calculus

\[ \Gamma; \Delta \sim \Sigma; \Theta \text{ is provable } \iff \Gamma; \Delta \sim \Sigma; \Theta \text{ is true.} \]
Example

Consider the default theory $T$, formalising the Nixon diamond:

$W = \{q, r\};$

$D = \{(q : p/p), (r : \neg p/\neg p)\}.$

We know that $T$ has two extensions:

- one containing $p$ but not containing $\neg p$; and
- one containing $\neg p$ but not containing $p$.

Therefore, the following two default sequents are true:

$q, r; (q : p/p), (r : \neg p/\neg p) \vdash p; \neg p,$

$q, r; (q : p/p), (r : \neg p/\neg p) \vdash \neg p; p.$

Both sequents are derivable in the calculus.

We give a proof of the first sequent.
Example (ctd.)

Let $\alpha$ be the following LK$^c$-proof of $q, r, p; \emptyset \vdash \emptyset; \neg p$:

\[
\begin{align*}
q, r, p, p & \not\Rightarrow \neg r, \alpha \\
q, r, p & \not\Rightarrow \neg p \quad \text{l}_2 \\
q, r, p; \emptyset & \vdash \emptyset; \neg p
\end{align*}
\]

Then, the proof of $q, r; (q : p/p), (r : \neg p/\neg p) \vdash p, \neg p$, is as follows:

\[
\begin{align*}
p \Rightarrow p & \quad p \Rightarrow p \quad \text{wl, wl} \\
q, r, p & \Rightarrow p \quad \text{wl, wl} \\
q, r, p; \emptyset & \vdash p, \emptyset \quad \text{l}_1 \quad \alpha \\
q, r, p; \emptyset & \vdash p, \neg p \quad \text{cu} \\
q, r, p; \emptyset & \vdash \neg p, \neg p \quad \text{cu} \\
q, r, p; \emptyset & \vdash p, \neg p, \neg p \\
q, r; (q : p/p) & \vdash p, \neg p, \neg p \quad \text{d}_2 \\
q, r; (q : p/p), (r : \neg p/\neg p) & \vdash p, \neg p
\end{align*}
\]
Remarks

➤ An analogous calculus for *skeptical default reasoning* was also developed (Bonatti & Olivetti, ACM Transactions on Computational Logic 3(2): 226-278, 2002).

➤ A characterisation of extensions using a *tableau calculus* (for different systems of default reasoning) was introduced by Amati, Aiello, Gabbay & Pirri (1996).
3.7 Default Logic and Answer-Set Programming
Default Logic and Search Problems

> Default Logic can be used for solving search problems.

> Basic idea:
  - represent a given search problem $P$ in terms of a default theory $T$ s.t.
    the solutions of $P$ are given by the extensions of $T$. 
Example: The \( k \)-Colourability Problem

➢ Given:
  
  – a number \( k \geq 1 \);
  
  – a graph \( G = (V, E) \)
    
    (\( V \) are the nodes and \( E \) are the edges of the graph);
  
  – \( k \) colours \( C = \{ c_1, \ldots, c_k \} \).

➢ \( G \) is \textit{\( k \)-colourable} \iff there is an assignment of colours \( c \in C \) to each node of \( G \) s.t. adjacent nodes do not have the same colour.

➢ The \textit{\( k \)-colourability problem} is the task to decide whether a given graph is \( k \)-colourable.


   This problem is well-known to be NP-complete.
Example: The $k$-Colourability Problem (ctd.)

The $k$-colourability problem can be represented in terms of the following default theory $T = \langle W, D \rangle$:

- $W = \emptyset$.

- $D$ consists of the following defaults:
  
  * for each node $v \in V$ and each colour $c_j \in C$ ($1 \leq j \leq k$), the following default is contained in $D$:
    
    \[
    \frac{\neg \text{clr}(v, c_1), \ldots, \neg \text{clr}(v, c_{j-1}), \neg \text{clr}(v, c_{j+1}), \ldots, \neg \text{clr}(v, c_k)}{\text{clr}(v, c_j)};
    \]

  * for each edge $(v, w) \in E$ and each colour $c \in C$, the following default is contained in $D$:
    
    \[
    \frac{\text{clr}(v, c) \land \text{clr}(w, c)}{\neg \text{bad}}.
    \]

The extensions of $T$ describe the admissible colourings of $G$. 

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The Answer-Set Programming Paradigm

The idea of representing search problems in terms of default theories is a special instance of the general notion of answer-set programming (ASP).

By ASP one understands a declarative method to represent and solve search problems based on the following method:

- Problems are represented in terms of theories of formal systems with semantics s.t. it holds:
  * Solutions of a given search problem are determined by the models (“answer sets”) of the corresponding theory.
  * Requirement:
    - Both the encoding as well as the reconstruction of the solutions is feasible in polynomial time.
The Answer-Set Programming Paradigm

- The term “answer-set programming” was coined by Vladimir Lifschitz (Univ. of Texas at Austin) in the early nineties
  - enjoys growing popularity and importance since then.

- Most important instance of the ASP paradigm:
  - logic programs with nonmonotonic negation (default negation, “negation-as-failure”) under the stable-model semantics.

  The stable-model semantics is a computational adaption of default logic!
The Stable-Model Semantics


- Formally:
  
  - A *normal logic program* (NLP) is a finite set of rules of form

    \[ a \leftarrow b_1, \ldots, b_n, \text{not } c_1, \ldots, \text{not } c_m, \]

    * \( a, b_1, \ldots, b_n, c_1, \ldots, c_m \) are atoms from a finite vocabulary;
    * \( \text{not} \) denotes default negation (negation-as-failure).
The Stable-Model Semantics (ctd.)

Intuitive meaning:

\[ a \leftarrow b_1, \ldots, b_n, \text{not } c_1, \ldots, \text{not } c_m \]

“fires” if \( b_1, \ldots, b_n \) are derivable but \( c_1, \ldots, c_m \) are not derivable.

The stable-model semantics for NLPs is the result of associating logic programs with default theories in a canonical way:

- For rule \( r \) as above, let \( \delta(r) \) be the following default:

\[
\delta(r) = \frac{b_1 \land \cdots \land b_n : \neg c_1, \ldots, \neg c_m}{a}.
\]

The \textit{stable models} of an NLP \( P \) are in a one-to-one correspondence to the extensions of the default theory \( \langle \emptyset, \{ \delta(r) \mid r \in P \} \rangle \).
The Stable-Model Semantics (ctd.)

➤ Besides NLPs, other, more general, classes of logic programs have been defined, e.g.:
   - disjunctive logic programs
     * disjunctions of atoms in heads of rules are admitted;
   - nested logic programs
     * Boolean combinations of atoms in heads and bodies of rules are admitted.

➤ For all these classes, the stable-model semantics can be extended to the answer-set semantics
   - here, besides default negation, a second kind of negation, strong negation, may also be used.

➤ Most recently, the stable-model semantics has also been generalised to arbitrary theories of first-order logic (Ferraris, Lee, Lifschitz, 2007).
The Stable-Model Semantics (ctd.)

Several sophisticated solvers of the answer-set semantics are available, e.g.:

- DLV (TU Wien, Univ. of Calabria);
- Smodels, GnT (Aalto University, Finland);
- clasp (Univ. of Potsdam).

Further courses about ASP:

- LU Einführung in Wissensbasierte Systeme (WS);
- VL Logikorientierte Programmierung (SS).
§4 Autoepistemic Logic

4.1 Foundations
Language of Autoepistemic Logic

Autoepistemic logic (AEL) describes the beliefs of an ideally rational agent reasoning about his own knowledge.

AEL uses the language of modal logics to represent nonmonotonic inferences directly in the object language.

Formally:

- The alphabet of AEL consists of the alphabet of classical propositional logic, together with the unary operator $L$.
- Formulas of AEL are defined analogously as in propositional logic, except that $L$ may also be used to construct formulas, i.e.,
  - $L\alpha$ is a formula whenever $\alpha$ is a formula.
- Definition: $M\alpha = \neg L\neg \alpha$.
- Intuitive meaning:
  - $L\alpha$: $\alpha$ is believed.
  - $M\alpha$: $\alpha$ can be consistently assumed.
In AEL, knowledge is represented in terms of *AE-theories*. An AE-theory is an arbitrary set of formulas of AEL. An AE-theory represents the initial knowledge of an agent (i.e., it comprises the *premises* of the agent).

Problem:

- Which totality of knowledge is induced by a given AE-theory?

Answer:

- Notion of an *AE-extension*. 
An AE-extension $E$ of an AE-theory $T$ should satisfy the following intuitive conditions:

- $E$ should be closed under classical propositional logic.
- $E$ should contain the premisses, i.e., it should hold that $T \subseteq E$.
- $E$ should satisfy the introspection properties of the agent:
  - *Positive introspection*: if $A \in E$, then $LA \in E$.
  - *Negative introspection*: if $A \notin E$, then $\neg LA \in E$.
- The truth of the initial knowledge $T$ should imply the truth of the total knowledge $E$.

Formal definition is given later on!
Let $A$ be a formula of AEL.

- The modal depth, $m$-depth($A$), of $A$ is the maximal number of nested modal operators in $A$.

- Examples:
  * $m$-depth($L(p \land Mq)$) = 2;
  * $m$-depth($Lp \land Mq$) = 1.

- $A$ is objective if $m$-depth($A$) = 0.
Let $S$ be a set of formulas of AEL.

- We define:

$$
\bar{S} = \{A \mid A \notin S\};
\neg S = \{\neg A \mid A \in S\};
LS = \{LA \mid A \in S\};
S_i = \{A \mid A \in S \text{ and } m\text{-depth}(A) \leq i\}, \text{ for } i \geq 0.
$$

Therefore:

* $\neg L\bar{S} = \{\neg LA \mid A \notin S\}$.
* $S_0$ is the set of all objective formulas in $S$. 

Notation and Concepts (ctd.)
The semantics and proof theory of classical propositional logic can be extended to formulas of AEL in a direct manner.

To this end, simply view all formulas of form $L^A$ as atomic formulas such formulas are referred to as modal atoms.
Notation and Concepts (ctd.)

More specifically:

- **By a **PL-interpretation** we understand a function assigning each propositional atom and each modal atom an element of \( \{0, 1\} \).**

- **The truth value, \( V_{mp}(A) \), of a formula \( A \) under a PL-interpretation \( mp \) is then defined analogously as in classical logic, i.e.,**

  - \( V_{mp}(\top) = 1 \),
  - \( V_{mp}(p) = mp(p) \), for each propositional atom \( p \),
  - \( V_{mp}(LA) = mp(LA) \), for each modal atom \( LA \),
  - \( V_{mp}(\neg A) = 1 - V_{mp}(A) \),
  - \( V_{mp}(A \rightarrow B) = 1 \), if \( V_{mp}(A) \leq V_{mp}(B) \), otherwise \( V_{mp}(A \rightarrow B) = 0 \).
Let $mp$ be a PL-interpretation.

- $mp$ is a PL-model of a formula $A$ $\iff V^{mp}(A) = 1$;
- $mp$ is a PL-model of an AE-theory $T$ $\iff V^{mp}(A) = 1$, for all $A \in T$.

For a theory $T$ and a formula $A$ we define:

- $T \models A$ $\iff$ each PL-model of $T$ is a PL-model of $A$.

Analogously, “$\vdash$” and “$Th(\cdot)$” denote the classical derivability relation and the deductive-closure operator, respectively, defined over the language of AEL.

- $T \vdash A$ $\iff$ $T \models A$. 
4.2 AE-Extensions
AE-Interpretations

➢ PL-interpretations treat formulas of form $A$ and $LA$ as \textit{completely independent} of each other
  
  — e.g., it may hold that $V^{mp}(A) = 0$ but $V^{mp}(LA) = 1$.

➢ We expect however a connection between $A$ and $LA$, \textit{depending on the current beliefs of an agent}.

➢ We thus need a generalised notion of an interpretation, taking this modal aspect into account!

➢ To achieve this, we define the notion of an \textit{AE-interpretation}.
AE-Interpretations (ctd.)

➤ An **AE-interpretation** is an ordered pair \( I = \langle m, S \rangle \), where
  - \( m \) is an interpretation of classical propositional logic, and
  - \( S \) is a set of beliefs, i.e., a set of formulas of AEL.

We refer to \( I = \langle m, S \rangle \) also as an AE-interpretation **of** \( S \).

➤ Intuitive meaning:
  - \( \langle m, S \rangle \) is the description of a possible world, where \( m \) specifies what is **actually true in the world**, while \( S \) specifies what the agent **believes about the world**.
The truth value, \( V^I(A) \), of a formula \( A \) under an AE-interpretation \( I = \langle m, S \rangle \) is defined as follows:

- \( V^I(\top) = 1 \);
- \( V^I(p) = m(p) \), for each propositional atom \( p \);
- \( V^I(\neg A) = 1 - V^I(A) \);
- \( V^I(A \rightarrow B) = 1 \), if \( V^I(A) \leq V^I(B) \), otherwise \( V^I(A \rightarrow B) = 0 \);
- \( V^I(LA) = 1 \), if \( A \in S \), otherwise \( V^I(LA) = 0 \).
AE-Interpretations (ctd.)

Let $I$ be an AE-interpretation.

- $I$ is an AE-model of a formula $A \iff V^I(A) = 1$.
- $I$ is an AE-model of an AE-theory $T \iff V^I(A) = 1$, for each $A \in T$.

Let $A$ be a formula, $T$ an AE-theory, and $S$ a set of formulas.

- $A$ is an AE-consequence of $T$ relative to $S$, symbolically $T \models_S A$, iff
  * each AE-interpretation of $S$ which is an AE-model of $T$ is also an AE-model of $A$. 
Consider $I = \langle m, S \rangle$ with

- $m(p) = m(q) = 1$, and
- $S = \{p, Lp, Lq\}$.

We have:

- $V^I(p) = V^I(q) = 1$,
- $V^I(Lp) = 1$, $V^I(Lq) = 0$.

$\implies I$ is not an AE-model of $S$. 

Furthermore, the following properties hold:

- $\{p\} \models_S Lp$.
  (Since for every AE-interpretation $I' = \langle m', S \rangle$ of $S$ we have $V^{I'}(Lp) = 1$, given that $p \in S$.)

- $\{p\} \nvdash Lp$.
  (Since we can find a PL-Interpretation $mp$ s.t. $mp(p) = 1$ but $mp(Lp) = 0$.)

- $\{q\} \nvdash S Lq$.
  (Since there is an AE-interpretation of $S$, namely $I = \langle m, S \rangle$, such that $V^I(Lq) = 0$ but $V^I(q) = 1$. Note that $V^I(Lq) = 0$ since $q \notin S$, and $V^I(q) = 1$ since $m(q) = 1$ by definition).
AE-Extensions

Recall:
- given an AE-theory $T$, representing the initial beliefs of an agent, an AE-extension of $T$ should express the total beliefs of an agent.

Since the operator $L$ refers to the own beliefs of an agent, the definition of an AE-extension is realised as in default logic by means of a fixed-point condition.
AE-Extensions: Definition

Let $T$ be an AE-theory and $E$ a set of formulas.

- $E$ is an AE-extension of $T$ $\iff$ $E = \{A \mid T \models_E A\}$.

Intuitive meaning:

- $\{A \mid T \models_E A\}$ expresses the deductive closure of $T$ under AE-consequence, having the set $E$ as “context” beliefs.

- $E$ is an acceptable state of beliefs if it reproduces itself under this closure operation.
**AE-Extensions: Properties**

**Important concepts:**

Let $T$ be an AE-theory and $S$ a set of formulas.

- $S$ is **sound with respect to $T$** $\iff S \subseteq \{ A \mid T \models_S A \}$.
- $S$ is **semantically complete** $\iff \{ A \mid S \models_S A \} \subseteq S$.

For example, $S = \{ p, \text{L}p, \text{L}q \}$ is neither sound with respect to $\{ q \}$ (since $\text{L}q \in S$ but $\{ q \} \not\models_S \text{L}q$), nor semantically complete (since $S \models_S \text{L}\text{L}p$ but $\text{L}\text{L}p \notin S$).

**Theorem 4.1**

$E$ is an AE-extension of $T$ $\iff$ (i) $T \subseteq E$, (ii) $E$ is sound with respect to $T$, and (iii) $E$ is semantically complete.
Remarks

The characterisations of AE-extensions given so far yield no method to actually find AE-extensions.

**BUT:**

- We can show that certain AE-theories possess no AE-extensions.
Example

Let $T = \{Lp\}$.

We claim:

- $T$ has no AE-extension.

Proof of claim: Assume that $E$ is an AE-extension of $T$.

We distinguish between two cases:

1. $p \notin E$. Then, $V^I(Lp) = 0$, for every AE-interpretation $I$ of $E$.
   \[ \implies \{Lp\} \models_E A, \text{ for every formula } A. \]
   \[ \implies A \in E, \text{ for every formula } A. \]
   \[ \implies p \in E, \text{ contradiction.} \]

2. $p \in E$. Consider $I = \langle m, E \rangle$ s.t. $m(p) = 0$.
   Since $E = \{A \mid \{Lp\} \models_E A\}$ and $p \in E$ by hypothesis, we have
   $\{Lp\} \models_E p$. Moreover, since $p \in E$, $V^I(Lp) = 1$ holds.
   \[ \implies \] So, $V^I(p) = 1$ must hold, a contradiction. $\square$
AE-Extensions: Properties (ctd.)

Theorem 4.2

Let $T$ be a set of objective formulas. Then, $T$ has precisely one AE-extension $E$, given by $E = \bigcup_{i \geq 0} E^i$, where

$$E^0 = \{ A \mid T \models A \text{ and } m\text{-depth}(A) = 0 \},$$

$$E^{i+1} = \{ A \mid T \models_{E^i} A \text{ and } m\text{-depth}(A) \leq i + 1 \}.$$
Finding AE-Extensions

In what follows, we describe a method to determine AE-extensions, following Niemelä (1991).

We define the following concepts.

Let $T$ be an AE-theory.

- By $MA(T)$ we denote the set of all modal atoms occurring in $T$, i.e.,
  $$MA(T) = \{LA \mid LA \text{ is a subformula of a formula } B \in T\}.$$ 

- A set $\Lambda \subseteq MA(T) \cup \neg MA(T)$ is called $T$-full iff for all $LA \in MA(T)$ the following holds:
  * $T \cup \Lambda \models A \iff LA \in \Lambda$;
  * $T \cup \Lambda \not\models A \iff \neg LA \in \Lambda$.

For each $T$-full $\Lambda$, it holds that $|\Lambda| = |MA(T)|$.

(Reason: For each $T$-full $\Lambda$ and each $LA \in MA(T)$, it holds that either $LA \in \Lambda$ or $\neg LA \in \Lambda$, but not both.)
Finding AE-Extensions (ctd.)

Central result:

- Let $T$ be an AE-theory. Then, there is a one-to-one correspondence between the AE-extensions of $T$ and the $T$-full sets.

- In particular:
  - For a given $T$-full set $\Lambda$, the corresponding AE-extension, $SE_T(\Lambda)$, is determined by the following condition:
    \[
    \ast \ A \in SE_T(\Lambda) \iff T \cup \Lambda \cup \{LB \mid LB \in M(A) \text{ and } B \in SE_T(\Lambda)\} \cup \{\neg LB \mid LB \in M(A) \text{ and } B \notin SE_T(\Lambda)\} \models A,
    \]
    where $M(A)$ denotes the set of all modal atoms of $A$ which are not proper subformulas of a modal atom of $A$. 

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Finding AE-Extensions (ctd.)

Consequence:

- Each AE-extension $E$ of an AE-theory $T$ is necessarily of the form

$$E = SE_T(\Lambda),$$

where $\Lambda \subseteq MA(T) \cup \neg MA(T)$ and $|\Lambda| = |MA(T)|$.

This property allows to select suitable candidates for being an AE-extension of a given AE-theory!
Example: Tweety

Let $T = \{b, ((b \land Mf) \rightarrow f)\}$.

Since $Mf = \neg L \neg f$, we obtain that $MA(T) = \{\neg L \neg f\}$, and hence $MA(T) \cup \neg MA(T) = \{\neg L \neg f, \neg L \neg f\}$.

There are two candidates for being a $T$-full set:

$$\Lambda_1 = \{\neg L \neg f\}; \quad \Lambda_2 = \{L \neg f\} = \{Mf\}.$$

It is easy to check that only $\Lambda_2$ is $T$-full.

$$E = SE_T(\Lambda_2) = SE_T(\{\neg L \neg f\}) = SE_T(\{Mf\})$$

is the only AE-extension of $T$, and we have that $f \in E$.

The AE-theory $T' = T \cup \{p, (p \rightarrow (b \land \neg f))\}$ has the unique AE-extension $E' = SE_{T'}(\{\neg L \neg f\})$. 
Example: Nixon Diamond

Let \( T = \{q, r, ((q \land Mp) \rightarrow p), ((r \land M\neg p) \rightarrow \neg p)\} \).

Then, \( MA(T) \cup \neg MA(T) = \{L\neg p, L\neg\neg p, \neg L\neg p, \neg L\neg\neg p\} \).

There are four candidates for \( T\)-full sets:

\[
\Lambda_1 = \{L\neg p, L\neg\neg p\}; \quad \Lambda_3 = \{\neg L\neg p, L\neg\neg p\};
\]

\[
\Lambda_2 = \{L\neg p, \neg L\neg\neg p\}; \quad \Lambda_4 = \{\neg L\neg p, \neg L\neg\neg p\}.
\]

Only \( \Lambda_2 \) and \( \Lambda_3 \) are \( T\)-full.

\( T \) has only two extensions, namely

\[
E = SE_T(\Lambda_2) = SE_T(\{L\neg p, \neg L\neg\neg p\}) = SE_T(\{L\neg p, M\neg p\}),
\]

\[
E' = SE_T(\Lambda_3) = SE_T(\{-L\neg p, L\neg\neg p\}) = SE_T(Mp, L\neg\neg p).
\]

\( \neg p \in E \) and \( p \in E' \).

N.B.: it holds that \( \neg p \in E \) and \( p \in E' \).
4.3 Stable Sets
We now give a *proof-theoretical* characterisation of AE-extensions.

Some concepts:

Let $S$ be a set of formulas.

- $S$ is *stable* iff the following conditions are met:
  * $S = \text{Th}(S)$;
  * if $A \in S$, then $LA \in S$ ($\iff L\bar{S} \subseteq S$);
  * if $A \notin S$, then $\neg LA \in S$ ($\iff \neg L\bar{S} \subseteq S$);

- $S$ is *grounded in* $T$ iff $S \subseteq \text{Th}(T \cup L\bar{S} \cup \neg L\bar{S})$. 

Syntactic Concepts
Properties

Theorem 4.3
Let $S$ be a set of formulas. Then:

1. $S$ is stable $\iff$ $S$ is semantically complete.

2. $S$ is grounded in $T$ $\iff$ $S$ is sound with respect to $T$.

Corollary 4.1
$E$ is an AE-extension of $T$ $\iff$ (i) $T \subseteq E$, (ii) $E$ is stable, and (iii) $E$ is grounded in $T$.

Theorem 4.4
$E$ is an AE-extension of $T$ $\iff$ $E = Th(T \cup LE \cup \neg LE)$. 
Properties (ctd.)

Theorem 4.5
Let $S, S'$ be stable sets. If $S \subseteq S'$ and $S'$ is consistent, then $S = S'$.

Proof:
Let $S, S'$ be stable sets such that $S \subseteq S'$ and $S'$ is consistent. Assume that $S \neq S'$.

Since $S \subseteq S'$ and $S \neq S'$, there must be some formula $A$ s.t. $A \in S'$ but $A \notin S$.

We then obtain the following:

- Since $A \in S'$ and $S'$ is stable, $L_A \in S'$ must hold.
- Since $A \notin S$ and $S$ is stable, $\neg L_A \in S$ must hold.
  \[\iff \neg L_A \in S', \text{ since } S \subseteq S'.\]

$\implies S'$ is inconsistent, contradiction. $\square$
Corollary 4.2 ("Maximality of AE-Extensions")

Let $E, E'$ be AE-extensions of $T$. If $E \subseteq E'$ and $E'$ is consistent, then $E = E'$.

Theorem 4.6

If $S, S'$ are stable sets such that $S_0 = S'_0$, i.e., $S$ and $S'$ coincide on their objective parts, then $S = S'$.

Theorem 4.7

Each stable set $S$ is the unique AE-extension of its objective part, i.e.,

$S = \{ A \mid S_0 \models_S A \}$. 
4.4 Relation to Default Logic
Basic Considerations

- We now analyse the relation between default logic and autoepistemic logic.

- Since the language of AEL is defined over a propositional alphabet, accordingly we consider only *propositional default theories*. 
Defaults as Formulas of AEL

Idea:

- It is suggestive to associate a (propositional) default $d = (A : B_1, \ldots, B_n/C)$ with a formula of AEL of the form

$$tr(d) = ((LA \land MB_1 \land \cdots \land MB_n) \rightarrow C).$$

We thus obtain a mapping assigning each (propositional) default theory $T = \langle W, D \rangle$ an AE-theory

$$tr(T) = W \cup \{tr(d) \mid d \in D\}.$$
We would like to have that $T$ and $tr(T)$ are in some sense “equivalent”.

More specifically:

- A translation $e$ assigning each propositional default theory $T$ an AE-theory $e(T)$ is faithful iff the following is met:
  - the objective parts of the AE-extensions of $e(T)$ coincide with the extensions of $T$,
    - i.e., the set of extensions of $T$ is given by
      $$\{E_0 \mid E \text{ is AE-extension of } e(T)\}.$$
 Defaults as Formulas of AEL (ctd.)

**BUT**: $tr$ is **not** faithful!

- Counterexample: Consider $T = \langle \emptyset, \{(p : /p)\}\rangle$.

  * $T$ has one extension, $Th(\emptyset)$, but $tr(T) = \{Lp \rightarrow p\}$ has **two** AE-extensions, namely $SE_T(\{\neg Lp\})$ and $SE_T(\{Lp\})$.

  → Only $SE_T(\{\neg Lp\})$ corresponds to $Th(\emptyset)$, since $(SE_T(\{\neg Lp\}))_0 = Th(\emptyset)$!
Discussion

The AE-extension $SE_T(\{Lp\})$ of $tr(T) = \{Lp \rightarrow p\}$ is in some sense problematic since it contains “self grounded” beliefs.

This is due to the following observations:

– $SE_T(\{Lp\})$ contains $p$, but from the premiss $Lp \rightarrow p$ alone, there is no reason to accept $p$.

– But $p$ is nonetheless accepted:
  * the agent can first assume $p$, i.e., taking $Lp$ into his belief set,
  * and afterwards justifying this assumption of $p$ by means of the premiss $Lp \rightarrow p$. 
Variants of AEL

➢ To avoid self grounded beliefs, stronger versions of AEL have been introduced, namely:
  
  – *minimal AE-extensions* (Konolige, 1988);
  – *strongly grounded AE-extensions* (Konolige, 1988);

➢ The following holds:
  
  – Extensions of a default theory $T$ coincide with the objective parts of the superstrongly grounded AE-extensions of $tr(T)$. 
A further stronger variant of AEL is the so-called *nonmonotonic logic N* (Marek & Truszczynski, 1990).

- In the nonmonotonic logic N, positive introspection is not realised by using assumptions of the form LA in the fixed-point condition,
  * rather it is encoded directly in the underlying monotonic logic.

- More specifically:
  * Let \( \vdash_N \) be the derivability relation resulting from the derivability relation of classical propositional logic by additionally assuming the *necessitation rule* \( \frac{A}{LA} \).
  * Let \( Th_N(T) = \{ A \mid T \vdash_N A \} \).
  \( E \) is an *N-extension* of \( T \) \( \iff \) \( E = Th_N(T \cup \neg \bar{E}) \).
Variants of AEL (ctd.)

The following was shown (Truszczynski, 1991):

– Extensions of a default theory $T = \langle W, D \rangle$ coincide with the objective parts of the N-extensions of $t(T) = W \cup \{t(d) \mid d \in D\}$, where

$$t(A : B_1, \ldots, B_n/C) = ((L_A \land \land L_{MB_1} \land \cdots \land L_{MB_n}) \rightarrow C).$$
Default Logic and Standard AEL

Problem:

– Can default logic also be faithfully translated into standard AEL?

Answer: Yes!

– Such a translation was given by Gottlob (1995).

– This translation consists of two parts:
  * First, the translation $t$ is applied to map default theories into nonmonotonic logic N in a faithful way.
  * Then, nonmonotonic logic N is translated faithfully into standard AEL.
Default Logic and Standard AEL (ctd.)

Important property:

- The translation by Gottlob is not modular.

  (A translation $e$ is modular iff $e(\langle W, D \rangle) = W \cup e(\langle \emptyset, D \rangle)$, for every default theory $\langle W, D \rangle$.)

- This is a consequence of the general property that there can be no translation of default logic into AEL which is both faithful and modular (Gottlob, 1995).