SAT Solving

Part 1: Motivation, normal form translations and easy problem classes

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May 2, 2010

Outline

Introduction and motivation

Translations to clausal normal form

Easy classes for the satisfiability problem

Concluding remarks for the first part

What is SAT and QSAT?

- Boolean satisfiability (SAT) and quantified B. SAT (QSAT)
  (we will consider formulas with Boolean quantifiers later)
- Operates on Boolean formulas (BFs) and quantified BFs (often restricted to normal forms; see later)
- Boolean case: Is a given BF satisfiable?
- Tautology, entailment, etc. checking reduced to SAT
- Looks easy, but gets hard very quickly as the size of the problem/formula increases

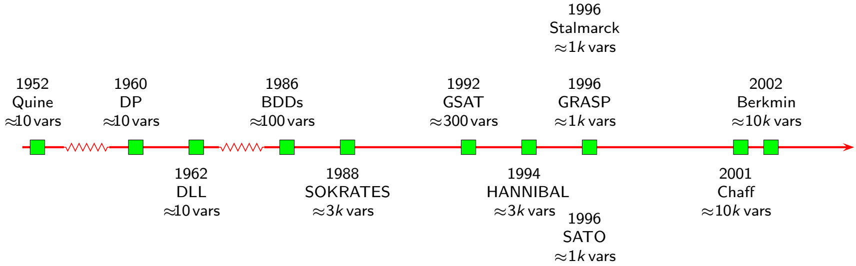
Why is SAT Important?

- Theoretical importance:
    - First NP-Complete problem discovered by S. A. Cook (The complexity of theorem proving procedures, Proc. 3rd ACM Symp. on the Theory of Computing,1971, 151-158)
- It is everywhere
    - Automatic Test Pattern Generation
    - Combinational Equivalence Checking
    - Various AI problems like, e.g., planning
    - Theorem Proving
    - Software/hardware modeling and verification, . . .
- SAT solvers available that can solve practical problems
    - SAT solving has been well studied for at least 40 years
    - Recent breakthroughs: Can handle over a million variables
    - Seen wide use in the industry, but still a lot of problems
    - Can we do better?

The Early Machines and SAT Solvers

- 1869: William Stanley Jevons: *Logical Piano*
  process logical problems faster than humans
- 1885: Allan Marquand proposed electrical version of Piano
- 1936: Benjamin Burack built the 1. electronic logic machine
- 1947: Theodore A. Kalin and William Burckhart
  built a machine to SAT check prop. formulas with $\leq$ 12 vars
- Computers arrived at the horizon and with them new opportunities

50 Years of SAT Solving Algorithms



1952
Quine
≈10 vars

1960
DP
≈10 vars

1962
DLL
≈10 vars

1986
BDDs
≈100 vars

1988
SOKRATES
≈3*k* vars

1992
GSAT
≈300 vars

1994
HANNIBAL
≈3*k* vars

1996
Stalmarck
≈1*k* vars

1996
GRASP
≈1*k* vars

1996
SATO
≈1*k* vars

2001
Chaff
≈10*k* vars

2002
Berkmin
≈10*k* vars

Transformation of BFs into Normal Forms

- Modern SAT solvers often restrict input to normal forms (expected: easier language allows more efficient data structures)
- We have already seen normal forms: NNF and CNF
  - NNF characterized by two conditions:
    1. Negation signs occur only in front of atoms
    2. The only connectives are $\wedge$ and $\vee$
  - NNF of ϕ (denoted by $nnf(\phi)$) and ϕ are equivalent!
- Often: Conjunctive Normal Form (CNF)
  - Conjunction (or set) of clauses (=disjunctions of literals)
  - Often: clauses as sets (causes problems sometimes)
- Different methods for translating BFs into CNFs yield different behaviour wrt proof search and proof complexity

The Traditional Approach for a CNF Translation

- Based on the application of distributivity laws
- Start with the formula ϕ and translate it to NNF
- Take *nnf*(ϕ) and replace the left side of the following equivalences by the right side (order does not matter!)

$$\begin{array}{lrcl} 1 & \phi \vee (\psi \wedge \chi) & \equiv & (\phi \vee \psi) \wedge (\phi \vee \chi) \\ 2 & (\psi \wedge \chi) \vee \phi & \equiv & (\psi \vee \phi) \wedge (\chi \vee \phi) \end{array}$$

- Observe that *nnf*(ϕ) $\equiv$ *cnf*(ϕ) $\equiv \phi$ holds

Exa: Transform ϕ: $(p \land q \to r) \to (q \to r)$ to CNF

$nnf(\phi)$: $(p \land q \land \neg r) \lor (\neg q \lor r)$

Formula	Rule
$(p \land q \land \neg r) \lor (\neg q \lor r)$	2
$(p \lor \neg q \lor r) \land (q \land \neg r \lor (\neg q \lor r))$	2
$(p \lor \neg q \lor r) \land (q \lor \neg q \lor r) \land (\neg r \lor \neg q \lor r)$	

Two disadvantages:

1. Disruption of the formula's structure
2. $cnf(\phi)$ can be exponentially longer than ϕ
   Take a formula in disjunctive NF and translate it to CNF!

Important concept for later: polarities of subformulas

Polarity Labels $(+, -, \pm)$ of Subformulas

$$
\begin{array}{lcll}
(\neg\phi_1)^{+} & \rightsquigarrow & (\neg\phi_1^{-})^{+} & \\
(\neg\phi_1)^{-} & \rightsquigarrow & (\neg\phi_1^{+})^{-} & \\
(\neg\phi_1)^{\pm} & \rightsquigarrow & (\neg\phi_1^{\pm})^{\pm} & \\
(\phi_1 \circ \phi_2)^{q} & \rightsquigarrow & (\phi_1^{q} \circ \phi_2^{q})^{q} & \text{for } q \in \{+,-,\pm\} \text{ and } \circ \in \{\vee, \wedge\} \\
(\phi_1 \rightarrow \phi_2)^{+} & \rightsquigarrow & (\phi_1^{-} \rightarrow \phi_2^{+})^{+} & \\
(\phi_1 \rightarrow \phi_2)^{-} & \rightsquigarrow & (\phi_1^{+} \rightarrow \phi_2^{-})^{-} & \\
(\phi_1 \rightarrow \phi_2)^{\pm} & \rightsquigarrow & (\phi_1^{\pm} \rightarrow \phi_2^{\pm})^{\pm} & \\
(\phi_1 \leftrightarrow \phi_2)^{q} & \rightsquigarrow & (\phi_1^{\pm} \leftrightarrow \phi_2^{\pm})^{q} & \text{for } q \in \{+,-,\pm\}
\end{array}
$$

$\Sigma^{q}(\phi)$: all subformula occurrences of ϕ occurring in polarity q

For simplicity, we restrict ourselves to input formulas without ↔

Structure-preserving (or Definitional) NFTs

The Basic Idea

- Well known in logic
  (occurred relatively late in ATP and theory (Tseitin 1968))
- Consider the input formula ϕ as a tree
- Label each subformula occurrence (SFO) with a new atom
  (atom neither occurs in ϕ nor is it introduced before)
- Construct equivalences of the form

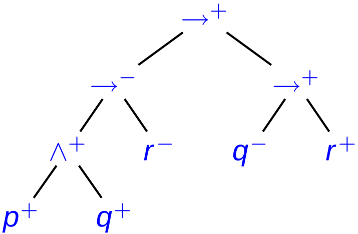
  $$l_\phi \leftrightarrow (l_{\phi_1} \circ l_{\phi_2}) \quad \text{for SFOs} \quad \phi_1 \circ \phi_2$$

  where l_ψ is the label for SF(O) ψ.
- Translate each $l_\phi \leftrightarrow (l_{\phi_1} \circ l_{\phi_2})$ to CNF using the NFT above

Example for a Translation: ϕ: $(p \wedge q \to r) \to (q \to r)$

Step 1: Label the Formula Tree

Tree of ϕ^+ with polarities



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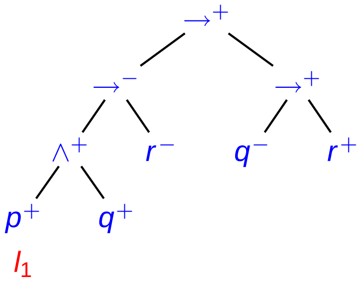

$l_1 \leftrightarrow p$

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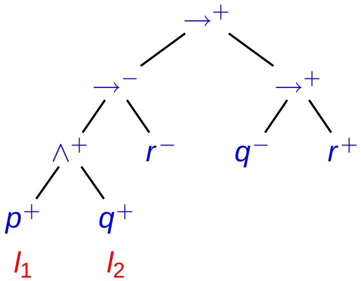

$l_1 \leftrightarrow p$

$l_2 \leftrightarrow q$

Example for a Translation: ϕ: $(p \land q \to r) \to (q \to r)$

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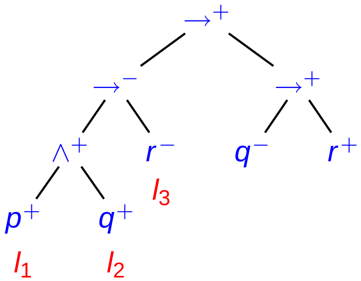

$l_1 \leftrightarrow p$

$l_2 \leftrightarrow q$

$l_3 \leftrightarrow r$

Example for a Translation: ϕ: $(p \wedge q \to r) \to (q \to r)$

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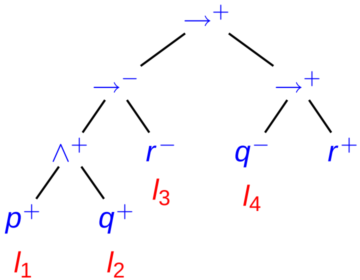


$l_1 \leftrightarrow p$

$l_2 \leftrightarrow q$

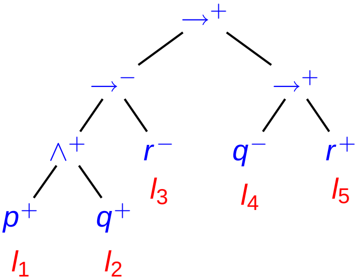
$l_3 \leftrightarrow r$

$l_4 \leftrightarrow q$

Example for a Translation: ϕ: $(p \wedge q \rightarrow r) \rightarrow (q \rightarrow r)$

Step 1: Label the Formula Tree

Tree of ϕ^+ with polarities



$l_1 \leftrightarrow p$

$l_2 \leftrightarrow q$

$l_3 \leftrightarrow r$

$l_4 \leftrightarrow q$

$l_5 \leftrightarrow r$

Example for a Translation: ϕ: $(p \wedge q \to r) \to (q \to r)$

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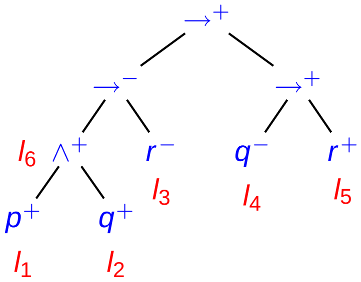
Tree of ϕ^+ with polarities



$l_1 \leftrightarrow p$

$l_2 \leftrightarrow q$

$l_3 \leftrightarrow r$

$l_4 \leftrightarrow q$

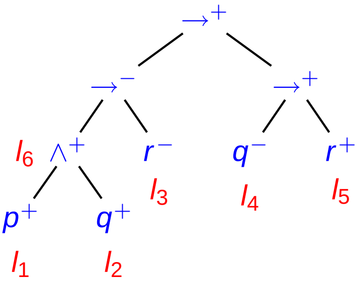
$l_5 \leftrightarrow r$

$l_6 \leftrightarrow p \wedge q$

Example for a Translation: ϕ: $(p \wedge q \rightarrow r) \rightarrow (q \rightarrow r)$

Step 1: Label the Formula Tree

Tree of ϕ^+ with polarities



$l_1 \leftrightarrow p$

$l_2 \leftrightarrow q$

$l_3 \leftrightarrow r$

$l_4 \leftrightarrow q$

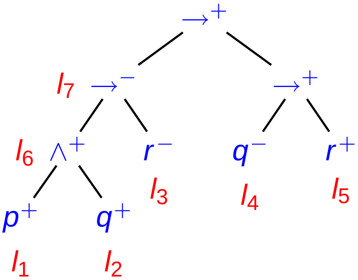
$l_5 \leftrightarrow r$

$l_6 \leftrightarrow l_1 \wedge l_2$

Example for a Translation: ϕ: $(p \wedge q \rightarrow r) \rightarrow (q \rightarrow r)$

Step 1: Label the Formula Tree

Tree of ϕ^+ with polarities



$l_1 \leftrightarrow p$

$l_2 \leftrightarrow q$

$l_3 \leftrightarrow r$

$l_4 \leftrightarrow q$

$l_5 \leftrightarrow r$

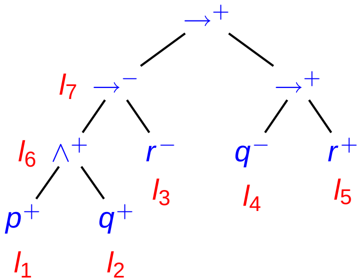
$l_6 \leftrightarrow l_1 \wedge l_2$

$l_7 \leftrightarrow p \wedge q \rightarrow r$

Example for a Translation: ϕ: $(p \wedge q \rightarrow r) \rightarrow (q \rightarrow r)$

Step 1: Label the Formula Tree

Tree of ϕ^+ with polarities



$l_1 \leftrightarrow p$

$l_2 \leftrightarrow q$

$l_3 \leftrightarrow r$

$l_4 \leftrightarrow q$

$l_5 \leftrightarrow r$

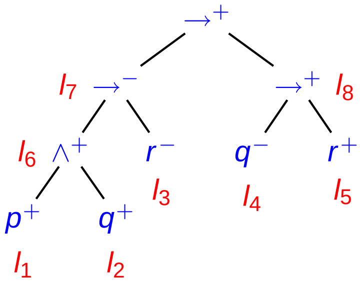
$l_6 \leftrightarrow l_1 \wedge l_2$

$l_7 \leftrightarrow l_6 \rightarrow l_3$

Example for a Translation: ϕ: $(p \wedge q \rightarrow r) \rightarrow (q \rightarrow r)$

Step 1: Label the Formula Tree

Tree of ϕ^+ with polarities



$l_1 \leftrightarrow p$

$l_2 \leftrightarrow q$

$l_3 \leftrightarrow r$

$l_4 \leftrightarrow q$

$l_5 \leftrightarrow r$

$l_6 \leftrightarrow l_1 \wedge l_2$

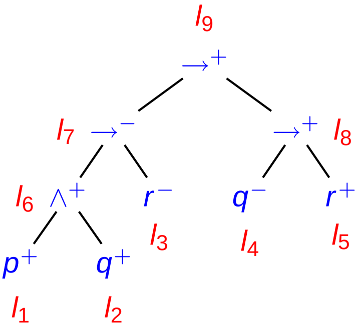
$l_7 \leftrightarrow l_6 \rightarrow l_3$

$l_8 \leftrightarrow l_4 \rightarrow l_5$

Example for a Translation: ϕ: $(p \wedge q \rightarrow r) \rightarrow (q \rightarrow r)$

Step 1: Label the Formula Tree

Tree of ϕ^+ with polarities



l_1	$\leftrightarrow$	p
l_2	$\leftrightarrow$	q
l_3	$\leftrightarrow$	r
l_4	$\leftrightarrow$	q
l_5	$\leftrightarrow$	r
l_6	$\leftrightarrow$	$l_1 \wedge l_2$
l_7	$\leftrightarrow$	$l_6 \rightarrow l_3$
l_8	$\leftrightarrow$	$l_4 \rightarrow l_5$
l_9	$\leftrightarrow$	$l_7 \rightarrow l_8$

Example for a Translation: ϕ: $(p \wedge q \rightarrow r) \rightarrow (q \rightarrow r)$

Step 2: Translate the “Labeling Formulas” to Clauses

Equivalences for SFOs in ϕ	Associated Clauses		
	$C_1(\phi)$	$C_2(\phi)$	$C_3(\phi)$
$l_1 \leftrightarrow p$	$\neg l_1 \vee p$	$l_1 \vee \neg p$	
$l_2 \leftrightarrow q$	$\neg l_2 \vee q$	$l_2 \vee \neg q$	
$l_3 \leftrightarrow r$	$\neg l_3 \vee r$	$l_3 \vee \neg r$	
$l_4 \leftrightarrow q$	$\neg l_4 \vee q$	$l_4 \vee \neg q$	
$l_5 \leftrightarrow r$	$\neg l_5 \vee r$	$l_5 \vee \neg r$	
$l_6 \leftrightarrow l_1 \wedge l_2$	$\neg l_6 \vee l_1$	$\neg l_6 \vee l_2$	$l_6 \vee \neg l_1 \vee \neg l_2$
$l_7 \leftrightarrow l_6 \rightarrow l_3$	$\neg l_7 \vee \neg l_6 \vee l_3$	$l_7 \vee l_6$	$l_7 \vee \neg l_3$
$l_8 \leftrightarrow l_4 \rightarrow l_5$	$\neg l_8 \vee \neg l_4 \vee l_5$	$l_8 \vee l_4$	$l_8 \vee \neg l_5$
$l_9 \leftrightarrow l_7 \rightarrow l_8$	$\neg l_9 \vee \neg l_7 \vee l_8$	$l_9 \vee l_7$	$l_9 \vee \neg l_8$

Defining the Translations

Formula ϕ^q	Associated Clauses		
	$C_1(\phi)^q$	$C_2(\phi)^q$	$C_3(\phi)^q$
p^+	$\neg l_p \vee p$		
p^-		$l_p \vee \neg p$	
$(\neg\phi_1)^+$	$\neg l_\phi \vee \neg l_{\phi_1}$		
$(\neg\phi_1)^-$		$l_\phi \vee l_{\phi_1}$	
$(\phi_1 \wedge \phi_2)^+$	$\neg l_\phi \vee l_{\phi_1}$	$\neg l_\phi \vee l_{\phi_2}$	
$(\phi_1 \wedge \phi_2)^-$			$l_\phi \vee \neg l_{\phi_1} \vee \neg l_{\phi_2}$
$(\phi_1 \vee \phi_2)^+$	$\neg l_\phi \vee l_{\phi_1} \vee l_{\phi_2}$		
$(\phi_1 \vee \phi_2)^-$		$l_\phi \vee \neg l_{\phi_1}$	$l_\phi \vee \neg l_{\phi_2}$
$(\phi_1 \rightarrow \phi_2)^+$	$\neg l_\phi \vee \neg l_{\phi_1} \vee l_{\phi_2}$		
$(\phi_1 \rightarrow \phi_2)^-$		$l_\phi \vee l_{\phi_1}$	$l_\phi \vee \neg l_{\phi_2}$

For any ϕ, C_ϕ^q denotes the clauses for ϕ^q ($q \in \{+, -\}$)

The Structure-preserving Normal Forms

The definitional form, $\delta(\phi)$, of ϕ is $\hat{\delta}(\phi) \cup \{l_\phi\}$ with

$$\hat{\delta}(\phi)\colon \{C^+_\psi, C^-_\psi \mid \psi \in \Sigma^+(\phi)\} \cup \{C^+_\psi, C^-_\psi \mid \psi \in \Sigma^-(\phi)\}$$

The p-definitional form, $\delta^+_p(\phi)$, of ϕ is $\hat{\delta}^+_p(\phi) \cup \{l_\phi\}$ with

$$\hat{\delta}^q_p(\phi)\colon \{C^q_\psi \mid \psi \in \Sigma^+(\phi)\} \cup \{C^r_\psi \mid \psi \in \Sigma^-(\phi)\}$$

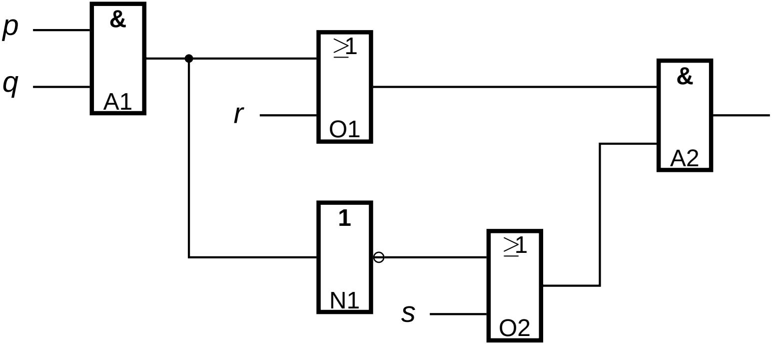
where $\{r\} = \{+, -\} \setminus \{q\}$)

- Sat-equivalence: ϕ has a model iff $\delta(\phi)$ ($\delta^+_p(\phi)$) has one
- Several variants and optimizations available, e.g.,
  no label for atoms or negations, don't translate clauses, etc.

Properties of the Translation

- Retains the structure of the formula (by labels for SFOs)
- For each SFOs, there are at most three clauses
- Each clause has at most three literals
- Normal form is linear-time computable
- ϕ and its definitional translation not logically equivalent
  (new labels change signature $\leadsto$ logical equivalence lost)
- ϕ has a model iff its definitional translation has one holds
  (this is the important prop.; cf 1st order ATP and Skolemization)
- Models of ϕ and its def. translation can be translated
- Generalizations used to extend calculi by extensions
  (resulting in stronger calculi which p-simulate, e.g., the cut rule)

An Application of the Translation: Circuits to CNF



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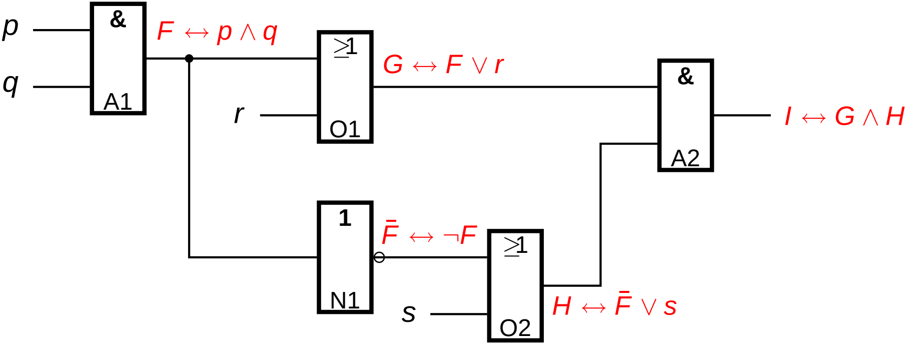

$F \leftrightarrow p \wedge q$

$G \leftrightarrow F \vee r$

$\bar{F} \leftrightarrow \neg F$

$H \leftrightarrow \bar{F} \vee s$

$I \leftrightarrow G \wedge H$

Use the definitional translation to get a sat-equivalent clause set

Historical Remarks on Structure-preserving NFTs

- As a starting point: Davis' article in Handbook on AR (HAR, edited by J.A. Robinson and A. Voronkov, 2001)
- Tseitin 1968: definit. translation + the extension principle
- Cook and Reckhow 1979 (JSL): limited extension (compare calculi with (limited) extension with calculi wo)
- Eder 1984, 1992: definitional translation for first-order logic
- Plaisted and Greenbaum 1986: mainly p-definitional translation for first-order logic + some optimizations
- Boy de la Tour 1992: mix of structure-preserving and traditional translations; goal: get short normal form
- Baaz et al.: Normal Form Transformations in HAR (implications of different NFTs to proof complexity in FOL)

Algorithms for 2-SAT

- Special case of a CNF: each clause has at most 2 literals
- Often used as a simplification procedure in (Q)SAT solvers
- Unlike 3-SAT, 2-SAT is decidable in polynomial time (degree of the polynomial depends on the method used)
- E.g., for resolution, there is a quadratic time procedure
- We discuss linear procedure based on implication graphs
  - Aspvall, Plass, Tarjan. A linear-time algorithm for testing the truth of certain quantified boolean formulas. IPL 8(3) 121-123, 1979 (Err. 14(4) 195, 1982)
  - A. del Val. Simplifying Binary Propositional Theories into Connected Components Twice as Fast. In R. Nieuwenhuis and A. Voronkov (Eds.): LPAR 2001, LNAI 2250, pp. 392-406, 2001.

Implications Associated to a 2-CNF

Clause C	Implication(s) $I(C)$	
p	$\neg p \rightarrow p$	
$\neg p$	$p \rightarrow \neg p$	
$p \vee q$	$\neg p \rightarrow q$	$\neg q \rightarrow p$
$p \vee \neg q$	$\neg p \rightarrow \neg q$	$q \rightarrow p$
$\neg p \vee q$	$p \rightarrow q$	$\neg q \rightarrow \neg p$
$\neg p \vee \neg q$	$p \rightarrow \neg q$	$q \rightarrow \neg p$

$u \rightarrow v$ means: "If literal u is true then literal q is also true"

Construction of the Implication Graph (IG)

Given a 2-CNF ϕ, construct the IG $\mathrm{G}(\phi) = (V_\phi, E_\phi)$:

$$\begin{aligned} V_\phi &= \{p, \neg p \mid \text{atom } p \text{ occurs in } \phi\} \\ E_\phi &= \{I(C) \mid C \in \phi\} \end{aligned}$$

Example: $C_1 : p \vee q$ $\quad C_2 : \neg p \vee q$

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Example: $C_1 : p \vee q$ $C_2 : \neg p \vee q$

p q

$\neg p$ $\neg q$

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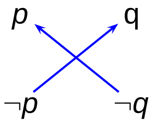

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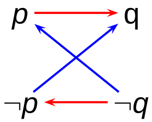


Next: Construct all strongly connected components of $\mathrm{G}(\phi)$

A linear 2-SAT Procedure

Reminder: Strongly connected components

- A directed graph is called strongly connected if each node is reachable from each other node via a path
- A strongly connected component (SCC) is a maximal strongly connected subgraph
- Tarjan (1972) provides a linear-time algorithm to compute all SCCs of a (directed) graph
- Important link and consequence:

A 2-CNF ϕ is SAT iff no p and $\neg p$ belong to the same SCC

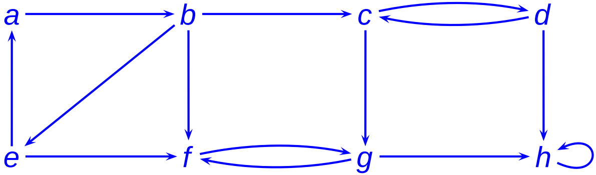
2-SAT is solvable in linear time

SCC(*G*): Compute all SCCs of Directed Graph *G*

Input: Directed graph G
Output: All SCCs of G

1. Call DFS(G) to compute finishing time $f(u)$ for each vertex u
2. Compute transpose G^t of G (reverse edge arrows)
3. Call DFS(G^t), but in the main loop, consider vertices in order of decreasing $f(u)$ (as computed in first DFS)
4. Output the vertices of each tree in DFS-forest obtained by the second DFS call as a separate SCC

SCC(G) on Example Graph *G*: Part 1



SCC(G) on Example Graph *G*: Part 1

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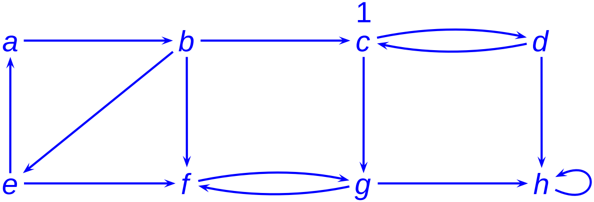
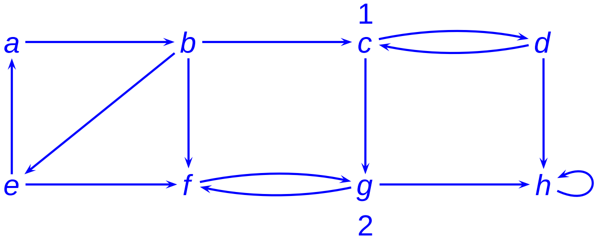


SCC(G) on Example Graph *G*: Part 1

1. Call DFS(*G*) to compute finishing time $f(u)$ for each vertex *u*



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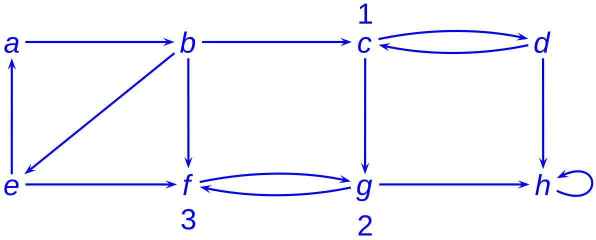

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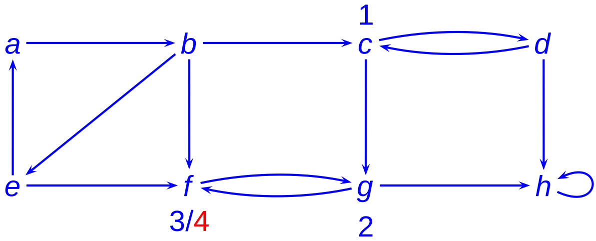

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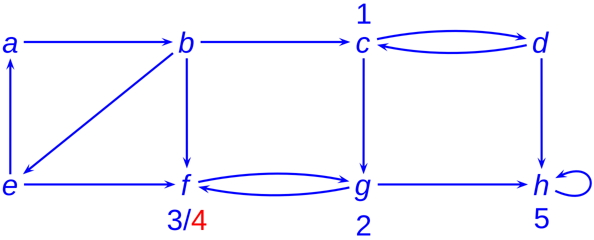

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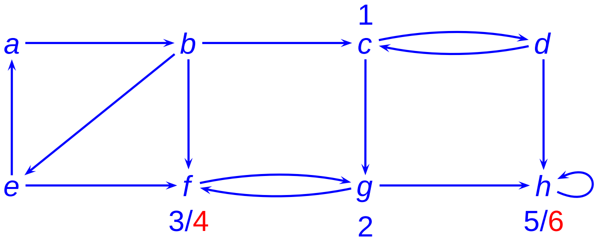

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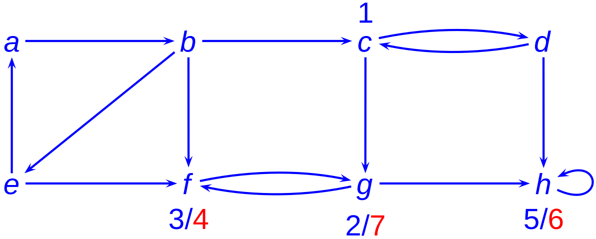

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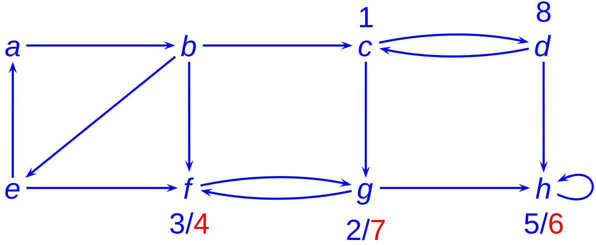

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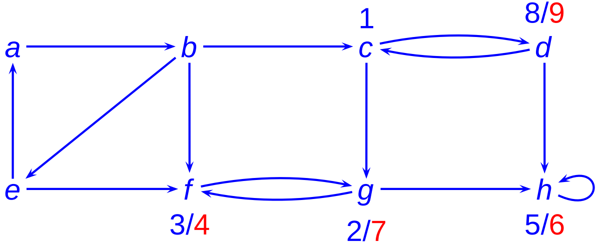

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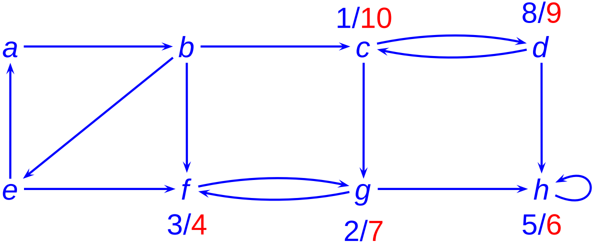

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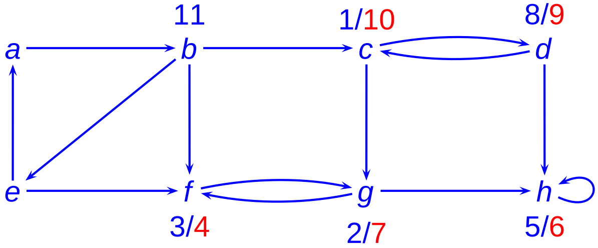

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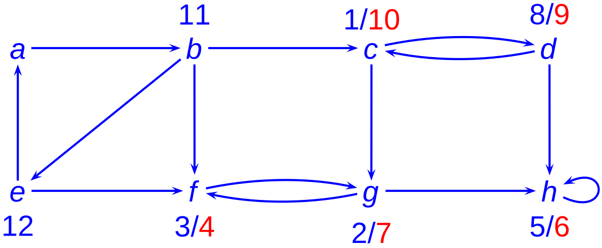

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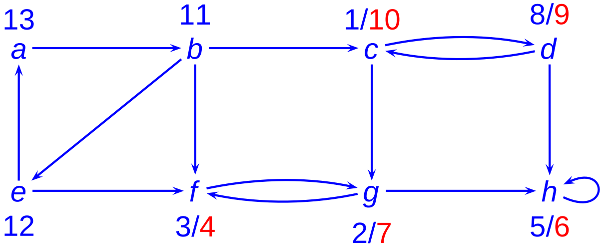

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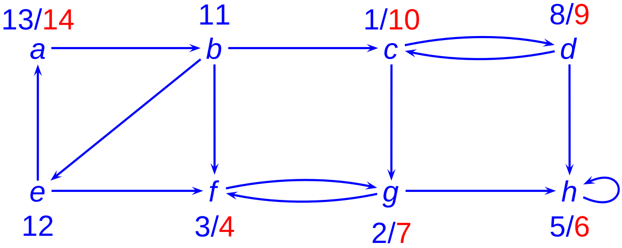

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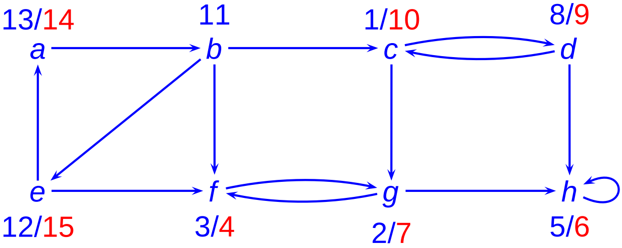

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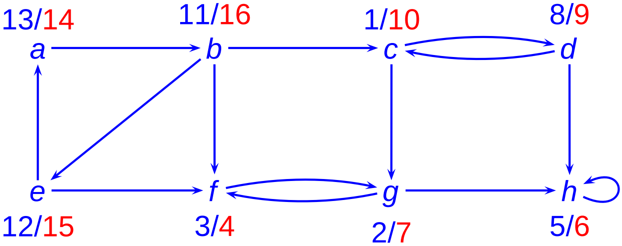
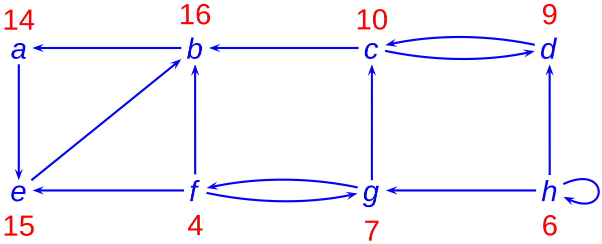


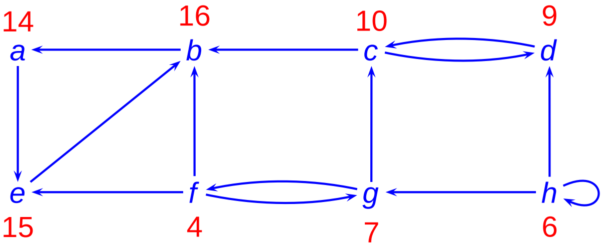
SCC(G) on Example Graph *G*: Part 2

2. Compute transpose G^t of G (reverse edge arrows)



SCC(G) on Example Graph *G*: Part 2

3. Call DFS(G^t), but in the main loop, consider vertices in order of decreasing $f(u)$ (as computed in first DFS)



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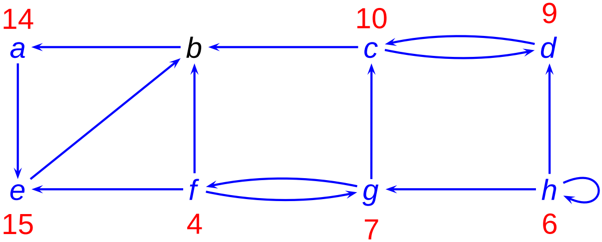

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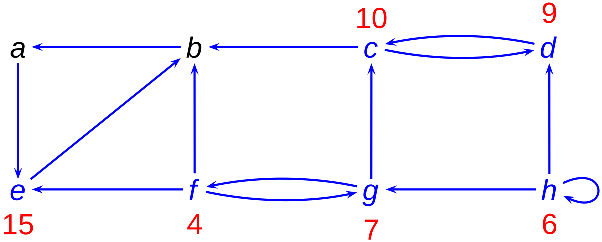

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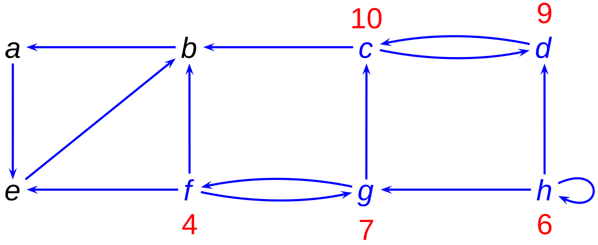

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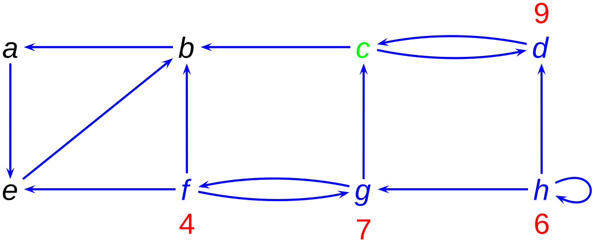

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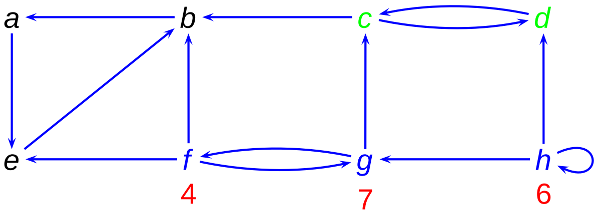

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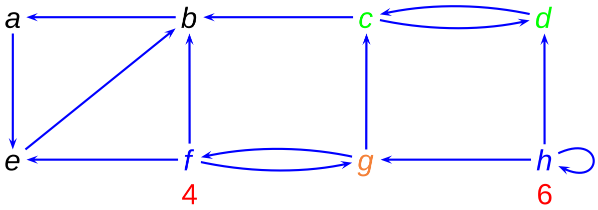

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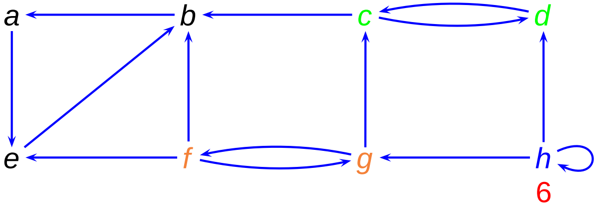
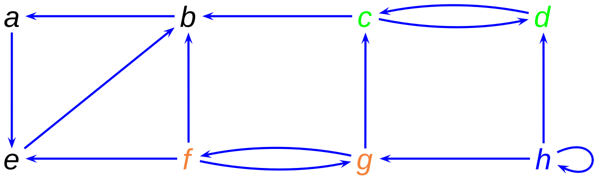


SCC(G) on Example Graph *G*: Part 2

4. Output the vertices of each tree in DFS-forest obtained by the second DFS call as a separate SCC: (*a*, *b*, *e*), (*c*,*d*), (*f*,*g*), (*h*)



BinSAT: Deciding 2-SAT Without SCCs

- The above algorithm uses SCCs to decide 2-SAT
- All elements in an SCC are equivalent
- Above algorithm requires two “runs”
- Can we avoid the SCC construction and get a faster procedure?

BinSAT: The Algorithm

Algorithm: TempPropUnit

```
Input: A literal $x$ to be tentatively assigned.
if tempval(x) = false then /* temporary conflict: $S \models \bar{x} \rightarrow x$ */
    Set $S$ := PropUnit($S \cup \{x\}$); return;
end
tempval($x$) := true; tempval($\bar{x}$) := false;
foreach $y\bar{x} \in S$ do
    if $\square \in S$ or permval(x) $\neq$ NIL then return;
    if permval(y) = NIL and tempval(y) $\neq$ true then TempPropUnit(y);
end
```

Algorithm: BinSAT

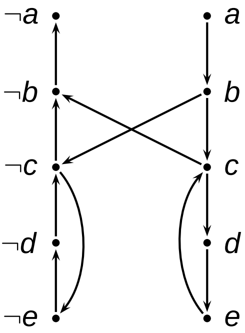
```
Input: A set $S$ of binary clauses
Result: Unsatisfiable or a model of $S$
foreach literal x in S do  tempval(x) := permval(x) := Nil;
$S$ := PropUnit(S);
while $\square \notin S$ and there exists a literal x with tempval(x) = permval(x) = Nil do
    TempPropUnit(x);
end
if $\square \in S$ then  return Unsatisfiable;
else  return GetModel();
```

BinSAT: Some Comments

- PropUnit: any implementation of unit resolution which
  - reports forced assignements in permval and
  - generates $\Box$ when a global contradiction occurs
- GetModel: for each variable x, return $permval(x)$ (if $\neq$ Nil) and $tempval(x)$ otherwise
- BinSAT handles tentative and permanent assignments
- Tentatively assign literal x and propagate consequences by TempPropUnit (=depth-first search applying unit resolution)
- If contradiction occurs, assign x permanently and compute consequences (=entailed literals) from this assignment

BinSAT: Example

$$S = \{\neg a \vee b, \neg b \vee c, \neg c \vee \neg b, \neg c \vee d, \neg d \vee e, \neg e \vee c\}$$



- Tentative assignment of *a* to true (by TempPropUnit(a)) results in a temp. conflict
- Propagate (by PropUnit($S \cup \{\neg b\}$))
- The resulting permvals are:
  $\neg b$ is true and $\neg a$ is true
- Backtrack (bt) and continoue the dfs . . .
- List exhausted ➥ bt and continoue from *b*
- Continouing this way yields the final result:
  permval($\neg a$) = permval($\neg b$) = true,
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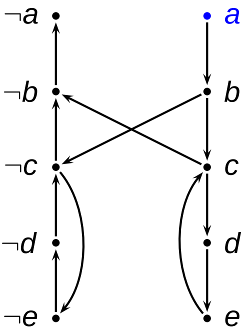

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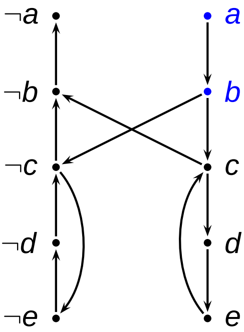

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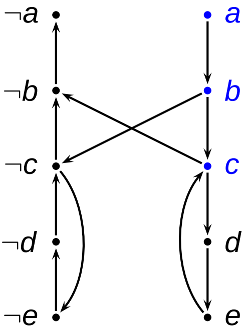

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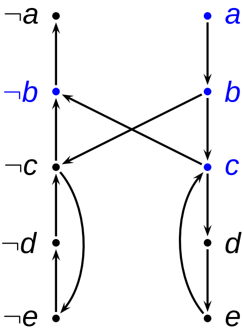

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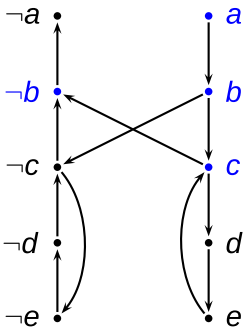

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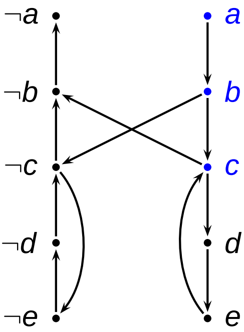

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BinSAT: Example

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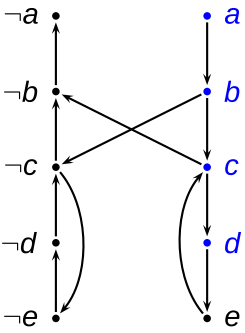

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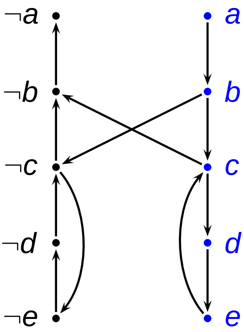

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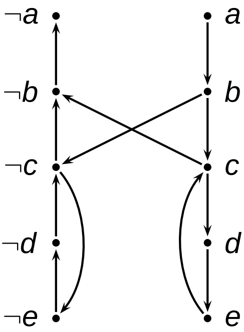

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Horn Satisfiability: The Graph Structure

- Horn clause: clause with at most one positive literal
- $p \leftarrow q_1, \ldots, q_n$ stands for $p \vee \neg q_1 \vee \cdots \vee \neg q_n$
- Restriction allows for efficient Horn-SAT algorithms (e.g., based on graphs)

$$
\begin{array}{lll}
p_1 & \leftarrow & \\
p_3 & \leftarrow & \\
p_6 & \leftarrow & \\
p_0 & \leftarrow & p_1, p_2, p_3 \\
p_2 & \leftarrow & p_1, p_3 \\
p_4 & \leftarrow & p_5, p_6 \\
p_5 & \leftarrow & p_0
\end{array}
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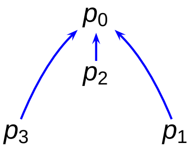

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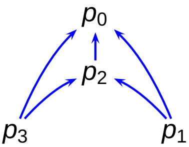
$$



Horn Satisfiability: The Graph Structure

- Horn clause: clause with at most one positive literal
- $p \leftarrow q_1, \ldots, q_n$ stands for $p \vee \neg q_1 \vee \cdots \vee \neg q_n$
- Restriction allows for efficient Horn-SAT algorithms (e.g., based on graphs)

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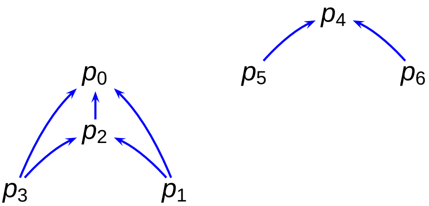

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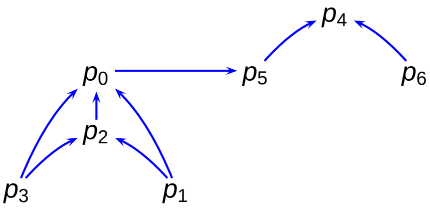

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- Clause set is now represented as a graph

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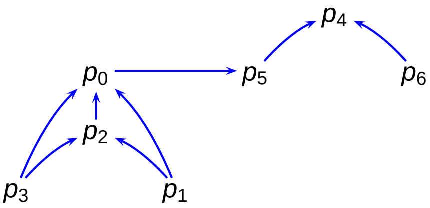

Horn Satisfiability: The Propagation Algorithm

- All red-marked atoms are true and derivable by punit res (i.e., res, where one parent is a positive unit clause (atom))
- Propagate consequences of true atoms through the graph

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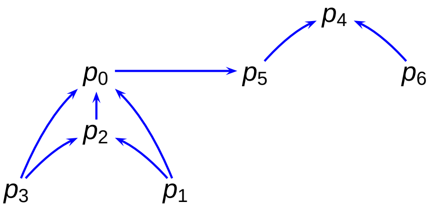

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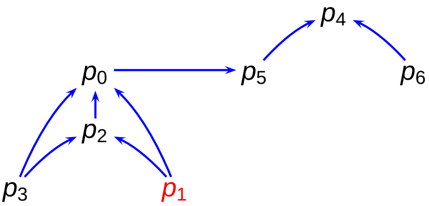

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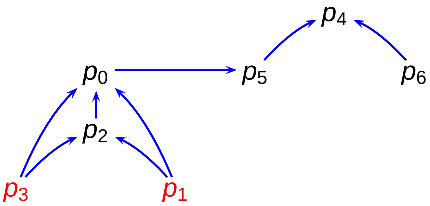

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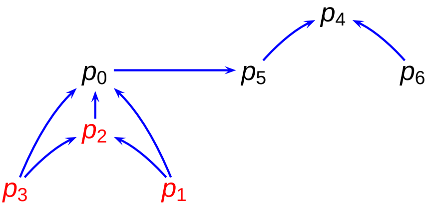

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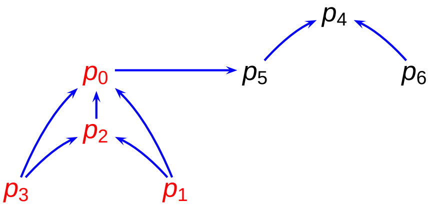

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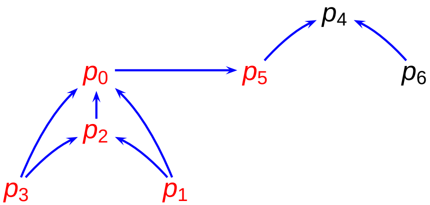

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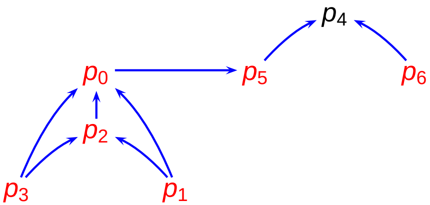

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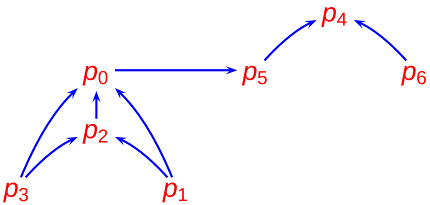


Horn Satisfiability: Handling Negative Clauses

- So far, all our Horn clauses were non-negative
  (and the clause set was SAT)
- UNSAT clause sets: at least one purely negative clause
- Clause set is UNSAT if,
  for a negative clause $\leftarrow q_1, \ldots, q_n$, all q_i are true

The XOR-SAT Problem

k-XOR Clauses and Clause Sets

A *k*-XOR clause, C, is a linear equation over the finite field $GF(2)$ using exactly k distinct variables, i.e.,

$$C = ((x_1 \oplus \ldots \oplus x_k) = \varepsilon) \quad \text{where } \varepsilon = 0 \text{ or } 1$$

A *k*-XOR formula (*k*-XOR clause set), φ, is a conjunction of not necessarily distinct *k*-XOR clauses

Truth Assignments for k-XOR Clauses

A truth assignment I is a mapping that assigns 0 or 1 to each variable in its domain

I satisfies an XOR clause $C = ((x_1 \oplus \ldots \oplus x_k) = \varepsilon)$ if and only if

$$I(C) := \left(\sum_{i=1}^{k} I(x_i) \right) \bmod 2 = \varepsilon.$$

I satisfies a formula φ if and only if it satisfies every clause in φ

XOR-SAT problem: Given an XOR formula φ, is there an assignment I which satisfies φ?

How to Solve XOR-SAT?

- Consider φ as a system of linear equations and write it as

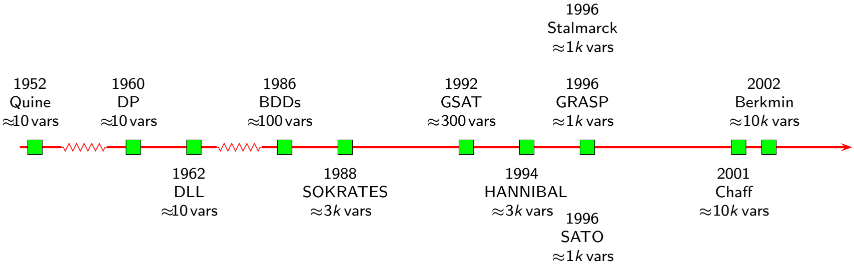
$$\mathbf{S} := (\mathbf{C} \mid \vec{e})$$

- **C** is the coefficient matrix from the lhs of the clauses, $\vec{e}$ is the column vector from the clauses' rhs
- Entries in **S** are 0 or 1
- Bring **S** to echelon form (by applying Gaussian elimination) (Coefficient arithmetic is performed in $GF(2)$!)
- Compute the rank of **S**
- Answer NO if rank of **S** $>$ rank of **C**; answer YES otherwise
- The runtime is in $O(l^2 n)$ (l no of clauses, n no of variables)

Concluding Remarks

- Horn-SAT, 2-SAT, XOR-SAT are easy (polynomial)
- Naive algorithms for Horn-, 2-SAT yield quadratic runtime
- For linear runtime, sophisticated data structures (and some restrictions on the input format) are necessary
- Extensions of the problems lead to NP-complete problems
    - Allow more than one positive literal in Horn clauses
    - Allow more than two literals per clause (yields k-CNF)
    - Mix Horn and 2-CNF
    - etc.

The Evolution of SAT Solving Algorithms



1952
Quine
$\approx$10 vars

1960
DP
$\approx$10 vars

1962
DLL
$\approx$10 vars

1986
BDDs
$\approx$100 vars

1988
SOKRATES
$\approx 3k$ vars

1992
GSAT
$\approx$300 vars

1994
HANNIBAL
$\approx 3k$ vars

1996
Stalmarck
$\approx 1k$ vars

1996
GRASP
$\approx 1k$ vars

1996
SATO
$\approx 1k$ vars

2001
Chaff
$\approx 10k$ vars

2002
Berkmin
$\approx 10k$ vars