Here-and-There with Constraints for Spatial Reasoning

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Abstract. In this paper, we present a possible implementation of the Declarative Spatial Reasoning (DSR) framework into the logic of Here-and-There with Constraints (HTc). The DSR framework enables spatial reasoning by relying on techniques from analytic geometry, where objects are described by means of parameters, and relations are defined through equations and inequalities involving these parameters. Recently, the logic of Here-and-There, which provides a clear and easy characterization of the non-monotonic stable model semantics, has been extended to capture constraints from external theories. Given the relevance of non-monotonicity for spatial reasoning, it is natural to investigate the implementation of DSR into HTc, as it would greatly simplify the formal characterization of DSR in a non-monotonic setting, and facilitate the application of the framework to any field involving spatial reasoning.

Keywords: Here-and-There with Constraints · Declarative Spatial Reasoning framework · Non-monotonic spatial reasoning

1 Introduction

This article integrates the Declarative Spatial Reasoning framework within the logic of Here-and-There with Constraints. The Declarative Spatial Reasoning framework (DSR) [1] is an approach that enables qualitative and quantitative spatial reasoning. It is generally considered as an interesting alternative to the widely used Qualitative Spatial Reasoning framework [4] by allowing more flexibility in the definition of relations, and more adaptability to real-world problems. DSR relies on techniques of analytic geometry for describing objects by means of parameters and relations through constraints involving these parameters. In order to enable non-monotonic reasoning in DSR, the approach has been implemented in the stable model semantics, and more specifically, within the characterization of stable models by means of circumscription [8].

In fact, many characterizations of stable models exist besides those on circumscriptions [3]. The one in terms of Equilibrium Logic is a characterization that offers a great simplicity, as it is based on the intuitionistic logic of Here-and-There [5]. An additional advantage of this approach is its fully logical nature, which means that additional features can be easily defined by extending the logic. Recently, the logic of Here-and-There has been extended to tackle constraints of external theories, resulting in the logic of Here-and-There with Constraints (HTc) [2]. It is therefore natural to investigate the implementation of DSR into this extension, as it would provide a great clarity of the framework, and moreover, an easy way to apply DSR techniques to other fields where non-monotonic spatial reasoning is necessary.

We start in Section 2 by defining spatial domains in line with those described in the DSR framework, with the additional specifications needed for our purposes. In Section 3, we show how a constraint satisfaction problem structure, as those used in HTc, can be constructed based on a spatial domain. Section 4 illustrates how spatial information can be processed in order to form a spatial domain and a constraint satisfaction problem according to it. These structures enable the formation of a theory that can be used to make inferences on the spatial information given. The section ends by showing how the stable models of a theory can be represented in a convenient and useful way. Section 5 concludes the paper and outlines possible future work.

2 Spatial Domains

A spatial domain $\mathfrak{S}$ defines all the object names, geometric shapes and relations, as well as the space considered. Each object name is assigned a geometric shape, and the relations receive their spatial meaning by means of polynomial constraints. More formally, a spatial domain consists of the following structure.

Definition 1. *A spatial domain is a structure* $\mathfrak{S} = \langle \mathcal{O}, \mathcal{G}, f, \mathcal{R}, \mathcal{S} \rangle$ *where*

- $\mathcal{O}$ *is a set of object names,*
- $\mathcal{G}$ *is a set of geometric shapes defined by means of pairs* $(g, \{x, ..., z\})$*,*
- $f : \mathcal{O} \rightarrow \mathcal{G}|_{shape}$ *is a function mapping each object name to a shape,*
- $\mathcal{R}$ *is a set of relations defined by means of pairs*
  $$(r_{g_1...g_k}\ ,\ \{\{s_{1.1},\ s_{1.2}, ...\}, \{s_{2.1},\ s_{2.2}, ...\}, ...\}),$$
- $\mathcal{S}$ *is the space considered.*

We detail each of these elements in turn. Set $\mathcal{O}$ simply contains the object names as symbolic constants. We will usually use the letters $a, b, c, ...$ for the object names.

Set $\mathcal{G}$ of geometric shapes contains pairs of the form

$$(g, \{x, ..., z\}) \tag{1}$$

where g is the name of a geometric shape (e.g. *point*, *circle*, *rectangle*,...), and $\{x, ..., z\}$ is a finite set of variable names. Note that $\mathcal{G}$ is constructed such that there is no pair (g, X) and (g, Y) with $X \neq Y$, that is, each shape is paired with only one set of variables. A geometric shape will sometimes be abbreviated by a single letter, such as 'p' for point, 'c' for circle and so on.

The construction of the set of variables $\{x, ..., z\}$ for a geometric shape relies on the way this shape can be characterized in an analytic geometrical way, that is, by means of parameters that allow the representation of the object in a coordinate system. For example, it is possible to characterize

- a two dimensional (2D) *point* by the set of variables $\{x, y\}$ for its coordinates,
- a *circle* on the plane by the set of variables $\{x, y, r\}$ for the coordinate of its center, and for its radius,
- a *rectangle* on the plane by the set of variables $\{x^{min}, y^{min}, x^{max}, y^{max}\}$ for the coordinates of the bottom left and top right points,
- a *polygon* on the plane by the set of variables $\{x^1, y^1, ..., x^n, y^n\}$ with n points for the vertices of the polygon.

Any other type of geometric shape can be considered, as long as it receives an unambiguous characterization in terms of parameters [6]. Additional shapes can be found in [1].

For each geometric shape g, we define symbol $\overline{g}$ as the set of variables paired with g in set $\mathcal{G}$. For instance, $\overline{circle} = \{x, y, r\}$. Finally, we construct set $\mathcal{G}|_{shape}$, which only contains the names of the shapes in $\mathcal{G}$, as follows.

$$\mathcal{G}|_{shape} = \{g \mid (g, \{x, ..., z\}) \in \mathcal{G}\} \tag{2}$$

Function $f : \mathcal{O} \to \mathcal{G}|_{shape}$ maps each object name $o \in \mathcal{O}$ to a geometric shape $g \in \mathcal{G}|_{shape}$. For example, an object a being a point is specified by $f(a) = point$.

Regarding set $\mathcal{R}$ of relations, it contains pairs of the form

$$(r_{g_1 \dots g_k}, \{\{s_{1.1}, s_{1.2}, ...\}, \{s_{2.1}, s_{2.2}, ...\}, ...\}) \tag{3}$$

where the first element is the name of a relation and the second element is a set of sets containing polynomial equations and inequalities. The sequence $g_1...g_k$ is used as an index of a relation r and corresponds to a tuple $(g_1, ..., g_k) \in \mathcal{G}|^1_{shape} \times ... \times \mathcal{G}|^k_{shape}$ with k the arity of the relation. For the sake of succinctness, we allow removing parenthesis and commas of these tuples, and abbreviate the shape names to a single letter as mentioned above, with no letter similar for two distinct geometric shapes. As a result, possible indexes are '*pp*' for two points, '*pr*' for point-rectangle, etc. These sequences are part of the relation names in order to specify the geometric shapes the relation applies to, and moreover, to distinguish homonym relations that apply to different shapes, e.g., $inside_{pc}$ and $inside_{cc}$. We will simply denote a relation by r instead of $r_{g_1 \dots g_k}$ when the sequence of shapes does not play a role in the formalism.

On the right hand of each pair in $\mathcal{R}$ is a set of sets containing polynomial equations and inequalities. A polynomial equation or inequality s is constructed by means of the usual symbols $<, >, \leq, \geq, =, \neq$, and involves terms that are formed by the usual operators $+$, -, *, **. The variables occurring in these terms are those used to characterize the geometric shapes defined in the spatial domain. More specifically, for a relation $r_{g_1 \dots g_k}$, the constraints paired with it will involve variables among $\overline{g_1} \cup ... \cup \overline{g_k} = \{x_1, ..., z_1, ..., x_k, ..., z_k\}$, where index $i \in [1, k]$ embeds the object order into the variable names while making them unique. As an example, a binary relation over a point and a circle (i.e. with index '*pc*') will be paired with constraints where the variables are $\overline{p}_1 \cup \overline{c}_2 = \{x_1, y_1, x_2, y_2, r_2\}$.

The idea for defining the spatial meaning of relations then follows the Declarative Spatial Reasoning framework, that is, the possible configurations of objects for a relation are described by means of constraints over the parameters of these objects. More specifically, spatial relations are defined by a set of joint conditions (or clauses), each of which corresponds to a single constraint or a disjunction of constraints. This is formally obtained by considering all the subsets in $\{\{s_{1.1}, s_{1.2}, ...\}, \{s_{2.1}, s_{2.2}, ...\}, ...\}$ conjunctively, whereas the constraints inside each subset are considered disjunctively. This construction simply corresponds, as will become clear, to the set-theoretic representation of a conjunctive normal form. Whenever a subset is a singleton, we simply remove the braces around it, i.e., $\{s_1, s_2, ...\}$.

Examples of relations that can be defined in $\mathcal{R}$ based on these ideas are presented below (note the relation $outside_{pr}$ for which a disjunction of constraints is needed):

$$(left_{pp}\ , \{x_1 < x_2\}) \tag{4}$$

$$(strict\_left_{pp}\ , \{x_1 < x_2\ ,\ y_1 = y_2\}) \tag{5}$$

$$(outside_{pr}\ , \{\{x_1 < x_2^{min}\ ,\ y_1 < y_2^{min}\ ,\ x_1 > x_2^{max}\ ,\ y_1 > y_2^{max}\}\}) \tag{6}$$

$$(disconnected_{cc}\ , \{(x_1 - x_2)^2 + (y_1 - y_2)^2 > (r_1 + r_2)^2\}) \tag{7}$$

$$(inside_{cc}\ , \{r_1 < r_2\ ,\ \Big((x_1 - x_2)^2 + (y_1 - y_2)^2 \leq (r_1 - r_2)^2\Big)\}) \tag{8}$$

Although only binary relations are presented, there is no limit to the arity of the relations that can be defined. Additional relations can be found in [1].

In a similar way as above, we define symbol $\overline{r}$ as the set $\{\{s_{1.1}, s_{1.2}, ...\}, \{s_{2.1}, s_{2.2}, ...\}, ...\}$ associated with relation r. We also construct set $\mathcal{R}|_{name}$ of all the relation names occurring in $\mathcal{R}$ as

$$\mathcal{R}|_{name} = \{r_{g_1 \ldots g_k} \mid (r_{g_1 \ldots g_k}, \{\{s_{1.1}, s_{1.2}, ...\}, \{s_{2.1}, s_{2.2}, ...\}, ...\}) \in \mathcal{R}\} \tag{9}$$

Finally, a spatial domain is always defined for a specific space $\mathcal{S}$ (e.g. $\mathbb{R}^2$, $\mathbb{R}^3$, etc), as the characterization of objects and relations depends on it. In our case, we only consider the two dimensional Euclidean space $\mathbb{R}^2$ and make it implicit in subsequent spatial domains. However, it is worth noting that there is no formal limit to develop our approach in a three-dimensional space, and even with the addition of time [7].

In general, the definition of geometric shapes and relations will remain the same among spatial domains defined with the same space $\mathcal{S}$. What will change from one structure to another is the set of shapes and relations that are considered, that is, the extensions of $\mathcal{G}$ and $\mathcal{R}$. Set $\mathcal{O}$ of object names will also change for each structure.

3 The Logic of *Here-and-There with Constraints* over Spatial Domains

The logic of Here-and-There with Constraints (HTc) extends the intuitionistic logic of Here-and-There in order to capture constraints expressed in one or more external theories. The syntax of HTc is based on a Constraint Satisfaction Problem (CSP), and its semantics is characterized through a denotational approach.

In this section, we first describe how to construct the CSP of a given spatial domain $\mathfrak{S}$. As will be seen, this construction is straightforward in the sense that each spatial domain $\mathfrak{S}$ will always produce the same CSP. Formulas can be formed based on the elements described in a CSP. Secondly, we describe how these formulas can be evaluated.

3.1 Syntax

We recall that a Constraint Satisfaction Problem (CSP) is a structure $\langle \mathcal{X}, \mathcal{D}, \mathcal{C} \rangle$, where $\mathcal{X}$ is a set of variables, $\mathcal{D}$ is the domain of the variables, and $\mathcal{C}$ is a set of constraints that limit the values from $\mathcal{D}$ that can be taken by the variables in $\mathcal{X}$.

We also adopt a structure $\langle \mathcal{X}, \mathcal{D}, \mathcal{C} \rangle$, with the particularity of constructing it based on the elements of the spatial domain. For doing so, it is first needed to form the Herbrand Base and the set of parameters relatively to this spatial domain.

The Herbrand Base $\mathcal{A}$ contains all the relations $r \in \mathcal{R}|_{name}$ grounded with elements $o \in \mathcal{O}$ that respect the geometric shapes of the relations. This corresponds to construct set $\mathcal{A}$ as

$$\begin{aligned} \mathcal{A} = \{ r_{g_1 \dots g_k}(o_1, ..., o_k) \quad | \quad & r_{g_1 \dots g_k} \in \mathcal{R}|_{name} \ , \ o_1, ..., o_k \in \mathcal{O} \\ & \mathit{and} \ f(o_1)...f(o_k) = g_1...g_k \} \end{aligned} \tag{10}$$

For instance, with $\mathcal{R}|_{name} = \{\mathit{left}_{pp}, \mathit{inside}_{pc}\}$, the set of objects $\mathcal{O} = \{a, b\}$, and the function f defined as $f(a) = \mathit{point}$ and $f(b) = \mathit{circle}$, we form the Herbrand Base $\mathcal{A} = \{\mathit{left}_{pp}(a, a) \ , \ \mathit{inside}_{pc}(a, b)\}$.

Set $\mathcal{P}$ of parameters contains, for each object in the domain, the variables associated with the shape of the object, but augmented with the name of the object as an index. More precisely, considering a set of object names $\mathcal{O}$ and a function f, the corresponding set $\mathcal{P}$ of parameters is constructed as follows.

$$\mathcal{P} = \{x_o, ..., z_o \mid o \in \mathcal{O} \ \mathit{and} \ (x, ..., z) = \overline{f(o)}\} \tag{11}$$

Note that $\overline{f(o)}$ combines the two functions defined so far, with priority given to the function f that returns the geometric shape of object o. As an illustration, consider a set $\mathcal{O} = \{a, b\}$ and a function f defined as $f(a) = \mathit{point}$ and $f(b) = \mathit{circle}$. The sets of variables once corresponds to $\overline{f(a)} = \overline{\mathit{point}} = \{x, y\}$ and $\overline{f(b)} = \overline{\mathit{circle}} = \{x, y, r\}$. The resulting set of parameters is $\mathcal{P} = \{x_a, y_a, x_b, y_b, r_b\}$. Note that following the construction of set $\mathcal{P}$, no two parameters can receive the same name.

The Herbrand Base and the set of parameters constructed on a spatial domain enable us to define the constraint satisfaction problem of this spatial domain as follows.

Definition 2. *Given a spatial domain $\mathfrak{S}$, we construct the Constraint Satisfaction Problem of $\mathfrak{S}$, noted $\langle \mathcal{X}, \mathcal{D}, \mathcal{C} \rangle_{\mathfrak{S}}$, by forming the sets*

1. $\mathcal{X} = \mathcal{A} \cup \mathcal{P}$ containing the Hebrand Base and the set of parameters,
2. $\mathcal{D} = \{\mathbf{t}\} \cup \mathbb{S}$ containing the special symbol $\mathbf{t}$ for true, as well as the base of space $\mathcal{S}$ (here considered as $\mathbb{R}$ for simplicity)
3. $\mathcal{C} = \mathcal{A} \cup \mathcal{T}$ composed by a set of regular atoms and a set of theory atoms.

The main specificity of a CSP in HTc, compared to a classical CSP, is that a constraint $C \in \mathcal{C}$ is called an *atom* (or *constraint atom*), since it corresponds to an atomic formula in the logic. More specifically for our approach, a constraint atom $C \in \mathcal{C}$ is either a regular atom $\mathtt{a} \in \mathcal{A}$ or a theory atom $\mathtt{s} \in \mathcal{T}$. We define each of these atoms in turn.

The set of regular atoms is composed by all the atoms of the form ($\mathtt{a} = \mathbf{t}$), where $\mathtt{a}$ is an element of the Herbrand Base. An example of regular atom is $(\mathit{left}(a, b) = \mathbf{t})$. We often abbreviate regular atoms by only conserving the element of the Herbrand Base, that is, $\mathit{left}(a, b)$. This abbreviation justifies why set $\mathcal{A}$ can be used both in sets $\mathcal{X}$ and $\mathcal{C}$ of the CSP.

The set of theory atoms is composed by all the equations and inequalities occurring in $\mathcal{R}$, but where the parameters have been associated with the objects from $\mathcal{O}$. Note that similarly to the formation of the Herbrand base, these replacements have to respect the geometric shapes the relations are about. We can give a formal description of the construction of $\mathcal{T}$ by first describing the equations and inequalities it would contain for a single relation $r_{g_1 \dots g_k}$, this set being noted $\mathcal{T}|_{r_{g_1 \dots g_k}}$. Considering a pair $(r_{g_1 \dots g_k}, \{\{s_{1.1}, s_{1.2}, ...\}, \{s_{2.1}, s_{2.2}, ...\}, ...\}) \in \mathcal{R}$, a set $\mathcal{O}$ of object names, and a function f mapping a geometric shape to each object, the set of theory atoms constructed for $r_{g_1 \dots g_k}$ is

$$\mathcal{T}|_{r_{g_1 \cdots g_k}} = \bigcup_{s \in A,\ A \in \overline{r}} s\ [x_1/x_{o_1}, ..., z_1/z_{o_1}]...[x_k\ /\ x_{o_k}, ..., z_k/z_{o_k}] \tag{12}$$

$$\textit{with } o_1, ..., o_k \in \mathcal{O}, \quad \textit{and } f(o_1)...f(o_k) = g_1...g_k$$

Note that the subset relationship from $\overline{r}$ does not appear in the construction of $\mathcal{T}|_r$, as it is formed with all $s \in A$ and $A \in \overline{r}$. As an example, consider the relation $(\mathit{left}_{pp}, \{x_1 < x_2\})$ and the set $\mathcal{O} = \{a, b, c\}$ composed of the two points a, b, and circle c. The four different theory atoms that can be formed for $\mathcal{T}|_{\mathit{left}_{pp}}$ are $x_a < x_a$, $x_a < x_b$, $x_b < x_a$ and $x_b < x_b$.

The whole set of theory atoms is simply obtained by creating a set of theory atoms for each relation in $\mathcal{R}$, and then taking the reunion of all these sets. Formally,

$$\mathcal{T} = \bigcup_{r \in \mathcal{R}|_{name}} \mathcal{T}|_r \tag{13}$$

It is clear that a CSP in our spatial approach is defined by means of two sub CSPs, that is, $\langle \mathcal{A}, \{\mathbf{t}\}, \mathcal{A} \rangle$ and $\langle \mathcal{P}, \mathbb{R}, \mathcal{T} \rangle$. The interaction between these two sub CSPs is made possible by constructing *formulas*, which can connect regular and theory atoms from $\mathcal{C}$ by means of the usual logical connectives $\wedge, \vee, \rightarrow$ and $\bot$. The connectives $\phi \leftrightarrow \psi \stackrel{def}{=} (\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$ and $\neg\phi \stackrel{def}{=} \phi \rightarrow \bot$ are defined as usual. Finally, a set of formulas is called a *theory* and is denoted Γ.

3.2 Semantics

Following the HTc semantics, we define a *partial valuation* v as a function $v : \mathcal{X} \rightarrow \mathcal{D} \cup \{\mathbf{u}\}$ that assigns to elements of $\mathcal{X}$, either an element of $\mathcal{D}$ respecting the domain of the variable (i.e. $\mathcal{A} \rightarrow \{\mathbf{t}\}$ and $\mathcal{P} \rightarrow \mathbb{R}$), or the specific symbol $\mathbf{u}$ standing for "undefined". Note that symbol $\mathbf{u} \notin \mathcal{D}$ is not an element of the domain.

A valuation can also be seen as a set that does not contain any two pairs (x, d) and (x, e) with $d \neq e$, where $x \in \mathcal{X}$ and $d, e \in \mathcal{D}$. Also, a valuation as a set does not include any pair $(x, .)$ where $v(x) = \mathbf{u}$. As an example, consider valuation $v_1(x_a) = 2$, $v_1(x_b) = \mathbf{u}$, $v_1(left(a, b)) = \mathbf{t}$. This valuation corresponds to set $v_1 = \{(x_a, 2),\ left(a, b)\}$ where no pair appears for the variable x_b which is undefined. Note that we also apply the abbreviation mentioned above for writing regular atoms in valuations, that is, we simply write $left(a, b)$ instead of $(left(a, b), \mathbf{t})$. Finally, we denote the set of all valuations for a CSP $\langle \mathcal{X}, \mathcal{D}, \mathcal{C} \rangle_{\mathfrak{S}}$ as $\mathcal{V}_{\mathcal{X},\mathcal{D}}$ and remove the subindices when clear from the context.

In order to decide the valuations that satisfy a constraint atom C, we use a *denotation* function of the form $[\![\cdot]\!] : \mathcal{C} \rightarrow 2^{\mathcal{V}}$ mapping each atom C to a set of partial valuations. Examples of denotation for theory atoms $\mathsf{s} \in \mathcal{T}$ are

$$[\![x_{o_1} < x_{o_2}]\!] \stackrel{def}{=} \{v \in \mathcal{V} \mid \ v(x_{o_1}) < v(x_{o_2}), \quad v(x_{o_1}), v(x_{o_2}) \in \mathbb{R}\} \tag{14}$$

$$[\![(x_{o_1} - r_{o_1}) = z_{o_2}]\!] \stackrel{def}{=} \{v \in \mathcal{V} \mid \ (v(x_{o_1}) - v(r_{o_1})) = v(z_{o_2}), \quad v(x_{o_1}), v(r_{o_1}), v(z_{o_2}) \in \mathbb{R}\} \tag{15}$$

$$[\![x_{o_1} \geq x_{o_2}]\!] \stackrel{def}{=} \{v \in \mathcal{V} \mid \ v(x_{o_1}) \geq v(x_{o_2}), \quad v(x_{o_1}), v(x_{o_2}) \in \mathbb{R}\} \tag{16}$$

where every $x_{o_i}, ..., z_{o_i} \in \mathcal{P}$. Note that a valuation can only satisfy a constraint C if none of the variables occurring in this constraint are assigned the value $\mathbf{u}$. This restriction corresponds to the condition $v(x_{o_i}), ..., v(z_{o_i}) \in \mathbb{R}$ occurring for each denotation. The relations $<, >, \leq, \geq, =, \neq$ used in the constraints follow their usual mathematical semantics.

Regarding regular atoms in $\mathcal{A}$, their denotations always have the same general form

$$[\![r(o_1, ..., o_k)]\!] \stackrel{def}{=} \{v \in \mathcal{V} \mid \ v(r(o_1, ..., o_k)) = \mathbf{t}\} \tag{17}$$

where k is the arity of the relation, and $r(o_1, ..., o_k)$ corresponds to an element $\mathsf{a} \in \mathcal{A}$. As a general remark, note that the denotation of an atom lets valuations vary freely on the variables not occurring in the atom.

We define an *interpretation* as a pair $\langle h, t \rangle$ of partial valuations such that $h \subset t$. The satisfaction relation is defined as follows.

Definition 3. *An interpretation $\langle h, t\rangle$ satisfies a formula ϕ, written $\langle h, t\rangle \models \phi$, if:*

1. $\langle h, t\rangle \not\models \bot$
2. $\langle h, t\rangle \models C$ *iff* $h \in [\![C]\!]$
3. $\langle h, t\rangle \models \varphi \wedge \psi$ *iff* $\langle h, t\rangle \models \varphi$ *and* $\langle h, t\rangle \models \psi$
4. $\langle h, t\rangle \models \varphi \vee \psi$ *iff* $\langle h, t\rangle \models \varphi$ *or* $\langle h, t\rangle \models \psi$
5. $\langle h, t\rangle \models \varphi \rightarrow \psi$ *iff for both* $v = h$ *and* $v = t$ *it holds:* $\langle v, t\rangle \not\models \varphi$ *or* $\langle v, t\rangle \models \psi$

The definition of an equilibrium model follows in a straightforward way.

Definition 4. *The equilibrium model of a theory Γ is an interpretation such that:*

1. $\langle t, t\rangle \models \Gamma$
2. *there is no* $h \subset t$ *such that* $\langle h, t\rangle \models \Gamma$

In this case, valuation t will also be called a *stable model* of program Γ. We note $\mathcal{M}$ the set of all the stable models for a theory Γ, that is, all the valuations that satisfy the formulas of the theory.

4 Theories of Spatial Information

Spatial information can be represented by a set Π containing the pieces of information $\pi^1, ..., \pi^\lambda$, where λ specifies the size of Π. Elements of Π can be seen as the premises of a problem or the known facts about a situation. Each $\pi^i \in \Pi$ corresponds to a relational expression of the form $r^i(o^i_1, ..., o^i_k)$, where r^i is the name of a relation and $o^i_1, ..., o^i_k$ are the names of the objects stated in the relation, with k its arity.

As an example of spatial information, consider the description

" Circle A is inside Circle B,
Circle B is disconnected from Circle C,
Circle C is smaller than Circle B."

The formalized version of this description corresponds to the following set

$$\Pi_1 = \{inside_{cc}(a, b), \quad dc_{cc}(b, c), \quad smaller_{cc}(c, b)\} \tag{18}$$

where dc stands for the relation *disconnected*. For some spatial information, the choice for the geometric shapes of the objects or the exact relations stated will not be as explicit as in our example. We do not cover such difficulties as it would involve linguistic knowledge that would deviate us from the main goal of this paper. For our purpose, we will only consider spatial information where the geometric shapes of the objects and the relations are clearly specified.

Processing and reasoning on the spatial information in Π first requires to define the spatial domain associated with it, which is presented next.

4.1 The Spatial Domain of Spatial Information, and its CSP

Considering a set of spatial information $\Pi = \{\pi^1, ..., \pi^\lambda\}$, where each $\pi^i = r^i(o^i_1, ..., o^i_k)$, the spatial domain $\mathfrak{S} = \langle \mathcal{O}, \mathcal{G}, f, \mathcal{R} \rangle$ is built as

- $\mathcal{O} = \{o^1_1, ..., o^1_k\} \cup ... \cup \{o^\lambda_1, ..., o^\lambda_k\}$ containing all the object names occurring in Π,
- $\mathcal{G}$ containing all the pairs that define the geometric shapes relevant for the problem,
- $f : \mathcal{O} \rightarrow \mathcal{G}|_{shape}$ associating each object name $o \in \mathcal{O}$ with a geometric shape $g \in \mathcal{G}|_{shape}$,
- $\mathcal{R}$ containing all the pairs that define the relations, where $\mathcal{R}|_{name} = \{r^1\} \cup ... \cup \{r^\lambda\}$ contains all the relation names occurring in Π.

Note that most of the object names will appear several times in the spatial information. Also, function f is entirely defined within Π, since the indexes of the relations specify the geometric shape of the objects.

Considering the spatial description above and its formal version Π_1, we construct the following spatial domain $\mathfrak{S}$:

- $\mathcal{O} = \{a, b, c\}$
- $\mathcal{G} = \{(circle, \{x, y, r\})\}$
- $f(a) = f(b) = f(c) = circle$
- $\mathcal{R} = \left\{ \begin{array}{ll} \big(dc_{cc} & , \{(x_1 - x_2)^2 + (y_1 - y_2)^2 > (r_1 + r_2)^2\}\big), \\ \big(inside_{cc} & , \{r_1 < r_2, ((x_1 - x_2)^2 + (y_1 - y_2)^2 \leq (r_1 - r_2)^2)\}\big) \\ \big(smaller_{cc} & , \{r_1 < r_2\}\big) \end{array} \right\}$

As explained in the previous section, once the spatial domain is defined, the construction of the corresponding constraint satisfaction problem is straightforward. For a set of formalized spatial information Π and a spatial domain $\mathfrak{S}$, we note the CSP constructed as $\langle \mathcal{X}, \mathcal{D}, \mathcal{C} \rangle^{\Pi}_{\mathfrak{S}}$. For instance, the structure constructed for spatial information Π_1 is noted $\langle \mathcal{X}, \mathcal{D}, \mathcal{C} \rangle^{\Pi_1}_{\mathfrak{S}}$.

It is worth noticing that a structure $\langle \mathcal{X}, \mathcal{D}, \mathcal{C} \rangle^{\Pi}_{\mathfrak{S}}$ will always be determined by set Π, since the latter enables the construction of a spatial domain in a quite straightforward way, and that the CSP directly follows from this spatial domain. It is now possible to construct a spatial theory based on this structure.

4.2 Spatial Theories

As a recall, a theory is simply a set of formulas Γ constructed over a structure $\langle \mathcal{X}, \mathcal{D}, \mathcal{C} \rangle^{\Pi}_{\mathfrak{S}}$. To ease its presentation, we more specifically define a spatial theory as a set $\Gamma = \Pi \cup \mathcal{E}$, where Π contains the formulas standing for the spatial information known and $\mathcal{E}$ is a set containing formulas defining the spatial meaning of each relation.

We start by describing set $\mathcal{E}$, as it contains the spatial meaning of all the relations used in Γ. Set $\mathcal{E}$ contains, for each regular atom $\mathtt{a} \in \mathcal{A}$, an equivalence formula that connects this regular atom to one or more theory atoms $\mathtt{s} \in \mathcal{T}$. The equivalence formula constructed for an atom $\mathtt{a}$ is noted $\varepsilon_{\mathtt{a}}$ and corresponds to

$$\mathtt{a} \leftrightarrow \big((\mathtt{s}_{1.1} \vee \mathtt{s}_{1.2} \vee ...) \wedge (\mathtt{s}_{2.1} \vee \mathtt{s}_{2.2} \vee ...) \wedge ...\big) \tag{19}$$

where $\mathtt{a} = r_{o_1,\ldots,o_k}$, each $\mathtt{s}_{i,j} = s_{i,j}[x_1/x_{o_1},\ldots,z_1/z_{o_1}]\ldots[x_k/x_{o_k},\ldots,z_k/z_{o_k}]$, and with $s_{i,j} \in A_i$ and $A_i \in \overline{r}$. As can be seen, these equivalence formulas are the exact point where the two sub-CSPs of our approach become intertwined, since the satisfiability of a regular atom becomes linked to the satisfiability of theory atoms. For instance, with the structure $\langle \mathcal{X},\mathcal{D},\mathcal{C}\rangle^{\Pi_1}_{\mathfrak{S}}$ defined above and the ground relation $\mathit{inside}_{cc}(a,c)$ from $\mathcal{A}$, the equivalence formula $\varepsilon_{\mathit{inside}_{cc}(a,c)}$ is

$$\mathit{inside}_{cc}(a,c) \quad \leftrightarrow \quad r_a < r_c \wedge \Big((x_a - x_c)^2 + (y_a - y_c)^2 \leq (r_a - r_c)^2\Big) \tag{20}$$

Set $\mathcal{E}$ is then simply obtained by creating an equivalence formula for each regular atom $\mathtt{a} \in \mathcal{A}$, and combining all these formulas, that is,

$$\mathcal{E} = \bigcup_{\mathtt{a}\in\mathcal{A}} \varepsilon_{\mathtt{a}} \tag{21}$$

It is worth noting that $\mathcal{E}$ is constructed based on all the relations occurring in set $\mathcal{A}$ of the CSP, and not only those occurring in Π.

By integrating set $\mathcal{E}$ within a spatial theory, any valuation v assigning the value $\mathbf{t}$ to a relation without satisfying its corresponding theory atoms, and vice versa, is rejected as a potential stable model of the theory. Only valuations where both a relation and its theory atoms are simultaneously true or not will be considered potential stable models. Moreover, the minimality conditions specific to the stable model semantics will make the case where both sides of the equivalence are false as the only possible stable model, at least when formulas of $\mathcal{E}$ only are considered. As a result of this minimality, the formulas in $\mathcal{E}$ alone have no direct effect on the regular atoms that can be deduced, and their single stable models will always be the empty set where no value is assigned to any of the variables.

Regarding set Π in the definition of a theory $\Gamma = \Pi \cup \mathcal{E}$, it exactly corresponds to the set constructed for the formalization of the spatial information, with the difference that elements of Π now correspond to regular atoms of the structure $\langle \mathcal{X},\mathcal{D},\mathcal{C}\rangle^{\Pi}_{\mathfrak{S}}$. Therefore, introducing Π into a theory Γ amounts to declaring that all the corresponding elements of $\mathcal{A}$ must be satisfied. This integration represents the fact that the pieces of information given should be satisfied in the stable models of the theory. In more general terms, for a set Π and a structure $\langle \mathcal{X},\mathcal{D},\mathcal{C}\rangle^{\Pi}_{\mathfrak{S}}$ constructed based on it, set Π will always represent a subset of all the ground relations, that is, $\Pi \subseteq \mathcal{A}$.

When Π is considered in combination with $\mathcal{E}$ in a theory Γ, all the equivalences from $\mathcal{E}$ that involve the ground relations mentioned in Π can no longer be satisfied by letting its elements undefined, since Π forces the regular atoms to be true. As a result, all the parameters involved in the equations and inequalities of the corresponding relations must receive a value, and moreover, a value that satisfy the equations and inequalities.

As an example, let us consider again theory $\Gamma_1 = \Pi_1 \cup \mathcal{E}_1$, where $\Pi_1 = \{\mathit{inside}_{cc}(a,b),\ \mathit{dc}_{cc}(b,c),\ \mathit{smaller}_{cc}(c,b)\}$ and $\mathcal{E}_1$ contains the formulas for the spatial meaning of relations as defined above. The stable models of theory Γ_1 constructed over $\langle \mathcal{X}, \mathcal{D}, \mathcal{C} \rangle_{\mathfrak{S}}^{\Pi_1}$ correspond to all the valuations $v \in \mathcal{V}$ where $v(\mathit{inside}_{cc}(a,b)) = \mathbf{t}$, $v(\mathit{dc}_{cc}(b,c)) = \mathbf{t}$ and $v(\mathit{smaller}_{cc}(c,b)) = \mathbf{t}$, and moreover, where all the corresponding parameters have received a value that satisfy the equations and inequalities of these relations. In a set-theoretic notation, this corresponds to all the valuations v where $\{\mathit{inside}_{cc}(a,b), \mathit{dc}_{cc}(b,c), \mathit{smaller}_{cc}(c,b)\} \subseteq v$, and also containing the pairs (x,d) with $x \in \mathcal{P}$ and $d \in \mathbb{R}$ wrt the equations and inequalities that must be satisfied. We continue by introducing a useful distinction that will ease the presentation of stable models.

4.3 Classes and Instances in Stable Models

Let us consider a structure $\langle \mathcal{X}, \mathcal{D}, \mathcal{C} \rangle_{\mathfrak{S}}^{\Pi}$ and a theory Γ constructed based on it. For a stable model t of theory Γ, we define the *class* of t as the set $A_t = t \cap \mathcal{A}$ containing only its regular atoms. It is relevant to notice that the class of a stable model is always a subset of the Herbrand Base, that is, $A \subseteq \mathcal{A}$. Similarly, we isolate the assignments for parameters contained in a stable model t by defining set $I_t = t \backslash \mathcal{A}$. For a stable model t, we call I_t an *instance* of class A_t, as it contains the values of the parameters that satisfy all the theory atoms associated with the elements of class A. It is clear that in our definition of a CSP, we have that $A_t \cup\ I_t = t$, and that $A_t \cap\ I_t = \emptyset$, for any stable model t. For both symbols A_t and I_t, we elude writing subscript t when clear from the context.

Considering set $\mathcal{M}$ containing all the stable models of a theory Γ over a structure $\langle \mathcal{X}, \mathcal{D}, \mathcal{C} \rangle_{\mathfrak{S}}^{\Pi}$, we collect all the different classes present in the different stable models by means of set $\mathfrak{C} = \{A_t \mid t \in \mathcal{M}\ \}$. Regarding instances, we will be more interested in obtaining all the instances of a class, rather than all the instances in $\mathcal{M}$ in general. Therefore, we construct the set of all the instances contained in $\mathcal{M}$ for a class $A \in \mathfrak{C}$ by the set $A^{\mathcal{I}} = \{\ I_t \mid t \in \mathcal{M},\ A_t = A\}$. We denote a specific instance of a class A by means of A^{I_i}, where $A^{I_i} \in A^{\mathcal{I}}$, and simply write A^I when we refer to the first instance A^{I_1} of a class.

We illustrate these concepts in Table 1 by showing the stable models contained in $\mathcal{M}_1$ for theory Γ_1. Each row corresponds to a stable model, with the class A_i on the left separated from its instances $A_i^{I_j}$ on the right. Classes are written only once in order to distinct them more easily, and the premises stated in Π are written in bold.

As can be seen, only three different classes compose the stable models of theory Γ_1, but an infinite number of instances exist for each of these classes, as suggested by the ellipsis. In general, it is clear that any class A will always have an infinite number of instances when working in dense spaces. On the contrary, there will only be a finite number of classes.

Table 1. The three classes and some of their instances for the stable models of theory Γ_1. The relations used in the set of premises Π are in bold. Those appearing in all the classes, but not in the premises, are underlined.

#	Classes	Instances	#
A_1	$\mathbf{inside(a,b)}$, $\mathbf{dc(b,c)}$, $\mathbf{smaller(c,b)}$, $\underline{dc(a,c)}$, $\underline{dc(c,b)}$, $\underline{dc(c,a)}$, $\underline{smaller(a,b)}$ $smaller(a,c)$	$x_a = 3$, $y_a = 4$, $r_a = 1$, $x_b = 3$, $y_b = 4$, $r_b = 2.1$, $x_c = 7$, $y_c = 5$, $r_c = 1.4$	$A_1^{I_1}$
		$x_a = 3$, $y_a = 4$, $r_a = 1$, $x_b = 3$, $y_b = 4$, $r_b = 2.1$, $x_c = 7$, $y_c = 5$, $r_c = 1.5$	$A_1^{I_2}$
		...	...
A_2	$\mathbf{inside(a,b)}$, $\mathbf{dc(b,c)}$, $\mathbf{smaller(c,b)}$, $\underline{dc(a,c)}$, $\underline{dc(c,b)}$, $\underline{dc(c,a)}$, $\underline{smaller(a,b)}$ $smaller(c,a)$	$x_a = 3$, $y_a = 4$, $r_a = 1$, $x_b = 3$, $y_b = 4$, $r_b = 2.1$, $x_c = 7$, $y_c = 5$, $r_c = 0.7$	$A_2^{I_1}$
		...	...
A_3	$\mathbf{inside(a,b)}$, $\mathbf{dc(b,c)}$, $\mathbf{smaller(c,b)}$, $\underline{dc(a,c)}$, $\underline{dc(c,b)}$, $\underline{dc(c,a)}$, $\underline{smaller(a,b)}$	$x_a = 3$, $y_a = 4$, $r_a = 1$, $x_b = 3$, $y_b = 4$, $r_b = 2.1$, $x_c = 7$, $y_c = 5$, $r_c = 1$	$A_3^{I_1}$
		...	...

4.4 Reasoning within a Spatial Theory

In our example, each class contains at least the elements of Π_1 as expected, but also additional relations that were not stated in the premises. More interestingly, some of these new relations appear in all the classes (i.e. the four relations $dc(a,c)$, $dc(c,b)$, $dc(c,a)$, $smaller(a,b)$ that are underlined in Table 1). These relations correspond to pieces of information that are logically implied by the stated ones. Indeed, when a new relation appears in all of the possible classes of the stable models of a theory, it means that regardless of the assignments that have been found for the satisfaction of the premises, each of these assignments have always satisfied the theory atoms of the new relation. In other words, the values given to the parameters turn out to also satisfy the equations and inequalities of other relations in $\mathcal{A}$. This indicates that some pieces of spatial information were implicit in the given set of premises, and that they logically follow from them.

Formally, if we consider a theory Γ over a structure $\langle \mathcal{X}, \mathcal{D}, \mathcal{C} \rangle_{\mathfrak{S}}^{\Pi}$ and a set $\mathfrak{C}$ containing the different classes of the stable models of Γ, we form the set $Cons(\Gamma)$ containing all the ground relations that are consequences of theory Γ as

$$Cons(\Gamma) = A_1 \cap ... \cap A_n \setminus \Pi \qquad with\ A_i \in \mathfrak{C} \tag{22}$$

Note that Π is directly removed from $Cons(\Gamma)$, as the relations in it were explicitly given and do not reflect the idea of deduction. For each atom $\mathtt{a} \in Cons(\Gamma)$, we say that $\mathtt{a}$ is a *consequential constraint* of theory Γ under the structure $\langle \mathcal{X}, \mathcal{D}, \mathcal{C} \rangle^{\Pi}_{\mathfrak{S}}$. Since the spatial domain suffices to fully characterize the structure $\langle \mathcal{X}, \mathcal{D}, \mathcal{C} \rangle^{\Pi}_{\mathfrak{S}}$, and that Π is contained in Γ, a consequential constraint can alternatively be written as

$$\Gamma \vdash_{\mathfrak{S}} \mathtt{a} \tag{23}$$

Regarding set $\mathcal{M}_1$ of stable models for theory Γ_1, four new relations are common to the three classes, namely, $dc(a, c)$, $dc(c, b)$, $dc(c, a)$ and $smaller(a, b)$. This means that the spatial information in Π_1 cannot be stated without making these additional relations true.

On the contrary, when no possible valuation can be found for the theory atoms corresponding to the relations stated in Π, the theory is said to be *unsatisfiable*. In such a case, set $\mathfrak{C}$ of classes is simply empty, and no stable model can be found. This can be written as

$$\Gamma \vdash_{\mathfrak{S}} \bot \tag{24}$$

4.5 Graph of a Class, Diagrams of its Instances

Each stable model t contains a class A and an instance I of this class, and both of them can be represented in a more convenient way than by sets.

When only binary relations compose the spatial domain, we can represent the class A of a stable model by means of a *graph* $G = (\mathcal{O}, A)$, where $\mathcal{O}$ is used for the set of nodes and A contains all the pairs for the edges. For each relation $r(o_1, o_2) \in A$, the name of the relation r is used as the label of an edge that goes from o_1 to o_2. The graphs of the three classes in Table 1 are represented in Figure 1 on the left.

Regarding the representation of instances, the parameters of objects can be used to represent these objects in a coordinate system, as specified in the definition of spatial domains. We call this representation of an instance, a *diagram*. The diagrams for the first instances of each class in Table 1 are represented in Figure 1 on the right.

It may happen that some sets of spatial information Π create a theory for which not all the parameters are assigned a value in the stable models. This is the case, for instance, with the very simple set of spatial information $\Pi = \{smaller_{cc}(a, b)\}$, where no value will be provided for x_a, y_a, x_b and y_b in the stable models. These cases occur due to the minimality conditions combined with the fact that the definitions of relations in $\mathcal{E}$ do not necessarily involve all the parameters associated to the objects in the equations and inequalities.

From a semantic point of view, it does not create any problem because valuations in HTc are defined as partial functions. However, one may want to obtain a value for each parameter in order to construct the diagrams of the instances occurring in the stable models. These values can be obtained by simply adding to the spatial theory, an axiom of the form

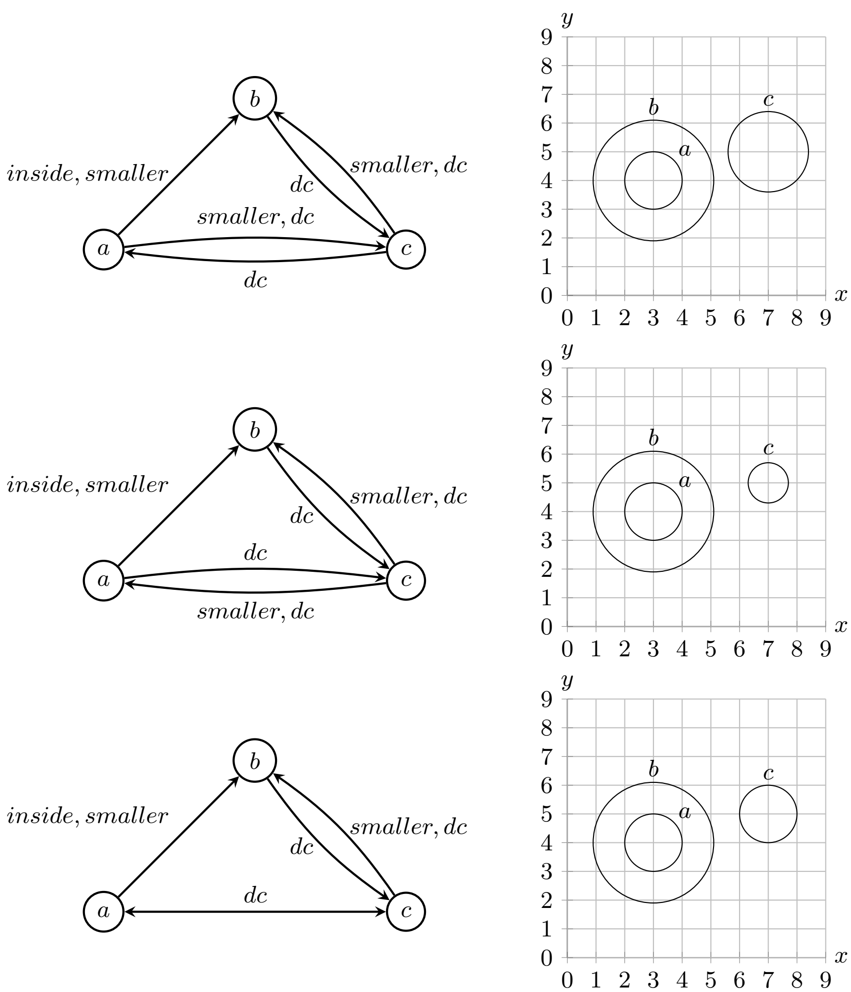

Fig. 1. Representations of the classes and instances of theory Γ_1 by means of graphs and diagrams.

$$x = x \qquad for\ each\ x \in \mathcal{P} \tag{25}$$

By means of (25), we ensure that variable x cannot remain undefined in the stable models, as the denotation of $x = x$ requires x to have a value in order to satisfy the constraint.

In fact, it is even possible to ground only a specific object $o \in \mathcal{O}$ if the relation *equal* is defined in the spatial domain, and consequently, appears in set $\mathcal{E}$. Whatever the geometric shapes of the objects, the relation *equal* will always generate equivalence formulas in $\mathcal{E}$ of the form

$$equal(o, o) \leftrightarrow x_o = x_o \wedge ... \wedge z_o = z_o \tag{26}$$

where $x_o, ..., z_o$ are all the parameters associated with object o. Each time the relation *equal* is stated in set Π for a certain object, all the parameters of this object will receive a value in the stable models. However, stating this relation will never change the spatial information given.

5 Conclusion

In this article, we have investigated a possible implementation of the Declarative Spatial Reasoning framework into the logic of Here-and-There with Constraints. Given a set of spatial information, our approach formally describes how the constraint satisfaction problem can be constructed relatively to this spatial information. This structure can then be used to define theories and apply the techniques developed within the Declarative Spatial Reasoning framework in order to make inferences about this spatial information.

As previously announced, one of the main goals of this implementation is to embed the Declarative Spatial Reasoning framework into a clear and simple logical characterization that also allows non-monotonicity. Indeed, non-monotonicity is essential to the field of spatial reasoning, as it allows expressing preferences, setting default values, or declaring inertia rules. A possible application of the current research is to add to a theory Γ, an additional subset that contains preferences expressed by means of strong and default negations. This will allow some models to be chosen over others (e.g. for modeling the psychology of diagrammatic reasoning), without preventing any default relations from being falsified by additional information.

Finally, it is worth mentioning that extending HTc with new syntactic constructions and possibilities is an active trend of research in the community. Any formal extension of HTc will automatically also extend the expressive possibilities of spatial theories presented here. An essential extension to consider for future work is the combination of HTc with temporal logics, as it would provide a convenient formalism for reasoning about spatiotemporal information.

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