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# ON EQUIVALENCES IN ANSWER-SET PROGRAMMING BY COUNTERMODELS IN THE LOGIC OF HERE-AND-THERE

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# On Equivalences in Answer-Set Programming by Countermodels in the Logic of Here-and-There

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**Abstract.** Different notions of equivalence, such as the prominent notions of strong and uniform equivalence, have been studied in Answer-Set Programming, mainly for the purpose of identifying programs that can serve as substitutes without altering the semantics, for instance in program optimization. Such semantic comparisons are usually characterized by various selections of models in the logic of Here-and-There (HT). For uniform equivalence however, correct characterizations in terms of HT-models can only be obtained for finite theories, respectively programs. In this article, we show that a selection of countermodels in HT captures uniform equivalence also for infinite theories. This result is turned into coherent characterizations of the different notions of equivalence by countermodels, as well as by a mixture of HT-models and countermodels (so-called equivalence interpretations). Moreover, we generalize the so-called notion of relativized hyperequivalence for programs to propositional theories, and apply the same methodology in order to obtain a semantic characterization which is amenable to infinite settings. This allows for a lifting of the results to first-order theories under a very general semantics given in terms of a quantified version of HT. We thus obtain a general framework for the study of various notions of equivalence for theories under answer-set semantics. Moreover, we prove an expedient property that allows for a simplified treatment of extended signatures, and provide further results for non-ground logic programs. In particular, uniform equivalence coincides under open and ordinary answer-set semantics, and for finite non-ground programs under these semantics, also the usual characterization of uniform equivalence in terms of maximal and total HT-models of the grounding is correct, even for infinite domains, when corresponding ground programs are infinite.

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# Contents



# 1 Introduction

Answer-Set Programming (ASP) is a fundamental paradigm for nonmonotonic knowledge representation [1] that encompasses logic programming under the answer-set semantics. It is distinguished by a purely declarative semantics and efficient solvers, such as, e.g., DLV [23], Smodels [35], clasp [15], GnT [21], and ASSAT [26]. Initially providing a semantics for rules with default negation in the body, the answer-set semantics (or stable-model semantics) [16] has been continually extended in terms of expressiveness and syntactic freedom. Starting with disjunctive rules, allowing for disjunctions in rule heads, negation in rule heads was considered and the development continued by allowing nested expressions, i.e., implication-free propositional formulas in the head and the body. Eventually, arbitrary propositional theories were given a non-classical minimal model semantics as their answer sets, which has recently been lifted to a general answer-set semantics for first-order theories [12].

In a different line of research, the restriction to Herbrand domains for programs with variables, i.e., non-ground programs, has been relaxed in order to cope with open domains [18], which is desirable for certain applications, e.g., in conceptual modelling and Semantic Web reasoning. The resulting open answer-set semantics has been further generalized by dropping the unique names assumption [17] for application settings where it does not apply, for instance, when combining ontologies with nonmontonic rules [5].

As for a logical characterization of the answer-set semantics, the logic of Here-and-There (HT), a non-classical logic extending intuitionistic logic, served as a basis. Equilibrium Logic selects certain minimal HT-models for characterizing the answer-set semantics for propositional theories and programs. It has recently been extended to Quantified Equilibrium Logic (QEL) for first-order theories on the basis of a quantified version of Here-and-There (QHT) [30, 31]. Equilibrium Logic serves as a viable formalism for the study of semantic comparisons of theories and programs, like different notions of equivalence [7, 24, 38, 10, 11, 19]. The practical relevance of this research originates in program optimization tasks that rely on modifications that preserve certain properties [6, 25, 22, 20, 34].

In previous work [13], we complemented this line of research by solving an open problem concerning uniform equivalence of propositional theories and programs. Intuitively, two propositional logic programs are uniformly equivalent if they have the same answer sets under the addition of an arbitrary set of atoms to both programs. Former characterizations of uniform equivalence, i.e., selections of HT-models based on a maximality criterion [8], failed to capture uniform equivalence for infinite propositional programs—a problem that becomes relevant when turning to the non-ground setting, respectively first-order theories, where infinite domains, such as the natural numbers, are encountered in many application domains. In [13], this has been remedied resorting to countermodels in HT.

In this article, we extend the former work beyond the basic notions of strong and uniform equivalence. So-called relativized notions thereof have been considered in order to capture more fine-grained semantical comparisons (see e.g., [8, 28]). Intuitively, these notions restrict the alphabet to be considered for potential additions, i.e., programs or sets of facts, respectively. A further refinement distinguishes the alphabet for atoms allowed in rule heads of an addition from the alphabet for atoms allowed in rule bodies [38]. The various notions of equivalence that can be formalized this way have recently been called *relativized hyperequivalence* [36, 37].

Similarly as for uniform equivalence, semantic characterizations of relativized hyperequivalence have been obtained by means of a maximality criterion so far, and only for finite propositional settings. We address this issue and apply the same methods as for uniform equivalence in order to obtain alternative characterizations. They can be stated without any finiteness restrictions and easily lift to first-order settings over infinite domains.

The new contributions compared to [13] can be summarized as follows:

- We provide full proofs for the characterizations of uniform equivalence, but also classical equivalence, answer-set equivalence, and strong equivalence, in terms of countermodels in HT, respectively in terms of *equivalence interpretations*, developed in [13].

- We extend these ideas to relativized settings of equivalence and generalize the notion of relativized hyperequivalence to propositional theories. Abstracting from the notions of rule head and rule body, we obtain respective notions of relativization for theories. We provide novel semantical characterizations in terms of equivalence interpretations for this generalized setting, again without any finiteness restrictions.

- We lift these results to first-order theories by means of QHT, essentially introducing, besides uniform equivalence, relativized hyperequivalence for first-order theories under the most general form of answer-set semantics currently considered.

- We correct an informal claim that has been made in connection with a property which allows for a simplified treatment of extended signatures and holds for QHT countermodels. Based on an erroneous example (Example 5 in [13]), it was claimed that this property does not hold for QHT-models, which is not the case.

- Eventually, we reconsider logic programs and prove, using the established characterization, that uniform equivalence coincides for open and ordinary answer-set semantics, as well as other results which have been stated without proof in [13].

Our results provide an elegant, uniform model-theoretic framework for the characterization of the different notions of equivalence considered in ASP. They generalize to first-order theories without finiteness restrictions, and are relevant for practical ASP systems that handle finite non-ground programs over infinite domains.

In particular the consideration of relativized notions of equivalence is of relevance in practice. For instance, program composition from modular parts is an issue of increasing interest in ASP [4, 22]. It usually hinges on semantic properties specified for an interface (input/output for 'calling' or connecting modules), i.e., properties that require compliance on a subset of the underlying language. Our results might be exploited to provide correctness guarantees for specific compositions.

Another benefit comes with the generalization to first-order theories. It facilitates and simplifies the study of combinations of ASP with other formalisms, or means for external data access, in a unifying formalism. Especially the combination of nonmonotonic rules with description logics is a highly relevant instance of such a combination. Our results can initiate or reduce difficulties in the study of modularity and optimization for such combined settings. (cf. [14] for preliminary work in this direction).

For the sake of presentation, the technical content is split into two parts, discussing the propositional case first, and addressing first-order theories and nonground programs in a second part. In particular, the organization is as follows: Section 2 introduces essential preliminaries for the treatment of the propositional case. In Section 3, we develop a characterization of uniform equivalence by means of countermodels in HT, and proceed with an alternative characterization in terms of equivalence interpretations, before we turn to generalizing and characterizing relativized hyperequivalence for propositional theories. After some introductory background on quantified HT, Section 4 deals with generalizations of previous results to first-order theories under generalized answer-set semantics. In Section 5, we apply our characterization of uniform equivalence to logic programs under various extended semantics in comparison with the traditional semantics over Herbrand domains, before we draw some conclusions in Section 6.

# 2 Preliminaries

We start with the propositional setting and briefly summarize the necessary background. Corresponding first-order formalisms will be introduced when discussing first-order theories, respectively non-ground logic programs.

## 2.1 Propositional Here-and-There

In the propositional case we consider formulas of a propositional signature $\mathcal{L}$, i.e., a set of propositional variables, and the connectives $\wedge$, $\vee$, $\rightarrow$, and $\bot$ for conjunction, disjunction, implication, and falsity, respectively. Furthermore we make use of the following abbreviations: $\phi \equiv \psi$ for $(\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$; $\neg\phi$ for $\phi \rightarrow \bot$; and $\top$ for $\bot \rightarrow \bot$. A formula is said to be *factual*[1] if it is built using $\wedge$, $\vee$, $\bot$, and $\neg$ (i.e., implications of the form $\phi \rightarrow \bot$), only. A theory $\Gamma$ is factual if every formula of $\Gamma$ has this property.

The logic of here-and-there is an intermediate logic between intuitionistic logic and classical logic. Like intuitionistic logic it can be semantically characterized by Kripke models, in particular using just two worlds, namely "*here*" and "*there*" (assuming that the *here* world is ordered before the *there* world). Accordingly, interpretations (HT-interpretations) are pairs $(X, Y)$ of sets of atoms from $\mathcal{L}$, such that $X \subseteq Y$. An HT-interpretation is *total* if $X = Y$. The intuition is that atoms in $X$ (the *here* part) are considered to be true,

[1] When uniform equivalence of theories is considered, then factual theories can be considered instead of facts—hence the terminology—see also the discussion at the end of this section.

atoms not in $Y$ (the *there* part) are considered to be false, while the remaining atoms (from $Y \setminus X$) are undefined.

We denote classical satisfaction of a formula $\phi$ by an interpretation $X$, i.e., a set of atoms, as $X \models \phi$, whereas satisfaction in the logic of here-and-there (an HT-model), symbolically $(X, Y) \models \phi$, is defined recursively:

1. $(X, Y) \models a$ if $a \in X$, for any atom $a$,
2. $(X, Y) \not\models \bot$,
3. $(X, Y) \models \phi \land \psi$ if $(X, Y) \models \phi$ and $(X, Y) \models \psi$,
4. $(X, Y) \models \phi \lor \psi$ if $(X, Y) \models \phi$ or $(X, Y) \models \psi$,
5. $(X, Y) \models \phi \rightarrow \psi$ if $(i)$ $(X, Y) \not\models \phi$ or $(X, Y) \models \psi$, and $(ii)$ $Y \models \phi \rightarrow \psi$[2].

An HT-interpretation $(X, Y)$ satisfies a theory $\Gamma$, iff it satisfies all formulas $\phi \in \Gamma$. For an axiomatic proof system see, e.g., [24].

A total HT-interpretation $(Y, Y)$ is called an *equilibrium model* of a theory $\Gamma$, iff $(Y, Y) \models \Gamma$ and for all HT-interpretations $(X, Y)$, such that $X \subset Y$, it holds that $(X, Y) \not\models \Gamma$. An interpretation $Y$ is an *answer set* of $\Gamma$ iff $(Y, Y)$ is an equilibrium model of $\Gamma$.

We will make use of the following simple properties: if $(X, Y) \models \Gamma$ then $(Y, Y) \models \Gamma$; and $(X, Y) \models \neg\phi$ iff $Y \models \neg\phi$; as well as of the following lemma.

**Lemma 1 (Lemma 5 in [29])** *Let $\phi$ be a factual propositional formula. If $(X, Y) \models \phi$ and $X \subseteq X' \subseteq Y$, then $(X', Y) \models \phi$.*

## 2.2 Propositional Logic Programming

A (*disjunctive*) rule $r$ is of the form

$$a_1 \lor \cdots \lor a_k \lor \neg a_{k+1} \lor \cdots \lor \neg a_l \ \leftarrow \ b_1, \ldots, b_m, \neg b_{m+1}, \ldots, \neg b_n, \tag{1}$$

where $a_1, \ldots, a_l, b_1, \ldots, b_n$ are atoms of a propositional signature $\mathcal{L}$, such that $l \geq k \geq 0$, $n \geq m \geq 0$, and $l+n > 0$. We refer to "$\neg$" as *default negation*. The *head* of $r$ is the set $H(r) = \{a_1, \ldots, a_k, \neg a_{k+1}, \ldots, \neg a_l\}$, and the *body* of $r$ is denoted by $B(r) = \{b_1, \ldots, b_m, \neg b_{m+1}, \ldots, \neg b_n\}$. Furthermore, we define the sets $H^+(r) = \{a_1, \ldots, a_k\}$, $H^-(r) = \{a_{k+1}, \ldots, a_l\}$, $B^+(r) = \{b_1, \ldots, b_m\}$, and eventually $B^-(r) = \{b_{m+1}, \ldots, b_n\}$. A *program* $\Pi$ (over $\mathcal{L}$) is a set of rules (over $\mathcal{L}$).

An interpretation $I$, i.e., a set of atoms, satisfies a rule $r$, symbolically $I \models r$, iff $I \cap H^+(r) \neq \emptyset$ or $H^-(r) \not\subseteq I$ if $B^+(r) \subseteq I$ and $B^-(r) \cap I = \emptyset$. Adapted from [16], the *reduct* of a program $\Pi$ with respect to an interpretation $I$, symbolically $\Pi^I$, is given by the set of rules

$$a_1 \lor \cdots \lor a_k \ \leftarrow \ b_1, \ldots, b_m,$$

obtained from rules in $\Pi$, such that $H^-(r) \subseteq I$ and $B^-(r) \cap I = \emptyset$.

An interpretation $I$ is called an *answer set* of $\Pi$ iff $I \models \Pi^I$ and it is subset minimal among the interpretations of $\mathcal{L}$ with this property.

[2]That is, $Y$ satisfies $\phi \rightarrow \psi$ classically.

### 2.3 Notions of Equivalence

For any two theories, respectively programs, and a potential extension by $\Gamma$, we consider the following notions of equivalence which have been shown to be the only forms of equivalence obtained by varying the logical form of extensions in the propositional case in [29].

**Definition 1** *Two theories $\Gamma_1, \Gamma_2$ over $\mathcal{L}$ are called*

- classically equivalent, $\Gamma_1 \equiv_c \Gamma_2$, *if and only if they have the same classical models;*
- answer-set equivalent, $\Gamma_1 \equiv_a \Gamma_2$, *if and only if they have the same answer sets, i.e., equilibrium models;*
- strongly equivalent, $\Gamma_1 \equiv_s \Gamma_2$, *if and only if, for any theory $\Gamma$ over $\mathcal{L}' \supseteq \mathcal{L}$, $\Gamma_1 \cup \Gamma$ and $\Gamma_2 \cup \Gamma$ are answer-set equivalent;*
- uniformly equivalent, $\Gamma_1 \equiv_u \Gamma_2$, *if and only if, for any factual theory $\Gamma$ over $\mathcal{L}' \supseteq \mathcal{L}$, $\Gamma_1 \cup \Gamma$ and $\Gamma_2 \cup \Gamma$ are answer-set equivalent.*

Emanating from a logic programming setting, uniform equivalence is usually understood wrt. sets of *facts* (i.e., atoms). Obviously, uniform equivalence wrt. factual theories implies uniform equivalence wrt. sets of facts. The converse direction has been shown as well for general propositional theories in [29](cf. Theorem 2). Therefore, in general there is no difference whether uniform equivalence is considered wrt. sets of facts or factual theories. The latter may be regarded as facts, i.e., rules with an empty body, of so-called nested logic program rules. One might also consider sets of disjunctions of atomic formulas and their negations (i.e., clauses), accounting for facts according to the definition of program rules in this article. Note that clauses constitute factual formulas and the classical transformation of clauses into implications is not valid under answer set semantics (respectively in HT).

## 3 Equivalence of Propositional Theories by HT-Countermodels

Uniform equivalence is usually characterized by so-called UE-models, i.e., total and maximal non-total HT-models, which fail to capture uniform equivalence for infinite propositional theories.

**Example 1 ([8])** Let $\Gamma_1$ and $\Gamma_2$ over $\mathcal{L} = \{a_i \mid i \geq 1\}$ be the following propositional theories

$$\Gamma_1 = \{a_i \mid i \geq 1\}, \quad \text{and} \quad \Gamma_2 = \{\neg a_i \rightarrow a_i,\ a_{i+1} \rightarrow a_i \mid i \geq 1\}.$$

Both, $\Gamma_1$ and $\Gamma_2$, have the single total HT-model $(\mathcal{L}, \mathcal{L})$. Furthermore, $\Gamma_1$ has no non-total HT-model $(X, \mathcal{L})$, i.e, such that $X \subset \mathcal{L}$, while $\Gamma_2$ has the non-total HT-models $(X_i, \mathcal{L})$, where $X_i = \{a_1, \ldots, a_i\}$ for $i \geq 0$. Both theories have the same total and maximal non-total (namely none) HT-models. But they are not uniformly equivalent as witnessed by the fact that $(\mathcal{L}, \mathcal{L})$ is an equilibrium model of $\Gamma_1$ but not of $\Gamma_2$. □

The reason for this failure is the inability of the concept of maximality to capture differences exhibited by an infinite number of HT-models.

### 3.1 HT-Countermodels

The above problem can be avoided by taking HT-countermodels that satisfy a closure condition instead of the maximality criterion.

**Definition 2** *An HT-interpretation* $(X, Y)$ *is an* HT-countermodel *of a theory* $\Gamma$ *if* $(X, Y) \not\models \Gamma$. *The set of HT-countermodels of a theory* $\Gamma$ *is denoted by* $C_s(\Gamma)$.

Intuitively, an HT-interpretation fails to be an HT-model of a theory $\Gamma$ when the theory is not satisfied at one of the worlds (*here* or *there*). Note that satisfaction at the *there* world amounts to classical satisfaction of the theory by $Y$. A simple consequence is that if $Y \not\models \Gamma$, then $(X, Y)$ is an HT countermodel of $\Gamma$ for any $X \subseteq Y$. At the *here* world, classical satisfaction is a sufficient condition but not necessary. For logic programs, satisfaction at the *here* world is precisely captured by the reduct of the program $\Pi$ wrt. the interpretation at the *there* world, i.e., if $X \models \Pi^Y$.

**Definition 3** *A total HT-interpretation* $(Y, Y)$ *is* total-closed *in a set* $S$ *of HT-interpretations if* $(X, Y) \in S$ *for every* $X \subseteq Y$. *We say that an HT-interpretation* $(X, Y)$ *is*

- closed *in a set* $S$ *of HT-interpretations if* $(X', Y) \in S$ *for every* $X \subseteq X' \subseteq Y$.
- there-closed *in a set* $S$ *of HT-interpretations if* $(Y, Y) \notin S$ *and* $(X', Y) \in S$ *for every* $X \subseteq X' \subset Y$.

A set $S$ of HT-interpretations is total-closed, if every total HT-interpretation $(Y, Y) \in S$ is total-closed in $S$. By the remarks on the satisfaction at the *there* world above, it is obvious that every total HT-countermodel of a theory is also total-closed in $C_s(\Gamma)$. Consequently, $C_s(\Gamma)$ is a total-closed set for any theory $\Gamma$. By the same argument, if $(X, Y)$ is an HT-countermodel such that $X \subset Y$ and $Y \not\models \Gamma$, then $(X, Y)$ is closed in $C_s(\Gamma)$. The more relevant cases concerning the characterization of equivalence are HT-countermodels $(X, Y)$ such that $Y \models \Gamma$.

**Example 2** Consider the theory $\Gamma_1$ in Example 1 and a non-total HT-interpretation $(X, \mathcal{L})$. Since $(X, \mathcal{L})$ is non-total, $X \subset \mathcal{L}$ holds, and therefore $(X, \mathcal{L}) \not\models a_i$, for some $a_i \in \mathcal{L}$. Thus, we have identified a HT-countermodel of $\Gamma_1$. Moreover the same argument holds for any non-total HT-interpretation of the from $(X', \mathcal{L})$ (in particular such that $X \subseteq X' \subset Y$). Therefore, $(X, \mathcal{L})$ is there-closed in $C_s(\Gamma_1)$. □

The intuition that, essentially, there-closed countermodels can be used instead of maximal non-total HT-models for characterizing uniform equivalence draws from the following observation. If $(X, Y)$ is a maximal non-total HT-model, then every $(X', Y)$, such that $X \subset X' \subset Y$, is a there-closed HT-countermodel. However, there-closed HT-countermodels are not sensitive to the problems that infinite chains cause for maximality.

Given a theory $\Gamma$, let $C_u(\Gamma)$ denote the set of there-closed HT-interpretations in $C_s(\Gamma)$.

**Theorem 1** *Two propositional theories* $\Gamma_1$, $\Gamma_2$ *are uniformly equivalent iff they have the same sets of there-closed HT-countermodels, in symbols* $\Gamma_1 \equiv_u \Gamma_2$ *iff* $C_u(\Gamma_1) = C_u(\Gamma_2)$.

**Proof**. For the only-if direction, assume that two theories, $\Gamma_1$ and $\Gamma_2$, are uniformly equivalent. Then they are classically equivalent, i.e., they coincide on total HT-models, and therefore also on total HT-countermodels. Since a total HT-interpretation $(Y, Y)$ is there-closed in $C_s(\Gamma)$ if $(Y, Y) \notin C_s(\Gamma)$, i.e., if $(Y, Y)$ is an HT-model of $\Gamma$, this proves that $\Gamma_1$ and $\Gamma_2$ coincide on total HT-interpretations that are there-closed in $C_s(\Gamma_1)$, respectively in $C_s(\Gamma_2)$.

To prove our claim, it remains to show that $\Gamma_1$ and $\Gamma_2$ coincide on non-total there-closed HT-countermodels $(X, Y)$, i.e., such that $(Y, Y)$ is an HT-model of both theories. Consider such a there-closed HT-countermodel of $\Gamma_1$. Then, $(Y, Y)$ is a total HT-model of $\Gamma_1 \cup X$ and no $X' \subset Y$ exists such that $(X', Y) \models \Gamma_1 \cup X$, either because it is an HT-countermodel of $\Gamma_1$ (in case $X \subseteq X' \subset Y$) or of $X$ (in case $X' \subset X$). Thus, $Y$ is an answer set of $\Gamma_1 \cup X$ and, by hypothesis since $X$ is factual, it is also an answer set of $\Gamma_2 \cup X$. The latter implies for all $X \subseteq X' \subset Y$ that $(X', Y) \not\models \Gamma_2 \cup X$. All these HT-interpretations are HT-models of $X$. Therefore we conclude that they all are HT-countermodels of $\Gamma_2$ and hence $(X, Y)$ is a there-closed HT-countermodel of $\Gamma_2$. Again by symmetric arguments, we establish the same for any there-closed HT countermodel $(X, Y)$ of $\Gamma_2$ such that $(Y, Y)$ is a common total HT-model. This proves that $\Gamma_1$ and $\Gamma_2$ have the same sets of there-closed HT countermodels.

For the if direction, assume that two theories, $\Gamma_1$ and $\Gamma_2$, have the same sets of there-closed HT-countermodels. This implies that they have the same total HT-models (since these are there-closed). Consider any factual theory $\Gamma'$ such that $Y$ is an answer set of $\Gamma_1 \cup \Gamma'$. We show that $Y$ is an answer set of $\Gamma_2 \cup \Gamma'$ as well. Clearly, $(Y, Y) \models \Gamma_1 \cup \Gamma'$ implies $(Y, Y) \models \Gamma'$ and therefore $(Y, Y) \models \Gamma_2 \cup \Gamma'$. Consider any $X \subset Y$. Since $Y$ is an answer set of $\Gamma_1 \cup \Gamma'$, it holds that $(X, Y) \not\models \Gamma_1 \cup \Gamma'$. We show that $(X, Y) \not\models \Gamma_2 \cup \Gamma'$. If $(X, Y) \not\models \Gamma'$ this is trivial, and in particular the case if $(X, Y) \models \Gamma_1$. So let us consider the case where $(X, Y) \not\models \Gamma_1$ and $(X, Y) \models \Gamma'$. By Lemma 1 we conclude from the latter that, for any $X \subseteq X' \subset Y$, $(X', Y) \models \Gamma'$. Therefore, $(X', Y) \not\models \Gamma_1$, as well. This implies that $(X, Y)$ is a there-closed HT-countermodel of $\Gamma_1$. By hypothesis, $(X, Y)$ is a there-closed HT-countermodel of $\Gamma_2$, i.e., $(X, Y) \not\models \Gamma_2$. Consequently, $(X, Y) \not\models \Gamma_2 \cup \Gamma'$. Since this argument applies to any $X \subset Y$, $(Y, Y)$ is an equilibrium model of $\Gamma_2 \cup \Gamma'$, i.e., $Y$ is an answer set of $\Gamma_2 \cup \Gamma'$. The argument with $\Gamma_1$ and $\Gamma_2$ interchanged, proves that $Y$ is an answer set of $\Gamma_1 \cup \Gamma'$ if it is an answer set of $\Gamma_2 \cup \Gamma'$. Therefore, the answer sets of $\Gamma_1 \cup \Gamma'$ and $\Gamma_2 \cup \Gamma'$ coincide for any factual $\Gamma'$, i.e., $\Gamma_1$ and $\Gamma_2$ are uniformly equivalent. □

**Example 3** Reconsider the theories in Example 1. Every non-total HT-interpretation $(X_i, \mathcal{L})$ is an HT-countermodel of $\Gamma_1$, and thus, each of them is there-closed in $C_s(\Gamma_1)$. On the other hand, none of these HT-interpretations is an HT-countermodel of $\Gamma_2$. Therefore, $\Gamma_1$ and $\Gamma_2$ are not uniformly equivalent. □

Countermodels have the drawback however, that they cannot be characterized directly in HT itself, i.e., as the HT-models of a 'dual' theory. The usage of "dual" here is non-standard compared to its application to particular calculi or consequence relations, but it likewise conveys the idea of a dual concept. In this sense HT therefore is non-dual:

**Proposition 1** *Given a theory $\Gamma$, in general there is no theory $\Gamma'$ such that $(X, Y)$ is an HT-countermodel of $\Gamma$ iff it is a HT-model of $\Gamma'$, for any HT-interpretation $(X, Y)$.*

**Proof**. As observed in [2], any theory has a total-closed set of countermodels. Consider the theory $\Gamma = \{a\}$ and suppose there exists a theory $\Gamma'$, such that $(X, Y)$ is an HT-countermodel of $\Gamma$ iff it is an HT-model of $\Gamma'$. Then, vice versa, $(X, Y)$ is an HT-countermodel of $\Gamma'$ iff it is an HT-model of $\Gamma$. Since for $Y = \{a\}$, $(Y, Y)$ is an HT-model of $\Gamma$, we conclude that $(Y, Y)$ is an HT-countermodel of $\Gamma'$. Because any theory has a total-closed set of countermodels, it follows that $(\emptyset, Y)$ is an HT-countermodel of $\Gamma'$, hence, an HT-model of $\Gamma$. Contradiction. □

### 3.2 Characterizing Equivalence by means of Equivalence Interpretations

The characterization of countermodels by a theory in HT essentially fails due to total HT-countermodels. However, total HT-countermodels of a theory are not necessary for characterizing equivalence, in the sense that they can be replaced by total HT-models of the theory for this purpose.

**Definition 4** *An HT-countermodel $(X, Y)$ of a theory $\Gamma$ is called a* here-countermodel *of $\Gamma$ if $Y \models \Gamma$.*

**Definition 5** *An HT-interpretation is an* equivalence interpretation *of a theory $\Gamma$ if it is a total HT-model of $\Gamma$ or a here-countermodel of $\Gamma$. The set of equivalence interpretations of a theory $\Gamma$ is denoted by $E_s(\Gamma)$.*

**Theorem 2** *Two theories $\Gamma_1$ and $\Gamma_2$ coincide on their HT-countermodels iff they have the same equivalence interpretations, symbolically $C_s(\Gamma_1) = C_s(\Gamma_2)$ iff $E_s(\Gamma_1) = E_s(\Gamma_2)$.*

**Proof**. For the only-if direction, assume that two theories, $\Gamma_1$ and $\Gamma_2$, have the same sets of HT-countermodels. This implies that they have the same here-countermodels. Furthermore, since the total HT-countermodels are equal, they also coincide on total HT-models. Consequently, $\Gamma_1$ and $\Gamma_2$ have the same equivalence interpretations.

For the if direction, assume that two theories, $\Gamma_1$ and $\Gamma_2$, coincide on their equivalence interpretations. Then they have the same total HT-models and hence the same total HT-countermodels. Since total HT-countermodels of every theory are total-closed in the set of HT-countermodels, the sets of HT-countermodels coincide on all HT-interpretations $(X, Y)$ such that $(Y, Y)$ is a (total) HT countermodel. All remaining HT-countermodels are here-countermodels and therefore coincide by hypothesis and the definition of equivalence interpretations. This proves the claim. □

As a consequence of this result, and the usual relationships on HT-models, we can characterize equivalences of propositional theories also by selections of equivalence interpretations, i.e., a mixture of non-total here-countermodels and total HT-models, such that the characterizations, in particular for uniform equivalence, are also correct for infinite theories.

**Definition 6** *Given a theory $\Gamma$, we denote by*

- *$C_c(\Gamma)$, respectively $E_c(\Gamma)$, the restriction to total HT-interpretations in $C_s(\Gamma)$, respectively in $E_s(\Gamma)$;*
- *$C_a(\Gamma)$ the set of there-closed HT-interpretations of the form $(\emptyset, Y)$ in $C_s(\Gamma)$, and by $E_a(\Gamma)$ the set of total-closed HT-interpretations in $E_s(\Gamma)$ (i.e., equilibrium models);*
- *$E_u(\Gamma)$ the set of closed HT-interpretations in $E_s(\Gamma)$.*

By means of the above sets of HT-countermodels, respectively equivalence interpretations, equivalences of propositional theories can be characterized as follows.

**Corollary 1** *Given two propositional theories $\Gamma_1$ and $\Gamma_2$, the following propositions are equivalent for $e \in \{c, a, s, u\}$:*

*(1)* $\Gamma_1 \equiv_e \Gamma_2$; *(2)* $C_e(\Gamma_1) = C_e(\Gamma_2)$; *(3)* $E_e(\Gamma_1) = E_e(\Gamma_2)$.

**Example 4** In our running example, $C_u(\Gamma_1) \neq C_u(\Gamma_2)$, as well as $E_u(\Gamma_1) \neq E_u(\Gamma_2)$, by the remarks on non-total HT-interpretations in Example 3. □

Since equivalence interpretations do not encompass total HT-countermodels, we attempt a direct characterization in HT.

**Lemma 2** *For any HT-interpretation $(X, Y)$ of signature $\mathcal{L}$ and $\tau_\epsilon = \{\neg\neg a \rightarrow a \mid a \in \mathcal{L}\}$, it holds that $(X, Y) \models \tau_\epsilon$ iff $X = Y$.*

**Proof**. $(X, Y) \models \tau_\epsilon$ for all $a \in \mathcal{L}$ iff $(X, Y) \models \neg\neg a \rightarrow a$ for all $a \in \mathcal{L}$ iff, for every $a \in \mathcal{L}$, it holds that $(X, Y) \not\models \neg\neg a$ or $(X, Y) \models a$, and $Y \models \neg\neg a \rightarrow a$. The latter is a tautology, and $(X, Y) \not\models \neg\neg a$ iff $a \notin Y$. We conclude that $(X, Y) \models \tau_\epsilon$ iff $(X, Y) \models a$ for all $a \in Y$, i.e., iff $X = Y$. □

By means of this lemma, we can use formulas of the form $\neg\neg a \rightarrow a$ to ensure for a given formula $\phi$ of $\Gamma$ that if $(X, Y) \models \phi$ then $X = Y$, i.e., that the HT-interpretation is total.

**Proposition 2** *Let $M$ be an HT-interpretation over $\mathcal{L}$. Then, $M \in E_s(\Gamma)$ for a theory $\Gamma$ iff $M \models \Gamma_\phi$ for some $\phi \in \Gamma$, where $\Gamma_\phi = \{\neg\neg\psi \mid \psi \in \Gamma\} \cup \{\phi \rightarrow (\neg\neg a \rightarrow a) \mid a \in \mathcal{L}\}$.*

**Proof**. For the only-if direction, assume $(X, Y)$ is an equivalence interpretation of $\Gamma$. Then $Y \models \psi$ for all $\psi \in \Gamma$ and therefore $(X, Y) \models \neg\neg\psi$ for all $\psi \in \Gamma$. If $X = Y$, then by Lemma 2, $(X, Y)$ also satisfies $\neg\neg a \rightarrow a$ for all $a \in \mathcal{L}$. In this case, $(X, Y) \models \Gamma_\phi$ for all $\phi \in \Gamma$. We continue with the case where $X \subset Y$. Then, $(X, Y)$ is a here-countermodel of $\Gamma$, i.e., there exists $\phi \in \Gamma$ such that $(X, Y) \not\models \phi$. This implies that $(X, Y) \models \phi \rightarrow (\neg\neg a \rightarrow a)$ for all $a \in \mathcal{L}$, i.e., $(X, Y) \models \Gamma_\phi$. This proves the claim for $X \subset Y$.

For the if direction, assume that $(X, Y) \models \Gamma_\phi$ for some $\phi \in \Gamma$. Then, $(X, Y) \models \neg\neg\psi$ for all $\psi \in \Gamma$, which implies $Y \models \psi$ for all $\psi \in \Gamma$. Consequently, $(X, Y)$ is an equivalence interpretation of $\Gamma$ if $X = Y$. If $X \subset Y$, we conclude that $(X, Y)$ does not satisfy $\neg\neg a \rightarrow a$ for some $a \in \mathcal{L}$ by Lemma 2. However, $(X, Y) \models \Gamma_\phi$ for some $\phi \in \Gamma$, hence $(X, Y) \models \phi \rightarrow (\neg\neg a \rightarrow a)$ for all $a \in \mathcal{L}$. Therefore, $(X, Y) \not\models \phi$ must hold for some $\phi \in \Gamma$. This proves, since $X \subset Y$, that $(X, Y)$ is a here-countermodel of $\Gamma$, i.e., an equivalence interpretation of $\Gamma$. □

For infinite propositional theories, we thus end up with a characterization of equivalence interpretations as the union of the HT-models of an infinite number of (infinite) theories. At least for finite theories, however, a characterization in terms of a (finite) theory is obtained (even for a potentially extended infinite signature).

If $\mathcal{L}' \supset \mathcal{L}$ and $M = (X, Y)$ is an HT-interpretation over $\mathcal{L}'$, then $M|_\mathcal{L}$ denotes the restriction of $M$ to $\mathcal{L}$: $M|_\mathcal{L} = (X|_\mathcal{L}, Y|_\mathcal{L})$. The restriction is *totality preserving*, if $X \subset Y$ implies $X|_\mathcal{L} \subset Y|_\mathcal{L}$.

**Proposition 3** *Let $\Gamma$ be a theory over $\mathcal{L}$, let $\mathcal{L}' \supset \mathcal{L}$, and let $M$ an HT-interpretation over $\mathcal{L}'$ such that $M|_\mathcal{L}$ is totality preserving. Then, $M \in C_s(\Gamma)$ implies $M|_\mathcal{L} \in C_s(\Gamma)$.*

**Proof**. Let $M = (X', Y')$, $M|_\mathcal{L} = (X, Y)$, and assume $M \not\models \Gamma$. First, suppose $M$ is total, hence, $Y' \not\models \Gamma$. Then, $Y \not\models \Gamma$, because otherwise $Y' \models \Gamma$ would hold, since $\Gamma$ is over $\mathcal{L}$. This proves the claim for total HT-countermodels, and since HT countermodels are total-closed, for any HT-countermodel $M = (X', Y')$, such that $Y' \not\models \Gamma$.

We continue with the case that $Y' \models \Gamma$. Then $X' \subset Y'$ holds, which means that $M$ is an equivalence interpretation of $\Gamma$. Therefore, $M \not\models \phi$ for some $\phi \in \Gamma$. Additionally, $M \models \neg\neg\psi$ for all $\psi \in \Gamma$ (recall that $Y' \models \Gamma$). This implies $M \models \Gamma_\phi$, where $\Gamma_\phi = \{\neg\neg\psi \mid \psi \in \Gamma\} \cup \{\phi \rightarrow (\neg\neg a \rightarrow a) \mid a \in \mathcal{L}\}$. Therefore, $M|_\mathcal{L} \models \Gamma_\phi$, i.e., $M|_\mathcal{L}$ is an equivalence interpretation of $\Gamma$. Since the restriction is totality preserving, $M|_\mathcal{L}$ is non-total. This proves $M|_\mathcal{L} \not\models \Gamma$. □

This eventually enables the characterization of the HT-countermodels of a finite theory by another finite theory, as stated in the next result.

**Theorem 3** *Let $\Gamma$ be a finite theory over $\mathcal{L}$, and let $M$ be an HT-interpretation. Then, $M \in E_s(\Gamma)$ iff $M|_\mathcal{L} \models \bigvee_{\phi \in \Gamma} \bigwedge_{\psi \in \Gamma_\phi} \psi$, and $M|_\mathcal{L}$ is totality preserving.*

**Proof**. For the only-if direction let $M \in E_s(\Gamma)$. If $M$ is total then $M|_\mathcal{L}$ is total and $M \models \Gamma$ implies $M|_\mathcal{L} \models \Gamma$. Hence, $M|_\mathcal{L} \in E_s(\Gamma)$ and $M|_\mathcal{L} \models \bigvee_{\phi \in \Gamma} \bigwedge_{\psi \in \Gamma_\phi} \psi$. So let $M$ be non-total. We show that $M|_\mathcal{L}$ is totality-preserving. Towards a contradiction assume the contrary. Then, $M|_\mathcal{L}$ is total. From $Y \models \Gamma$ we conclude $Y|_\mathcal{L} \models \Gamma$ and the same for $X|_\mathcal{L}$ by $X|_\mathcal{L} = Y|_\mathcal{L}$. Because $\Gamma$ is over $\mathcal{L}$, $X \models \Gamma$ follows, hence $M \models \Gamma$, which is a contradiction. Thus, $M|_\mathcal{L}$ is totality-preserving. Then $M|_\mathcal{L}$ is also non-total and in $C_s(\Gamma)$. Therefore $M|_\mathcal{L} \in E_s(\Gamma)$, which implies $M|_\mathcal{L} \models \bigvee_{\phi \in \Gamma} \bigwedge_{\psi \in \Gamma_\phi} \psi$.

For the if direction, consider any HT-interpretation $M$ such that $M|_\mathcal{L}$ satisfies the theory $\bigvee_{\phi \in \Gamma} \bigwedge_{\psi \in \Gamma_\phi} \psi$ and $M|_\mathcal{L}$ is totality preserving. If $M$ is total then $M|_\mathcal{L}$ is total and $M|_\mathcal{L} \models \Gamma$, which implies $M \models \Gamma$, since $\Gamma$ is over $\mathcal{L}$. If $M$ is non-total then $M|_\mathcal{L}$ is non-total and $M|_\mathcal{L} \not\models \Gamma$, which implies $M \not\models \Gamma$. □

**Example 5** Let $\Gamma = \{a\}$ over $\mathcal{L} = \{a\}$ and recall what the proof of Proposition 1 established: There is no theory $\Gamma'$ such that $(X, Y)$ is an HT-model of $\Gamma'$ iff it is an HT-countermodel of $\Gamma$. According to Theorem 3 however, we can characterize $E_s(\Gamma)$ by means of totality-preserving HT-models of the theory $\Gamma' = \{\neg\neg\, a \wedge (a \rightarrow (\neg\neg a \rightarrow a))\}$. Consider any HT-interpretation $(X, Y)$ over $\mathcal{L}' \supset \mathcal{L}$. It is easily verified that $(X, Y) \models \Gamma'$ iff $a \in Y$. If additionally $a \in X$ and $X \subset Y$, then $(X|_\mathcal{L}, Y|_\mathcal{L})$ is not totality preserving. Thus, $(X, Y)$ is a totality-preserving HT-model of $\Gamma'$ iff $a \in Y$ and either $X = Y$ or $a \notin X$. These interpretations respectively correspond to the total models and the here-countermodels, i.e., the equivalence interpretations of $\Gamma$ over $\mathcal{L}'$. □

### 3.3 Relativized Hyperequivalence for Propositional Theories

We now turn to the notion of relativized hyperequivalence. The term 'hyperequivalence' has been coined in the context of ASP, as a general expression for different forms of equivalence, which guarantee that the semantics is preserved under the addition of arbitrary programs (called *contexts*) from a particular class of programs [36]. Relativized hyperequivalence emanates from the study of relativized notions of equivalence by restricting contexts to particular alphabets (see e.g., [8, 28]). It has been generalized to the setting, where possibly different alphabets are used to restrict the head atoms and the body atoms allowed to appear in context rules [38].

While up to now relativized hyperequivalence has only been studied for finite programs, we aim at a generalization of relativized hyperequivalence for propositional theories under the answer-set semantics, without any finiteness restrictions. For this purpose, we first generalize the notions of 'head atom' and 'body atom' for theories.

The occurrence of an atom $a$ in a formula $\phi$ is called *positive* if $\phi$ is implication free, if $a$ occurs in the consequent of an implication in $\phi$, or if $\phi$ is of the form $(\phi_1 \rightarrow \phi_2) \rightarrow \phi_3$ and $a$ occors in $\phi_1$. An occurrence of $a$ is called *negative* if $a$ occurs in the antecedent of an implication. The notion of positive and negative occurrence is extended to (sub-)formulas in the obvious way. Note that any occurrence under negation therefore is a negative occurrence, and that the occurrence of an atom or subformula may be both positive and negative, for instance the occurrence of $b$ in $a \rightarrow (b \rightarrow \bot)$, viz. $a \rightarrow \neg b$.

A propositional theory $\Gamma$ over $A^+ \cup A^-$, where $A^+$ and $A^-$ are sets of propositional variables, is called an $A^+$-$A^-$*-theory* if every formula in $\Gamma$ has positive occurrences of atoms from $A^+$, and negative occurrences of atoms from $A^-$, only. Note that $\bot$ is always allowed to appear both, positively and negatively. An $A^+$-$A^-$-theory is called *extended*, if additionally factual formulas over $A^+$ are permitted.

By means of these notions, relativized hyperequivalence for propositional theories can be expressed as follows, which is a proper generalization of the logic programming setting.

**Definition 7** *Two propositional theories* $\Gamma_1, \Gamma_2$ *over* $\mathcal{L}$ *are called* relativized hyperequivalent wrt. $A^+$ and $A^-$*, symbolically* $\Gamma_1 \equiv^{A^+}_{A^-} \Gamma_2$*, iff for any* $A^+$*-*$A^-$*-theory* $\Gamma$ *over* $\mathcal{L}' \supseteq \mathcal{L}$*,* $\Gamma_1 \cup \Gamma$ *and* $\Gamma_2 \cup \Gamma$ *are answer-set equivalent.*

Towards a characterization of relativized hyperequivalence, our goal is to follow the same methodology that we used to characterize uniform equivalence, i.e., resorting to HT-countermodels and respective closure conditions. However, while in the logic programming setting such closure conditions may be obtained from certain monotonicity properties of the program reduct, we first have to establish corresponding properties for theories. A first property in this respect is the following. Note that although the next result is stated for extended $A^+$-$A^-$-theories (for reasons which will become clear later), it trivially also holds for any (non-extended) $A^+$-$A^-$-theory.

**Proposition 4** *Consider an extended propositional* $A^+$*-*$A^-$*-theory* $\Gamma$*, and an HT-interpretation* $(X, Y)$*. Then,* $(X, Y) \models \Gamma$ *implies* $(X', Y) \models \Gamma$*, for all* $X' \subseteq Y$ *such that* $X|_{A^+} \subseteq X'|_{A^+}$ *and* $X'|_{A^-} \subseteq X|_{A^-}$*.*

**Proof**. Consider any $A^+$-$A^-$-formula $\phi$ in $\Gamma$, i.e., any formula that has positive occurrences of atoms from $A^+$, and negative occurrences of atoms from $A^-$, only. We show by induction on the formula structure of $\phi$, that for all $X' \subseteq Y$ such that $X|_{A^+} \subseteq X'|_{A^+}$ and $X'|_{A^-} \subseteq X|_{A^-}$:

**(a)** $(X, Y) \models \phi$ implies $(X', Y) \models \phi$ if $\phi$ is a positive occurrence; and

**(b)** $(X, Y) \not\models \phi$ implies $(X', Y) \models \phi$ if $\phi$ is a negative occurrence.

For the base case, consider any atomic formula $\phi$, and suppose first that (a) the occurrence of $\phi$ is a positive occurrence. Then, $(X, Y) \models \phi$ implies that $\phi$ is not $\bot$, and thus is an atom $a$ from $A^+$ such that $a \in X$. Since $X|_{A^+} \subseteq X'|_{A^+}$ for all $X'$ under consideration, we conclude that $a \in X'$. Hence, $(X', Y) \models \phi$. Suppose (b) $\phi$ is a negative occurrence. If $(X, Y) \not\models \phi$, then either $\phi$ is $\bot$, and $(X', Y) \not\models \phi$ follows trivially. Otherwise, $\phi$ is an atom $b$ from $A^-$, such that $b \notin X|_{A^-}$. Since $X'|_{A^-} \subseteq X|_{A^-}$ for all $X'$ under consideration, we conclude that $b \notin X'$, i.e., $(X', Y) \not\models \phi$. This proves (a) and (b) for atomic formulas.

For the induction step, assume that (a) and (b) hold for any $A^+$-$A^-$-formula of connective nesting depth $n-1$, and let $\phi$ be a formula of connective nesting depth $n$. Consider the case where $\phi$ is of the form $\phi_1 \wedge \phi_2$, respectively $\phi_1 \vee \phi_2$. If $\phi$ is a positive occurrence (a), then so are $\phi_1$ and $\phi_2$, both of connective nesting depth depth $n-1$. From $(X, Y) \models \phi$ we conclude $(X, Y) \models \phi_1$ and (or) $(X, Y) \models \phi_2$. The induction hypothesis applies, proving $(X', Y) \models \phi_1$ and (or) $(X', Y) \models \phi_2$, for all $X' \subseteq Y$ such that $X|_{A^+} \subseteq X'|_{A^+}$ and $X'|_{A^-} = X|_{A^-}$ i.e., $(X', Y) \models \phi$ for all $X'$ under consideration. In case $\phi$ is a negative occurrence (b), then so are $\phi_1$ and $\phi_2$, both of connective nesting depth $n-1$. Then, $(X, Y) \not\models \phi$ implies $(X, Y) \not\models \phi_1$ or (and) $(X, Y) \not\models \phi_2$, and the same holds for any $(X', Y)$ under consideration by induction hypothesis. This proves $(X, Y) \not\models \phi$ implies $(X', Y) \models \phi$.

Finally, let $\phi$ be of the form $\phi_1 \rightarrow \phi_2$. Then, independent of whether $\phi$ occurs positively or negatively, $\phi_1$ is a negative occurrence and $\phi_2$ is a positive occurrence, both of connective nesting depth $n-1$. First, suppose that $\phi$ is a positive occurrence (a), as well as that $(X, Y) \models \phi$. Towards a contradiction assume that there exists $X' \subseteq Y$ such that $X|_{A^+} \subseteq X'|_{A^+}$, $X'|_{A^-} \subseteq X|_{A^-}$, and $(X', Y) \not\models \phi$. Since $(X, Y) \models \phi$ implies that $Y \models \phi$, we conclude that both, $(X', Y) \models \phi_1$ and $(X', Y) \not\models \phi_2$, hold. From the latter, since

$\phi_2$ is a positive occurrence of connective nesting depth $n-1$, it follows that $(X,Y) \not\models \phi_2$ (otherwise by induction hypothesis (a) $(X',Y) \models \phi_2$). This implies $(X,Y) \not\models \phi_1$ since $(X,Y) \models \phi$. However, $\phi_1$ is a negative occurrence of connective nesting depth $n-1$, thus by induction hypothesis (b) we conclude that $(X',Y) \not\models \phi_1$, a contradiction. Therefore, $(X',Y) \models \phi$ for all $X'$ under consideration, which proves (a). For (b), let $\phi$ be a negative occurrence and suppose $(X,Y) \not\models \phi$. If $Y \not\models \phi$, then also $(X',Y) \not\models \phi$ for all $X'$ under consideration. In case $Y \models \phi$, we conclude that $(X,Y) \models \phi_1$ and $(X,Y) \not\models \phi_2$. Since $\phi$ is a negative occurrence, not only $\phi_1$ but also $\phi_2$ is a negative occurrence, both of connective nesting depth $n-1$. Therefore, by induction hypothesis (b) we conclude that $(X',Y) \not\models \phi_2$. Moreover, also because $\phi$ is a negative occurrence, $\phi_1$ is a positive occurrence as well. Hence, by induction hypothesis (a) we conclude $(X',Y) \models \phi_1$ from $(X,Y) \models \phi_1$, viz. $(X',Y) \not\models \phi$, for all $X'$ under consideration. This concludes the inductive argument and proves (a) and (b) for $A^+$-$A^-$-formulas of arbitrary connective nesting.

Next, we turn to factual formulas $\psi$ in $\Gamma$, and prove by induction on the formula structure of $\psi$, that

**(c)** $(X,Y) \models \psi$ implies $(X',Y) \models \psi$, for all $X' \subseteq Y$ such that $X|_{A^+} \subseteq X'|_{A^+}$ and $X'|_{A^-} \subseteq X|_{A^-}$; and

**(d)** $(Y,Y) \not\models \psi$ implies $(X',Y) \not\models \psi$, for all $X' \subseteq Y$.

For the base case, consider any atomic formula $\psi$, and suppose first that (c) $(X,Y) \models \psi$. Then, $\psi$ is not $\bot$, but an atom $a$ from $A^+$ such that $a \in X$. Since $X|_{A^+} \subseteq X'|_{A^+}$ for all $X'$ such that $X|_{A^+} \subseteq X'|_{A^+}$ and $X'|_{A^-} \subseteq X|_{A^-}$, we conclude that $a \in X'$. Hence, $(X',Y) \models \psi$. For (d), assume $(Y,Y) \not\models \psi$. Then $\psi$ is $\bot$ or $\psi$ is a an atom not in $Y$. In the former case, $(X',Y) \not\models \psi$ follows trivially for all $X' \subseteq Y$. In the latter case, the atom also cannot be a member of any $X'$ such that $X' \subseteq Y$. Therefore, $(X',Y) \not\models \psi$, for all $X' \subseteq Y$. This proves (c) and (d) for atomic formulas.

For the induction step, assume that (c) and (d) hold for any factual formula of connective nesting depth $n-1$, and let $\psi$ be a factual formula of connective nesting depth $n$. Consider the case where $\psi$ is of the form $\psi_1 \wedge \psi_2$, respectively $\psi_1 \vee \psi_2$. Since $\psi$ is factual, so are $\psi_1$ and $\psi_2$, both of connective nesting depth depth $n-1$. In case (c), from $(X,Y) \models \psi$ we conclude $(X,Y) \models \psi_1$ and (or) $(X,Y) \models \psi_2$. The induction hypothesis applies, proving $(X',Y) \models \psi_1$ and (or) $(X',Y) \models \psi_2$, for all $X' \subseteq Y$ such that $X|_{A^+} \subseteq X'|_{A^+}$ and $X'|_{A^-} \subseteq X|_{A^-}$, i.e., $(X',Y) \models \psi$ for all $X'$ under consideration. Assume (d), i.e., $(Y,Y) \not\models \psi$. As a consequence, $(Y,Y) \not\models \psi_1$ or (and) $(Y,Y) \not\models \psi_2$, hence by induction hypothesis, for all $X' \subseteq Y$, it holds that $(X',Y) \not\models \psi_1$ or (and) $(X',Y) \not\models \psi_2$. Therefore, $(X',Y) \not\models \psi$, for all $X' \subseteq Y$.

Finally, let $\psi$ be of the form $\psi_1 \rightarrow \bot$. Then, $\psi_1$ is factual and of connective nesting depth depth $n-1$. In case (c), if $(X,Y) \models \psi$, then $Y \models \psi$, hence $Y \not\models \psi_1$, i.e., $(Y,Y) \not\models \psi_1$ and by induction hypothesis (d), the same holds for any $(X',Y)$ such that $X' \subseteq Y$. Thus, in particular for $X' \subseteq Y$ such that $X|_{A^+} \subseteq X'|_{A^+}$ and $X'|_{A^-} \subseteq X|_{A^-}$, it follows that $(X',Y) \not\models \psi_1$. Moreover, $Y \models \psi$, and therefore $(X',Y) \models \psi \rightarrow \bot$, for all $X' \subseteq Y$ such that $X|_{A^+} \subseteq X'|_{A^+}$ and $X'|_{A^-} \subseteq X|_{A^-}$. For (d), assume $(Y,Y) \not\models \psi$. Consequently $Y \not\models \psi$, and this implies $(X',Y) \not\models \psi$, for all $X' \subseteq Y$. This concludes the inductive argument and proves (c) and (d) for factual formulas over $A^+$ of arbitrary connective nesting.

Concerning the claim of the proposition, since $(X,Y) \models \Gamma$ implies $(X,Y) \models \phi$ and $(X,Y) \models \psi$, for every $A^+$-$A^-$-formula $\phi$ in $\Gamma$ and every factual formula $\psi$ in $\Gamma$, we conclude that $(X',Y) \models \phi$ and $(X',Y) \models \psi$, for all $X' \subseteq Y$ such that $X|_{A^+} \subseteq X'|_{A^+}$ and $X'|_{A^-} \subseteq X|_{A^-}$. This proves $(X',Y) \models \Gamma$, for all $X'$ under consideration. □

Complementary to this result, given a total HT-model of an (extended) $A^+$-$A^-$-theory, we can infer its satisfaction for the following class of non-total HT-interpretations.

**Proposition 5** *Consider an extended propositional $A^+$-$A^-$-theory $\Gamma$, and a total HT-interpretation $(Y,Y)$. Then, $(Y,Y) \models \Gamma$ implies $(X',Y) \models \Gamma$, for all $X' \subseteq Y$ such that $X'|_{A^+} = Y|_{A^+}$.*

**Proof**. Consider any $A^+$-$A^-$-formula $\phi$ in $\Gamma$, i.e., any formula that has positive occurrences of atoms from $A^+$, and negative occurrences of atoms from $A^-$, only. We show by induction on the formula structure of $\phi$, that for all $X' \subseteq Y$ such that $X'|_{A^+} = Y|_{A^+}$:

**(a)** $(Y, Y) \models \phi$ implies $(X', Y) \models \phi$ if $\phi$ is a positive occurrence; and

**(b)** $(Y, Y) \not\models \phi$ implies $(X', Y) \not\models \phi$ if $\phi$ is a negative occurrence.

For the base case, consider any atomic formula $\phi$, and suppose first (a) that $\phi$ is a positive occurrence such that $(Y, Y) \models \phi$. Then $\phi$ is not $\bot$, and thus is an atom $a$ from $A^+$ such that $a \in Y$. Since $X'|_{A^+} = Y|_{A^+}$ for all $X'$ under consideration, we conclude that $a \in X'$. Hence, $(X', Y) \models \phi$. Suppose (b) $\phi$ is a negative occurrence. If $(Y, Y) \not\models \phi$, then $\phi$ is either $\bot$, or an atom $b$ from $A^-$, such that $b \notin Y$. Since $X' \subseteq Y$ implies $X'|_{A^-} \subseteq Y|_{A^-}$ for all $X'$ under consideration, we conclude that $b \notin X'$. Hence, $(X', Y) \not\models \phi$.

For the induction step, assume that (a) and (b) hold for any $A^+$-$A^-$-formula of connective nesting depth $n-1$, and let $\phi$ be a formula of connective nesting depth $n$. Consider the case where $\phi$ is of the form $\phi_1 \wedge \phi_2$, respectively $\phi_1 \vee \phi_2$. If $\phi$ is a positive occurrence (a), then so are $\phi_1$ and $\phi_2$, both of connective nesting depth depth $n-1$. From $(Y, Y) \models \phi$ we conclude $(Y, Y) \models \phi_1$ and (or) $(Y, Y) \models \phi_2$. The induction hypothesis applies, proving $(X', Y) \models \phi_1$ and (or) $(X', Y) \models \phi_2$, for all $X' \subseteq Y$ such that $X'|_{A^+} = Y|_{A^+}$, i.e., $(X', Y) \models \phi$ for all $X'$ under consideration. In case $\phi$ is a negative occurrence (b), then so are $\phi_1$ and $\phi_2$, both of connective nesting depth $n-1$. Then, $(Y, Y) \not\models \phi$ implies $(Y, Y) \not\models \phi_1$ or (and) $(Y, Y) \not\models \phi_2$, and the same holds for any $(X', Y)$ under consideration by induction hypothesis. This proves $(X', Y) \not\models \phi$. Finally, let $\phi$ be of the form $\phi_1 \rightarrow \phi_2$. Then, independent of whether $\phi$ occurs positively or negatively, $\phi_1$ is a negative occurrence and $\phi_2$ is a positive occurrence, both of connective nesting depth $n-1$. First, suppose $(Y, Y) \models \phi$. Towards a contradiction assume that there exists $X' \subseteq Y$ such that $X'|_{A^+} = Y|_{A^+}$ and $(X', Y) \not\models \phi$. Since $(Y, Y) \models \phi$ implies that $Y \models \phi$, we conclude that both, $(X', Y) \models \phi_1$ and $(X', Y) \not\models \phi_2$, hold. From the latter, since $\phi_2$ is a positive occurrence of connective nesting depth $n-1$, it follows that $(Y, Y) \not\models \phi_2$ (otherwise by induction hypothesis (a) $(X', Y) \models \phi_2$). This implies $(Y, Y) \not\models \phi_1$ since $(Y, Y) \models \phi$. However, $\phi_1$ is a negative occurrence of connective nesting depth $n-1$, thus by induction hypothesis (b) we conclude that $(X', Y) \not\models \phi_1$, a contradiction. Therefore, $(X', Y) \models \phi$ for all $X'$ under consideration, which proves (a). For (b), let $\phi$ be a negative occurrence and suppose $(Y, Y) \not\models \phi$. Then $Y \not\models \phi$, hence also $(X', Y) \not\models \phi$ for all $X'$ under consideration. This concludes the inductive argument and proves (a) and (b) for $A^+$-$A^-$-formulas of arbitrary connective nesting.

Next, we turn to factual formulas $\psi$ in $\Gamma$, and prove by induction on the formula structure of $\psi$, that $(Y, Y) \models \psi$ implies $(X', Y) \models \psi$, for all $X' \subseteq Y$ such that $X'|_{A^+} = Y|_{A^+}$.

For the base case, consider any atomic formula $\psi$, and suppose that $(Y, Y) \models \psi$. Then, $\psi$ is not $\bot$, but an atom $a$ from $A^+$ such that $a \in Y$. Since $X'|_{A^+} = Y|_{A^+}$ for all $X'$ under consideration, we conclude that $a \in X'$. Hence, $(X', Y) \models \psi$, for all $X' \subseteq Y$ such that $X'|_{A^+} = Y|_{A^+}$.

For the induction step, assume that the claim holds for any factual formula of connective nesting depth $n-1$, and let $\psi$ be a factual formula of connective nesting depth $n$. Consider the case where $\psi$ is of the form $\psi_1 \wedge \psi_2$, respectively $\psi_1 \vee \psi_2$. Since $\psi$ is factual, so are $\psi_1$ and $\psi_2$, both of connective nesting depth depth $n-1$. From $(Y, Y) \models \psi$ we conclude $(Y, Y) \models \psi_1$ and (or) $(Y, Y) \models \psi_2$. The induction hypothesis applies, proving $(X', Y) \models \psi_1$ and (or) $(X', Y) \models \psi_2$, for all $X' \subseteq Y$ such that $X'|_{A^+} = Y|_{A^+}$, i.e., $(X', Y) \models \psi$ for all $X'$ under consideration. Finally, let $\psi$ be of the form $\psi_1 \rightarrow \bot$. Then, $\psi_1$ is factual and of connective nesting depth depth $n-1$. If $(Y, Y) \models \psi$, then $Y \models \psi$, hence $Y \not\models \psi_1$, i.e., $(Y, Y) \not\models \psi_1$ and by Case (d) in the proof of Proposition 4, the same holds for any $(X', Y)$ such that $X' \subseteq Y$. Thus, in particular for $X' \subseteq Y$ such that $X'|_{A^+} = Y|_{A^+}$, it follows that $(X', Y) \not\models \psi_1$. Moreover, $Y \models \psi$, and therefore $(X', Y) \models \psi \rightarrow \bot$, for all $X' \subseteq Y$ such that $X'|_{A^+} = Y|_{A^+}$. This concludes the inductive argument and proves the claim for factual formulas over $A^+$ of arbitrary connective nesting.

Concerning the claim of the proposition, since $(Y, Y) \models \Gamma$ implies $(Y, Y) \models \phi$ and $(Y, Y) \models \psi$, for every $A^+$-$A^-$-formula $\phi$ in $\Gamma$ and every factual formula $\psi$ in $\Gamma$, we conclude that $(X', Y) \models \phi$ and $(X', Y) \models \psi$, for all $X' \subseteq Y$ such that $X'|_{A^+} = Y|_{A^+}$. This proves $(X', Y) \models \Gamma$, for all $X'$ under consideration. □

Having established these properties of $A^+$-$A^-$-theories, we can state respective closure conditions for HT-interpretations referring to countermodels, or which we consider more convenient here, referring to equivalence interpretations.

**Definition 8** *Given a propositional theory $\Gamma$ over $\mathcal{L}$, sets of propositional variables $A^+ \subseteq \mathcal{L}'$, $A^- \subseteq \mathcal{L}'$, $\mathcal{L}' \supseteq \mathcal{L}$, and an HT-interpretation $(X, Y)$, we say that*

- $(Y, Y)$ *is* $A^+$-total *iff* $(Y|_{A^+}, Y)$ *is closed in* $E_s(\Gamma)$*;*
- $(X, Y)$ *is* $A^+$-closed *in* $E_s(\Gamma)$ *iff* $(X', Y) \in E_s(\Gamma)$*, for all* $X' \subseteq Y$ *such that* $X|_{A^+} \subseteq X'|_{A^+}$ *and* $X'|_{A^-} \subseteq X|_{A^-}$.

With these concepts, a semantic characterization of relativized hyperequivalence for propositional theories can be established by means of the following characteristic equivalence interpretations.

**Definition 9** *An HT-interpretation $(X, Y)$ is an* HT-hyperequivalence interpretation wrt. $A^+$ and $A^-$ *of a propositional theory $\Gamma$ iff $(Y, Y)$ is $A^+$-total and there exists an HT-interpretation $(X', Y)$ such that $X = X'|_{A^+ \cup A^-}$ and $(X', Y)$ is $A^+$-closed in $E_s(\Gamma)$.*
*The set of HT-hyperequivalence interpretations wrt. $A^+$ and $A^-$ of a propositional theory $\Gamma$ is denoted by $E^{A^+}_{A^-}(\Gamma)$.*

This definition intuitively generalizes the characterization of [38] for the logic programming setting to propositional theories. Note however, that rather than resorting to HT-models and a maximality criterion, the above definition refers to equivalence interpretations (i.e., HT-countermodels in case of non-totality) and respective closure conditions. As in the case of uniform equivalence, this not only simplifies the definition, but also avoids difficulties in infinite settings. The next result establishes that HT-hyperequivalence interpretations precisely characterize relativized hyperequivalence.

**Theorem 4** *Two propositional theories $\Gamma_1, \Gamma_2$ are relativized hyperequivalent wrt. $A^+$ and $A^-$ if and only if they coincide on their HT-hyperequivalence interpretations wrt. $A^+$ and $A^-$, symbolically $\Gamma_1 \overset{A^+}{\underset{A^-}{\equiv}} \Gamma_2$ iff $E^{A^+}_{A^-}(\Gamma_1) = E^{A^+}_{A^-}(\Gamma_2)$.*

**Proof**. In the following, we will use the following notational simplification: For any set of atoms $X$, we write $X_+$ for $X|_{A^+}$, and $X_-$ for $X|_{A^-}$.

For the only-if direction suppose $\Gamma_1 \overset{A^+}{\underset{A^-}{\equiv}} \Gamma_2$ and towards a contradiction assume that $E^{A^+}_{A^-}(\Gamma_1) \neq E^{A^+}_{A^-}(\Gamma_2)$. W.l.o.g. let $(X, Y) \in E^{A^+}_{A^-}(\Gamma_1)$ and $(X, Y) \notin E^{A^+}_{A^-}(\Gamma_2)$ (the other case is symmetric). Note that $(X, Y) \in E^{A^+}_{A^-}(\Gamma_1)$ implies that $(Y, Y)$ is $A^+$-total, i.e., $(Y_+, Y)$ is closed in $E_s(\Gamma_1)$. This implies that $(Y_+, Y)$ is in $E^{A^+}_{A^-}(\Gamma_1)$. Suppose $(Y_+, Y)$ is not in $E^{A^+}_{A^-}(\Gamma_2)$. Then, either $(Y, Y) \not\models \Gamma_2$, or there exists $Y_+ \subseteq X' \subset Y$ such that $(X', Y) \models \Gamma_2$. Let $\Gamma = Y_+$ and observe that in both cases $Y$ is not an answer set of $\Gamma_2 \cup \Gamma$. In the former case because $(Y, Y) \not\models \Gamma_2 \cup \Gamma$, in the latter because $X' \subset Y$ and $(X', Y) \models \Gamma_2 \cup \Gamma$ (note that $(X', Y) \models \Gamma$ by Proposition 5). However, $Y$ is an answer set of $\Gamma_1 \cup \Gamma$. Indeed, $(Y_+, Y)$ is closed in $E_s(\Gamma_1)$. And for any $X' \subset Y$ such that $Y_+ \not\subseteq X'_+$, obviously $(X', Y)$ is a non-total HT-countermodel of $\Gamma$. Consequently $(Y, Y)$ is total-closed in $E_s(\Gamma_1 \cup \Gamma)$. Because $\Gamma$ is an $A^+$-$A^-$-theory, this contradicts $\Gamma_1 \,{}^{A^+}_{A^-}\!\!\equiv\, \Gamma_2$. Thus, we conclude that $(Y_+, Y) \in E^{A^+}_{A^-}(\Gamma_2)$. Note that therefore $(Y, Y)$ is $A^+$-total for $\Gamma_2$, which implies that $(Y|_A, Y)$ is in $E^{A^+}_{A^-}(\Gamma_2)$, hence $X \subset Y|_A$ and $X_+ \subset Y_+$. Consider the following theory $\Gamma = X_+ \cup \{\alpha \rightarrow \beta \mid \alpha \in Y_- \setminus X_-, \beta \in Y_+ \setminus X_+\}$. We show that $Y$ is an answer set of $\Gamma_1 \cup \Gamma$. Obviously, $Y \models \Gamma$ because $X_+ \subset Y_+$ and $\beta \in Y$ for every $\beta \in Y_+ \setminus X_+$. Therefore, $(Y, Y) \models \Gamma_1 \cup \Gamma$. Towards a contradiction, assume that there exists $X' \subset Y$ such that $(X', Y) \models \Gamma_1 \cup \Gamma$. From $(X', Y) \models \Gamma$, we conclude that either $X'_+ = Y_+$, or that $X_+ \subseteq X'_+ \subset Y_+$ and $X'_- \subseteq X_-$. In both cases, $(X', Y) \not\models \Gamma_1$. In the former case because $(Y, Y)$ is $A^+$-total, i.e., $(Y_+, Y)$ is closed in $E_s(\Gamma_1)$. In the latter case, it is a consequence of the fact that $(X, Y) \in E^{A^+}_{A^-}(\Gamma_1)$, which implies $(X', Y) \not\models \Gamma_1$ by $A^+$-closure. This contradicts our assumption concerning the existence of $X' \subset Y$ such that $(X', Y) \models \Gamma_1 \cup \Gamma$, and proves that $Y$ is an answer set of $\Gamma_1 \cup \Gamma$. However, $Y$ is not an answer set of $\Gamma_2 \cup \Gamma$. To wit, since $(X, Y) \notin E^{A^+}_{A^-}(\Gamma_2)$, there exists $X' \subset Y$ such that $X_+ \subseteq X'_+$, $X'_- \subseteq X_-$, and $(X', Y) \models \Gamma_2$. Moreover, $(X', Y)$ is an HT-model of $\Gamma$. Observe that $X'_- \subseteq X_-$ implies that $(X', Y)$ is an HT-model of every formula of the form $\alpha \rightarrow \beta$ in $\Gamma$. Hence, $(X', Y) \models \Gamma_2 \cup \Gamma$, and since $X' \subset Y$, it follows that $Y$ is not an answer set of $\Gamma_2 \cup \Gamma$. Note that $\Gamma$ is an $A^+$-$A^-$-theory, which contradicts $\Gamma_1 \,{}^{A^+}_{A^-}\!\!\equiv\, \Gamma_2$. This proves $E^{A^+}_{A^-}(\Gamma_1) = E^{A^+}_{A^-}(\Gamma_2)$.

For the if direction, suppose $E^{A^+}_{A^-}(\Gamma_1) = E^{A^+}_{A^-}(\Gamma_2)$ and towards a contradiction assume that $\Gamma_1 \,{}^{A^+}_{A^-}\!\!\not\equiv \Gamma_2$. W.l.o.g. let $Y$ be an answer set of $\Gamma_1 \cup \Gamma$ for some $A^+$-$A^-$-theory $\Gamma$, such that $Y$ is not an answer set of $\Gamma_2 \cup \Gamma$ (the other case is symmetric). Then, $(Y, Y)$ is an equivalence interpretation of both, $\Gamma_1$ and $\Gamma$, and $(Y_+, Y)$ is closed in $E_s(\Gamma_1 \cup \Gamma)$, which implies (taking Proposition 5 into account) that $(Y, Y)$ is $A^+$-total for $\Gamma_1$ and $(Y|_A, Y)$ is in $E^{A^+}_{A^-}(\Gamma_1)$. Therefore, $(Y|_A, Y)$ is also in $E^{A^+}_{A^-}(\Gamma_2)$, with the consequence that $(Y, Y)$ is in $E_s(\Gamma_2)$, and thus $(Y, Y) \in E_s(\Gamma_2 \cup \Gamma)$. Since by assumption $Y$ is not an answer set of $\Gamma_2 \cup \Gamma$, there exists $X \subset Y$ such that $(X, Y) \notin E_s(\Gamma_2 \cup \Gamma)$, i.e., $(X, Y) \models \Gamma_2 \cup \Gamma$. Since $(Y|_A, Y) \in E^{A^+}_{A^-}(\Gamma_2)$, it holds that $X|_A \subset Y|_A$. Moreover, $X_+ \subset Y_+$ due to $A^+$-totality of $(Y, Y)$. Clearly, $(X|_A, Y)$ is not in $E^{A^+}_{A^-}(\Gamma_2)$ as witnessed by $(X, Y) \models \Gamma_2$, and thus $(X|_A, Y) \notin E^{A^+}_{A^-}(\Gamma_1)$ since $E^{A^+}_{A^-}(\Gamma_1) = E^{A^+}_{A^-}(\Gamma_2)$. From $(X|_A, Y) \notin E^{A^+}_{A^-}(\Gamma_1)$, we conclude that there exists $X' \subseteq Y$, such that $X_+ \subseteq X'_+$, $X'_- \subseteq X_-$, and $(X', Y) \notin E_s(\Gamma_1)$, i.e., $X' \subset Y$ and $(X', Y) \models \Gamma_1$. By Proposition 4, $(X, Y) \models \Gamma$ implies $(X', Y) \models \Gamma$. Consequently, $(X', Y) \models \Gamma_1 \cup \Gamma$, and since $X' \subset Y$, this contradicts our assumption that $Y$ is an answer set of $\Gamma_1 \cup \Gamma$, and proves $\Gamma_1 \,{}^{A^+}_{A^-}\!\!\equiv\, \Gamma_2$. □

Like in the logic programming setting, the framework obtained by the consideration of relativized hyperequivalence interpretations provides a general unified characterization of semantic characterizations of equivalence notions. In other words, the notions of equivalence considered in the previous subsection are obtained as special cases. For this purpose, one needs to refer to the universal alphabet (respectively signature), denoted by $\mathcal{A}$, explicitely. Then, by definition, setting $A^+ = A^- = \emptyset$ amounts to answer-set equivalence, $A^+ = A^- = \mathcal{A}$ yields strong equivalence, and $A^+ = \mathcal{A}$, $A^- = \emptyset$ characterizes uniform equivalence. The latter is not by definition but follows from two simple observations: every set of facts over $\mathcal{A}$ is a $\mathcal{A}$-$\emptyset$-theory, and every $\mathcal{A}$-$\emptyset$-theory is a factual theory modulo formulas of the form $\bot \rightarrow \phi$, which are tautologies in HT.

**Corollary 2** *Given two propositional theories $\Gamma_1$ and $\Gamma_2$ over $\mathcal{L} \subseteq \mathcal{A}$, the following propositions are equivalent for $e \in \{a, s, u\}$, $A^+(a) = A^-(a) = \emptyset$, $A^+(s) = A^-(s) = \mathcal{A}$, $A^+(u) = \mathcal{A}$, and $A^-(u) = \emptyset$:*

*(1)* $\Gamma_1 \equiv_e \Gamma_2$;  *(2)* $\Gamma_1 \,{}^{A^+(e)}_{A^-(e)}\!\!\equiv\, \Gamma_2$.

In these particular cases, relativized hyperequivalence interpretations coincide with the respective characteristic sets of equivalence interpretations.

**Proposition 6** *Let $\Gamma$ be a propositional theory over $\mathcal{L} \subseteq \mathcal{A}$, and let $e \in \{a, s, u\}$, $A^+(a) = A^-(a) = \emptyset$, $A^+(s) = A^-(s) = \mathcal{A}$, $A^+(u) = \mathcal{A}$, and $A^-(u) = \emptyset$. Then,*

$$E_e(\Gamma) = E^{A^+(e)}_{A^-(e)}(\Gamma).$$

**Proof**. First consider answer-set equivalence, i.e., $e = a$ and $A^+ = A^- = \emptyset$. Then for any HT-interpretation $(X, Y)$, it holds that $(X, Y) \in E^{A^+}_{A^-}(\Gamma) = E^{A^+(e)}_{A^-(e)}(\Gamma)$ iff $(\emptyset, Y)$ is there-closed in $E_s(\Gamma)$ and $X = \emptyset$. The former follows from the first condition in Definition 9 since $Y|_{A^+} = \emptyset$, and the latter from the second condition in Definition 9, i.e., from the existence of an $X'$ such that $X = X'|_{A^+}$ (since $X'|_\emptyset = \emptyset$ for any $X'$). Note that $X = \emptyset$ and $(\emptyset, Y)$ there-closed in $E_s(\Gamma)$ are exactly the requirements for $(X, Y) \in E_a(\Gamma)$. This proves $(X, Y) \in E^{A^+}_{A^-}(\Gamma)$ iff $(X, Y) \in E_a(\Gamma)$.

Turning to strong equivalence, let $e = s$ and $A^+ = A^- = \mathcal{A}$. Then for any HT-interpretation $(X, Y)$ over $\mathcal{A}$, it holds that $(X, Y) \in E^{A^+}_{A^-}(\Gamma) = E^{A^+(e)}_{A^-(e)}(\Gamma)$ iff $(Y, Y)$ in $E_s(\Gamma)$ and $(X, Y)$ in $E_s(\Gamma)$. The former follows from the first condition in Definition 9 since $Y|_\mathcal{A} = Y$, and the latter from the second condition in Definition 9, i.e., from the existence of an $X'$ such that $X = X'|_\mathcal{A}$ (which implies $X' = X$ since $X'|_\mathcal{A} = X'$ for any $X'$) and such that $X'' \in E_s(\Gamma)$ for all $X'' \subseteq Y$ where $X'|_\mathcal{A} = X''|_\mathcal{A}$ (i.e., for $X'' = X' = X$). Note that $(X, Y) \in E_s(\Gamma)$ implies $(Y, Y) \in E_s(\Gamma)$. Consequently, it holds that $(X, Y) \in E^{A^+}_{A^-}(\Gamma)$ iff $(X, Y) \in E_s(\Gamma)$.

Eventually consider uniform equivalence, i.e., $e = u$, $A^+ = \mathcal{A}$, and $A^- = \emptyset$. In this case, $(X, Y) \in E^{A^+}_{A^-}(\Gamma) = E^{A^+(e)}_{A^-(e)}(\Gamma)$, for any HT-interpretation $(X, Y)$ over $\mathcal{A}$, iff $(Y, Y)$ in $E_s(\Gamma)$ and $(X', Y)$ in $E_s(\Gamma)$ for all $X \subseteq X'' \subseteq Y$. The former follows from the first condition in Definition 9 since $Y|_\mathcal{A} = Y$, and the latter from the second condition in Definition 9, i.e., from the existence of an $X'$ such that $X = X'|_\mathcal{A}$ (which implies $X' = X$ since $X'|_\mathcal{A} = X'$ for any $X'$) and such that $X'' \in E_s(\Gamma)$ for all $X'' \subseteq Y$ where $X'|_\mathcal{A} \subseteq X''|_\mathcal{A}$ (i.e., for $X' = X \subseteq X'' \subseteq Y$). Note that this are exactly the requirements for $(X, Y)$ being closed in $E_s(\Gamma)$, thus for $(X, Y) \in E_u(\Gamma)$. Therefore, $(X, Y) \in E^{A^+}_{A^-}(\Gamma)$ iff $(X, Y) \in E_u(\Gamma)$, which proves the claim. □

Moreover, a setting where $A^+ = A^-$ is termed relativized strong equivalence, and $A^- = \emptyset$ denotes relativized uniform equivalence. A further remark is in place, however. While we proved for uniform equivalence of propositional theories, that it is indifferent to whether we restrict additions (contexts) to sets of atoms or whether we allow for factual theories, $\mathcal{A}$-$\emptyset$-theories syntactically do not encompass factual theories, since negation, i.e., formulas of the form $a \to \bot$, are not permitted. One question that this raises is: would allowing factual theories as contexts make a difference for relativized notions of uniform equivalence?

The answer is by inspection of the proof of Theorem 4 in connection with Proposition 4 and Proposition 5. Recall that the propositions have been stated for extended $A^+$-$A^-$-theories. Therefore, the only-if direction of Theorem 4 also holds for extended $A^+$-$A^-$-theories. Since the if direction just referred to $A^+$-$A^-$-theories (which, trivially, are extended $A^+$-$A^-$-theories too), we obtain the following.

**Corollary 3** *Two propositional theories $\Gamma_1, \Gamma_2$ are* relativized hyperequivalent wrt. extended $A^+$-$A^-$-theories *if and only if they coincide on their HT-hyperequivalence interpretations wrt. $A^+$ and $A^-$.*

Thus, also relativized uniform equivalence is independent of whether sets of atoms or factual theories are permitted as contexts. More generally, for any notion of relativized hyperequivalence, factual theories over

$A^+$ can be allowed in the context without altering the notion of equivalence captured. This holds essentially due to Proposition 4, which generalizes Lemma 1 (Lemma 5 in [29]) in this respect.

A final result establishes, that the notion of relativized hyperequivalence which has been introduced in this section is a proper generalization of the respective logic programming version to the more general case of propositional theories under answer-set semantics. It is a straight forward consequence of Theorem 4, since the $A^+$-$A^-$-theories in the proof of the if direction consist of formulas corresponding to rules with heads restricted to positive atoms from $A^+$ and body atoms from $A^-$. Let us say that two propositional programs $\Pi_1$ and $\Pi_2$ are relativized hyperequivalent wrt. $A^+$ and $A^-$ in the logic programming sense, in symbols $\Pi_1 \;{}^{A^+}_{A^-}\!\!\equiv_{lp} \Pi_2$, if and only if $\Pi_1 \cup \Pi \equiv_a \Pi_2 \cup \Pi$ for any program $\Pi$, such that $H^-(r) = \emptyset$, $H^+(r) \subseteq A^+$, and $B(r) \subseteq A^-$, for all $r \in \Pi$.

**Corollary 4** *Given two programs $\Pi_1$ and $\Pi_2$, let $A^+$ and $A^-$ be sets of propositional variables. Then, $\Pi_1 \;{}^{A^+}_{A^-}\!\!\equiv_{lp} \Pi_2$ if and only if $\Pi_1 \;{}^{A^+}_{A^-}\!\!\equiv \Pi_2$.*

# 4 Generalization to First-Order Theories

Since the characterizations, in particular of uniform equivalence, presented in the previous section capture also infinite theories, they pave the way for generalizing this notion of equivalence to non-ground settings without any finiteness restrictions. In this section we study first-order theories.

As first-order theories we consider sets of sentences (closed formulas) of a first-order signature $\mathcal{L} = \langle \mathcal{F}, \mathcal{P} \rangle$ in the sense of classical first-order logic. Hence, $\mathcal{F}$ and $\mathcal{P}$ are pairwise disjoint sets of function symbols and predicate symbols with an associated arity, respectively. Elements of $\mathcal{F}$ with arity 0 are called object constants. A 0-ary predicate symbol is a propositional constant. Formulas are constructed as usual and variable-free formulas or theories are called *ground*. A sentence is said to be *factual* if it is built using connectives $\wedge$, $\vee$, $\exists$, $\forall$, and $\neg$ (i.e., implications of the form $\phi \rightarrow \bot$), only. A theory $\Gamma$ is factual if every sentence of $\Gamma$ has this property. The abbreviations introduced for propositional formulas carry over: $\phi \equiv \psi$ for $(\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$; $\neg\phi$ for $\phi \rightarrow \bot$; and $\top$ for $\bot \rightarrow \bot$.

## 4.1 Static Quantified Logic of Here-and-There

Semantically we refer to the static quantified version of here-and-there with decidable equality as captured axiomatically by the system $\mathbf{QHT}^{\mathbf{s}}_{=}$ [30, 24, 31]. It is characterized by Kripke models of two worlds with a common universe (hence static) that interpret function symbols in the same way.

More formally, consider a first-order interpretation $I$ of a first-order signature $\mathcal{L}$ on a universe $\mathcal{U}$. We denote by $\mathcal{L}^I$ the extension of $\mathcal{L}$ obtained by adding pairwise distinct names $c_\varepsilon$ as object constants for the objects in the universe, i.e., for each $\varepsilon \in \mathcal{U}$. We write $\mathcal{C}_\mathcal{U}$ for the set $\{c_\varepsilon \mid \varepsilon \in \mathcal{U}\}$ and identify $I$ with its extension to $\mathcal{L}^I$ given by $I(c_\varepsilon) = \varepsilon$. Furthermore, let $t^I$ denote the value assigned by $I$ to a ground term $t$ (of signature $\mathcal{L}^I$), and let $\mathcal{L}_\mathcal{F}$ denote the restriction of $\mathcal{L}$ to function symbols (thus including object constants). By $\mathcal{B}_{\mathcal{P},\mathcal{C}_\mathcal{U}}$ we denote the set of atomic formulas built using predicates from $\mathcal{P}$ and constants $\mathcal{C}_\mathcal{U}$.

We represent a first-order interpretation $I$ of $\mathcal{L}$ on $\mathcal{U}$ as a pair $\langle I|_{\mathcal{L}_\mathcal{F}}, I|_{\mathcal{C}_\mathcal{U}}\rangle$,[3] where $I|_{\mathcal{L}_\mathcal{F}}$ is the restriction of $I$ on function symbols, and $I|_{\mathcal{C}_\mathcal{U}}$ is the set of atomic formulas from $\mathcal{B}_{\mathcal{P},\mathcal{C}_\mathcal{U}}$ which are satisfied in $I$. Correspondingly, classical satisfaction of a sentence $\phi$ by a first-order interpretation $\langle I|_{\mathcal{L}_\mathcal{F}}, I|_{\mathcal{C}_\mathcal{U}}\rangle$ is denoted by $\langle I|_{\mathcal{L}_\mathcal{F}}, I|_{\mathcal{C}_\mathcal{U}}\rangle \models \phi$. We also define a subset relation for first-order interpretations $I_1, I_2$ of $\mathcal{L}$ on $\mathcal{U}$ (ie., over the same domain) by $I_1 \subseteq I_2$ if $I_1|_{\mathcal{L}_\mathcal{F}} = I_2|_{\mathcal{L}_\mathcal{F}}$ and $I_1|_{\mathcal{C}_\mathcal{U}} \subseteq I_2|_{\mathcal{C}_\mathcal{U}}$.

[3] We use angle brackets to distinguish from HT-interpretations.

A QHT-interpretation of $\mathcal{L}$ is a triple $\langle I, J, K \rangle$, such that $(i)$ $I$ is an interpretation of $\mathcal{L}_\mathcal{F}$ on $\mathcal{U}$, and $(ii)$ $J \subseteq K \subseteq \mathcal{B}_{\mathcal{P},\mathcal{C}_\mathcal{U}}$.

The satisfaction of a sentence $\phi$ of signature $\mathcal{L}^I$ by a QHT-interpretation $M = \langle I, J, K \rangle$ (a QHT-model) is defined as:

1. $M \models p(t_1, \ldots, t_n)$ if $p(c_{t_1^I}, \ldots, c_{t_n^I}) \in J$;
2. $M \models t_1 = t_2$ if $t_1^I = t_2^I$;
3. $M \not\models \bot$;
4. $M \models \phi \wedge \psi$ if $M \models \phi$ and $M \models \psi$,
5. $M \models \phi \vee \psi$, if $M \models \phi$ or $M \models \psi$,
6. $M \models \phi \rightarrow \psi$ if $(i)$ $M \not\models \phi$ or $M \models \psi$, and $(ii)$ $\langle I, K \rangle \models \phi \rightarrow \psi$[4];
7. $M \models \forall x \phi(x)$ if $M \models \phi(c_\varepsilon)$ and $\langle I, K \rangle \models \phi(c_\varepsilon)$ for all $\varepsilon \in \mathcal{U}$;
8. $M \models \exists x \phi(x)$ if $M \models \phi(c_\varepsilon)$ for some $\varepsilon \in \mathcal{U}$;.

A QHT-interpretation $M = \langle I, J, K \rangle$ is called a *QHT-countermodel* of a theory $\Gamma$ iff $M \not\models \Gamma$; it is called *total* if $J = K$. A total QHT-interpretation $M = \langle I, K, K \rangle$ is called a *quantified equilibrium model* (*QEL-model*) of a theory $\Gamma$, iff $M \models \Gamma$ and $M' \not\models \Gamma$, for all QHT-interpretations $M' = \langle I, J, K \rangle$ such that $J \subset K$. A first-order interpretation $\langle I, K \rangle$ is an *answer set* of $\Gamma$ iff $M = \langle I, K, K \rangle$ is a QEL-model of a theory $\Gamma$.

In analogy to the propositional case, we will use the following simple properties.

**Lemma 3** *If* $\langle I, J, K \rangle \models \phi$ *then* $\langle I, K, K \rangle \models \phi$.

**Lemma 4** $\langle I, J, K \rangle \models \neg\phi$ *iff* $\langle I, K \rangle \models \neg\phi$.

### 4.2 Characterizing Equivalence by QHT-countermodels

We aim at generalizing uniform equivalence for first-order theories, in its most liberal form, which means wrt. factual theories. For this purpose, we first lift Lemma 1.

**Lemma 5** *Let* $\phi$ *be a factual sentence. If* $\langle I, J, K \rangle \models \phi$ *and* $J \subseteq J' \subseteq K$, *then* $\langle I, J', K \rangle \models \phi$.

**Proof**. The proof is by induction on the formula structure of $\phi$. Let $M = \langle I, J, K \rangle$, $M \models \phi$, and $M' = \langle I, J', K \rangle$ for some $J \subseteq J' \subseteq K$. For the base case, consider an atomic sentence $\phi$. If $\phi$ is of the form $p(t_1, \ldots, t_n)$, then $p(c_{t_1^I}, \ldots, c_{t_n^I}) \in J$ because $M \models \phi$. By the fact that $J' \supseteq J$ we conclude that $p(c_{t_1^I}, \ldots, c_{t_n^I}) \in J'$ and hence $M' \models \phi$. If $\phi$ is of the form $t_1 = t_2$ then $M \models \phi$ implies $t_1^I = t_2^I$, and thus $M' \models \phi$. Note also that $M \models \phi$ implies $\phi \neq \bot$. This proves the claim for atomic formulas.

For the induction step, assume that $M \models \phi$ implies $M' \models \phi$, for any sentence of depth $n - 1$, and let $\phi$ be a sentence of depth $n$. We show that $M \models \phi$ implies $M' \models \phi$. Suppose $\phi$ is the conjunction or disjunction of two sentences $\phi_1$ and $\phi_2$. Then $\phi_1$ and $\phi_2$ are sentences of depth $n - 1$. Hence, $M \models \phi_1$

[4] That is, $\langle I, K \rangle$ satisfies $\phi \rightarrow \psi$ classically.

implies $M' \models \phi_1$, and the same for $\phi_2$. Therefore, if $M$ models both or one of the sentences then so does $M'$, which implies $M \models \phi$ implies $M' \models \phi$ if $\phi$ is the conjunction or disjunction of two sentences. As for implication, since $\phi$ is factual we just need to consider the case where $\phi$ is of the form $\phi_1 \rightarrow \bot$, i.e., $\neg\phi_1$. By Lemma 4, $M \models \neg\phi_1$ iff $\langle I, K\rangle \models \neg\phi_1$ iff $M' \models \neg\phi_1$. This proves $M \models \phi$ implies $M' \models \phi$ if $\phi$ is an implication with $\bot$ as its consequence. Eventually, consider a quantified sentence $\phi$, i.e., $\phi$ is of the form $\forall x\phi_1(x)$ or $\exists x\phi_1(x)$. In this case, $M \models \phi$ implies $M \models \phi_1(c_\varepsilon)$ and $\langle I, K\rangle \models \phi_1(c_\varepsilon)$, for all $\varepsilon \in \mathcal{U}$, respectively $M \models \phi_1(c_\varepsilon)$, for some $\varepsilon \in \mathcal{U}$, in case of existential quantification. Since each of the sentences $\phi_1(c_\varepsilon)$ is of depth $n-1$, the same is true for $M'$ by assumption, i.e., $M' \models \phi_1(c_\varepsilon)$ and $\langle I, K\rangle \models \phi_1(c_\varepsilon)$, for all $\varepsilon \in \mathcal{U}$, respectively $M' \models \phi_1(c_\varepsilon)$, for some $\varepsilon \in \mathcal{U}$. It follows that $M \models \phi$ implies $M' \models \phi$ also for quantified sentences $\phi$ of depth $n$, and therefore, for any sentence $\phi$ of depth $n$. This proves the claim. $\Box$

The different notions of closure naturally extend to (sets of) QHT-interpretations. In particular, a total QHT-interpretation $M = \langle I, K, K\rangle$ is called *total-closed* in a set $S$ of QHT-interpretations if $\langle I, J, K\rangle \in S$ for every $J \subseteq K$. A QHT-interpretation $\langle I, J, K\rangle$ is *closed* in a set $S$ of QHT-interpretations if $\langle I, J', K\rangle \in S$ for every $J \subseteq J' \subseteq K$, and it is *there-closed* in $S$ if $\langle I, K, K\rangle \notin S$ and $\langle I, J', K\rangle \in S$ for every $J \subseteq J' \subset K$.

The first main result lifts the characterization of uniform equivalence for theories by HT-countermodels to the first-order case.

**Theorem 5** *Two first-order theories are uniformly equivalent iff they have the same sets of there-closed QHT-countermodels.*

**Proof**. For the only-if direction, assume that two theories, $\Gamma_1$ and $\Gamma_2$, are uniformly equivalent. We first show that they coincide on total QHT-models. Let $\langle I, K, K\rangle$ be a total QHT-model of $\Gamma_1$ then it is also a total QHT-model of $\Gamma_1 \cup K$ over $\mathcal{L}' = \langle \mathcal{F} \cup \mathcal{U}_\mathcal{C}, \mathcal{P}\rangle$. Furthermore, for any proper subset $J$ of $K$, it holds that $\langle I, J, K\rangle \not\models K$, since there exists some ground atomic formula $p(c_1, \ldots, c_n) \in K$, $c_i \in \mathcal{U}_\mathcal{C}$ for $1 \leq i \leq n$, such that $p(c_1, \ldots, c_n) \notin J$. Therefore, $\langle I, J, K\rangle \not\models \Gamma_1 \cup K$, and $\langle I, K, K\rangle$ is a QEL-model of $\Gamma_1 \cup K$. Consequently, $\langle I, K\rangle$ is an answer set of $\Gamma_1 \cup K$. By the hypothesis that the theories are uniformly equivalent and since $K$ is factual, we infer that $\langle I, K\rangle$ is an answer set of $\Gamma_2 \cup K$, i.e., $\langle I, K, K\rangle$ is a QEL-model of $\Gamma_2 \cup K$. Thus, $\langle I, K, K\rangle$ is a total QHT-model of $\Gamma_2$. The same argument can be symmetrically applied to any total QHT-model of $\Gamma_2$. This proves that both theories coincide on total models, and therefore also on total QHT-countermodels.

Since a total QHT-interpretation $\langle I, K, K\rangle$ is there-closed in $C_s(\Gamma)$ if $\langle I, K, K\rangle \notin C_s(\Gamma)$, i.e., if $\langle I, K, K\rangle$ is a QHT-model of $\Gamma$, this proves that $\Gamma_1$ and $\Gamma_2$ coincide on total QHT-interpretations that are there-closed in $C_s(\Gamma_1)$, respectively in $C_s(\Gamma_2)$.

To prove our claim, it remains to show that $\Gamma_1$ and $\Gamma_2$ coincide on non-total there-closed QHT-countermodels $\langle I, J, K\rangle$, i.e., such that $\langle I, K, K\rangle$ is a QHT-model of both theories. Consider such a there-closed QHT-countermodel of $\Gamma_1$. Then, $\langle I, K, K\rangle$ is a total QHT-model of $\Gamma_1 \cup J$ over $\mathcal{L}' = \langle \mathcal{F} \cup \mathcal{U}_\mathcal{C}, \mathcal{P}\rangle$, and no $J' \subset K$ exists such that $\langle I, J', K\rangle \models \Gamma_1 \cup J$, either because it is a QHT-countermodel of $\Gamma_1$ (in case $J \subseteq J' \subset K$) or of $J$ (in case $J' \subset J$). Thus, $\langle I, K\rangle$ is an answer set of $\Gamma_1 \cup J$ and, by hypothesis since $J$ is factual, it is also an answer set of $\Gamma_2 \cup J$. The latter implies for all $J \subseteq J' \subset K$ that $\langle I, J', K\rangle \not\models \Gamma_2 \cup J$. All these QHT-interpretations are obviously QHT-models of $J$. Therefore we conclude that they all are QHT-countermodels of $\Gamma_2$. For this reason, $\langle I, J, K\rangle$ is a there-closed QHT-countermodel of $\Gamma_2$. Again by symmetric arguments, we establish the same for any there-closed QHT-countermodel $\langle I, J, K\rangle$ of $\Gamma_2$, such that $\langle I, K, K\rangle$ is a common total QHT-model. This proves that $\Gamma_1$ and $\Gamma_2$ have the same sets of there-closed QHT-countermodels.

For the if direction, assume that two theories, $\Gamma_1$ and $\Gamma_2$, have the same sets of there-closed QHT-countermodels. This implies that they have the same total QHT-models (because they are there-closed). Consider any factual theory $\Gamma'$ such that $\langle I, K\rangle$ is an answer set of $\Gamma_1 \cup \Gamma'$. We show that $\langle I, K\rangle$ is an answer set of $\Gamma_2 \cup \Gamma'$ as well. Clearly, $\langle I, K, K\rangle \models \Gamma_1 \cup \Gamma'$ implies $\langle I, K, K\rangle \models \Gamma'$ and therefore $\langle I, K, K\rangle \models \Gamma_2 \cup \Gamma'$. Consider any $J \subset K$. Since $\langle I, K\rangle$ is an answer set of $\Gamma_1 \cup \Gamma'$, it holds that $\langle I, J, K\rangle \not\models \Gamma_1 \cup \Gamma'$. We show that $\langle I, J, K\rangle \not\models \Gamma_2 \cup \Gamma'$ follows. If $\langle I, J, K\rangle \not\models \Gamma'$ this is trivial. In particular this is the case if $\langle I, J, K\rangle \models \Gamma_1$. So let us consider the case where $\langle I, J, K\rangle \not\models \Gamma_1$ and $\langle I, J, K\rangle \models \Gamma'$. By Lemma 5 we conclude from the latter that, for any $J \subseteq J' \subset K$, it holds that $\langle I, J', K\rangle \models \Gamma'$. Therefore, $\langle I, J', K\rangle \not\models \Gamma_1$, has to hold as well. This implies that $\langle I, J, K\rangle$ is a there-closed QHT-countermodel of $\Gamma_1$. By hypothesis, $\langle I, J, K\rangle$ also is a there-closed QHT-countermodel of $\Gamma_2$, i.e., $\langle I, J, K\rangle \not\models \Gamma_2$. Consequently, $\langle I, J, K\rangle \not\models \Gamma_2 \cup \Gamma'$. Since this argument applies to any $J \subset K$, $\langle I, K, K\rangle$ is a QEL-model of $\Gamma_2 \cup \Gamma'$, i.e., $\langle I, K\rangle$ is an answer set of $\Gamma_2 \cup \Gamma'$. The same argument with $\Gamma_1$ and $\Gamma_2$ interchanged, proves that $\langle I, K\rangle$ is an answer set of $\Gamma_1 \cup \Gamma'$ if it is an answer set of $\Gamma_2 \cup \Gamma'$. Therfore, the answer sets of $\Gamma_1 \cup \Gamma'$ and $\Gamma_2 \cup \Gamma'$ coincide for any factual theory $\Gamma'$, i.e., $\Gamma_1$ and $\Gamma_2$ are uniformly equivalent. □

We next turn to an alternative characterization by a mixture of QHT-models and QHT-countermodels as in the propositional case. A QHT-countermodel $\langle I, J, K\rangle$ of a theory $\Gamma$ is called QHT here-countermodel of $\Gamma$ if $\langle I, K\rangle \models \Gamma$. A QHT-interpretation $\langle I, J, K\rangle$ is an QHT equivalence-interpretation of a theory $\Gamma$, if it is a total QHT-model of $\Gamma$ or a QHT here-countermodel of $\Gamma$. In slight abuse of notation, we reuse the notation $S_e$, $S \in \{C, E\}$ and $e \in \{c, a, s, u\}$, for respective sets of QHT-interpretations, and arrive at the following formal result:

**Theorem 6** *Two theories coincide on their QHT-countermodels iff they have the same QHT equivalence-interpretations, in symbols $C_s(\Gamma_1) = C_s(\Gamma_2)$ iff $E_s(\Gamma_1) = E_s(\Gamma_2)$.*

**Proof**. For the only-if direction, assume that two theories $\Gamma_1, \Gamma_2$ have the same sets of QHT-countermodels. This implies that they have the same QHT here-countermodels. Furthermore, since the total QHT-countermodels are equal, they also coincide on total QHT-models. Consequently, $\Gamma_1$ and $\Gamma_2$ have the same QHT equivalence-interpretations.

For the if direction, assume that two theories, $\Gamma_1$ and $\Gamma_2$, coincide on their QHT equivalence-interpretations. Then they have the same total QHT-models and hence the same total QHT-countermodels. Since the total QHT-countermodels of every theory are total-closed in the set of QHT-countermodels, the sets of QHT-countermodels coincide on all QHT-interpretations $\langle I, J, K\rangle$ such that $\langle I, K, K\rangle$ is a (total) QHT-countermodel. All remaining QHT-countermodels are QHT here-countermodels and therefore coincide by hypothesis and the definition of QHT equivalence-interpretations. This proves the claim. □

As a consequence of these two main results, we obtain an elegant, unified formal characterization of the different notions of equivalence for first-order theories under generalized answer-set semantics.

**Corollary 5** *Given two first-order theories $\Gamma_1$ and $\Gamma_2$, the following propositions are equivalent for $e \in \{c, a, s, u\}$: $\Gamma_1 \equiv_e \Gamma_2$; $C_e(\Gamma_1) = C_e(\Gamma_2)$; $E_e(\Gamma_1) = E_e(\Gamma_2)$.*

Moreover, lifting the characterization of HT-countermodels provided in Proposition 2 to the first-order setting, allows us to prove a property, which simplifies the treatment of extended signatures.

**Proposition 7** *Let $M$ be a QHT-interpretation over $\mathcal{L}$ on $\mathcal{U}$. Then, $M \in E_s(\Gamma)$ for a theory $\Gamma$ iff $M \models \Gamma_\phi(M)$ for some $\phi \in \Gamma$, where $\Gamma_\phi(M) = \{\neg\neg\psi \mid \psi \in \Gamma\} \cup \{\phi \rightarrow (\neg\neg a \rightarrow a) \mid a \in \mathcal{B}_{\mathcal{P},\mathcal{C}_\mathcal{U}}\}$.*

**Proof**. We first show that for a QHT-interpretation $M = \langle I, J, K \rangle$ over $\mathcal{U}$ and $M \models \{\neg\neg a \rightarrow a \mid a \in \mathcal{B}_{\mathcal{P},\mathcal{C}_{\mathcal{U}}}\}$ iff $J = K$. $M \models \neg\neg a \rightarrow a$ for all $a \in \mathcal{B}_{\mathcal{P},\mathcal{C}_{\mathcal{U}}}$ iff, for every $a \in \mathcal{B}_{\mathcal{P},\mathcal{C}_{\mathcal{U}}}$, $M \not\models \neg\neg a$ or $M \models a$, and $\langle I, K \rangle \models \neg\neg a \rightarrow a$. The latter is a tautology, and $M \not\models \neg\neg a$ iff $a \notin K$. We conclude that $M \models \neg\neg a \rightarrow a$ iff $M \models a$ for all $a \in K$, i.e., iff $J = K$.

For the only-if direction, assume that $M \models \Gamma_\phi(M)$ for some $\phi \in \Gamma$. Then, $M \models \neg\neg\psi$ for all $\psi \in \Gamma$, which implies $\langle I, K \rangle \models \psi$ for all $\psi \in \Gamma$. Consequently, $M$ is a QHT equivalence-interpretation of $\Gamma$ if $J = K$. If $J \subset K$, we conclude that $M$ does not satisfy $\neg\neg a \rightarrow a$ for some $a \in \mathcal{B}_{\mathcal{P},\mathcal{C}_{\mathcal{U}}}$ as shown above. However, $M \models \Gamma_\phi(M)$ for some $\phi \in \Gamma$, hence $M \models \phi \rightarrow (\neg\neg a \rightarrow a)$ for all $a \in \mathcal{B}_{\mathcal{P},\mathcal{C}_{\mathcal{U}}}$. Therefore, $M \not\models \phi$ holds for some $\phi \in \Gamma$. This proves, since $J \subset K$, that $M$ is a QHT here-countermodel of $\Gamma$, and thus a QHT equivalence-interpretation of $\Gamma$.

For the if direction, assume $M$ is a QHT equivalence-interpretation of $\Gamma$. Then $\langle I, K \rangle \models \psi$ for all $\psi \in \Gamma$ and therefore $M \models \neg\neg\psi$ for all $\psi \in \Gamma$. If $J = K$, then as shown above, $M$ also satisfies $\neg\neg a \rightarrow a$ for all $a \in \mathcal{B}_{\mathcal{P},\mathcal{C}_{\mathcal{U}}}$. In this case, $M \models \Gamma_\phi(M)$ for all $\phi \in \Gamma$. We continue with the case where $J \subset K$. Then, $M$ is a QHT here-countermodel of $\Gamma$, i.e., there exists $\phi \in \Gamma$ such that $M \not\models \phi$. This implies that $M \models \phi \rightarrow (\neg\neg a \rightarrow a)$ for all $a \in \mathcal{B}_{\mathcal{P},\mathcal{C}_{\mathcal{U}}}$, i.e., $M \models \Gamma_\phi(M)$. This proves the claim for $J \subset K$. □

For QHT-models it is known that $M \models \Gamma$ implies $M|_{\mathcal{L}} \models \Gamma$ (cf. e.g., Proposition 3 in [5]), hence $M|_{\mathcal{L}} \not\models \Gamma$ implies $M \not\models \Gamma$, i.e., $M|_{\mathcal{L}} \in C_s(\Gamma)$ implies $M \in C_s(\Gamma)$. The converse direction holds for totality preserving restrictions.

**Theorem 7** *Let $\Gamma$ be a theory over $\mathcal{L}$, let $\mathcal{L}' \supset \mathcal{L}$, and let $M$ a QHT-interpretation over $\mathcal{L}'$ such that $M|_{\mathcal{L}}$ is totality preserving. Then, $M \in C_s(\Gamma)$ implies $M|_{\mathcal{L}} \in C_s(\Gamma)$.*

**Proof**. Let $M = \langle I', J', K' \rangle$, $M|_{\mathcal{L}} = \langle I, J, K \rangle$, and assume $M \not\models \Gamma$. First, suppose $\langle I', K', K' \rangle \not\models \Gamma$, i.e., there exists a sentence $\phi \in \Gamma$, such that $\langle I', K', K' \rangle \not\models \phi$. We show that $\langle I, K, K \rangle \not\models \phi$ by induction on the formula structure of $\phi$.

Let us denote $\langle I, K, K \rangle$ by $N$ and $\langle I', K', K' \rangle$ by $N'$. For the base case, consider an atomic sentence $\phi$. If $\phi$ is of the form $p(t_1, \ldots, t_n)$, then $p(c_{t_1^{I'}}, \ldots, c_{t_n^{I'}}) \notin K'$ because $N' \not\models \phi$. Since $t_1, \ldots, t_n$ are terms in $\mathcal{L}$, it holds that $p(c_{t_1^{I'}}, \ldots, c_{t_n^{I'}}) = p(c_{t_1^{I}}, \ldots, c_{t_n^{I}})$, and by the fact fact that $K \subseteq K'$ we conclude that $p(c_{t_1^{I}}, \ldots, c_{t_n^{I}}) \notin K$ and hence $N \not\models \phi$. If $\phi$ is of the form $t_1 = t_2$ then $N' \not\models \phi$ implies $t_1^{I'} \neq t_2^{I'}$, and since $t_1$, $t_2$ are terms in $\mathcal{L}$, it follows that $t_1^I \neq t_2^I$ and thus $N \not\models \phi$. If $\phi$ is $\perp$ then $N' \not\models \phi$ and $N \not\models \phi$. This proves the claim for atomic formulas.

For the induction step, assume that $N' \not\models \phi$ implies $N \not\models \phi$, for any sentence of depth $n-1$, and let $\phi$ be a sentence of depth $n$. We show that $M|_{\mathcal{L}} \models \phi$ implies $M \models \phi$. Suppose $\phi$ is the conjunction or disjunction of two sentences $\phi_1$ and $\phi_2$. Then $\phi_1$ and $\phi_2$ are sentences of depth $n-1$. Hence, $N' \not\models \phi_1$ implies $N \not\models \phi_1$, and the same for $\phi_2$. Therefore, if $N'$ is a QHT-countermodel of one or both of the sentences then so is $N$, which implies $N' \not\models \phi$ implies $N \not\models \phi$ if $\phi$ is the conjunction or disjunction of two sentences. As for implication, let $\phi$ be of the form $\phi_1 \rightarrow \phi_2$. In this case, $N' \not\models \phi$ implies $N' \models \phi_1$ and $N' \not\models \phi_2$. Therefore, $N \models \phi_1$ by the usual sub-model property for QHT-models, and $N \not\models \phi_2$ by assumption. Hence, $N \not\models \phi$. Eventually, consider a quantified sentence $\phi$, i.e., $\phi$ is of the form $\forall x \phi_1(x)$ or $\exists x \phi_1(x)$. In this case, $N' \not\models \phi$ implies $N' \not\models \phi_1(c_\varepsilon)$ for some, respectively all, $\varepsilon \in \mathcal{U}$. (Note that $\langle I', K' \rangle \not\models \phi_1(c_\varepsilon)$ iff $N' \not\models \phi_1(c_\varepsilon)$.) Since each of the sentences $\phi_1(c_\varepsilon)$ is of depth $n-1$, the same is true for $N$ by assumption. It follows that $N' \not\models \phi$ implies $N \not\models \phi$ also for quantified sentences $\phi$ of depth $n$, and therefore, for any sentence $\phi$ of depth $n$. This concludes the inductive argument and proves the claim for total QHT-countermodels.

Moreover, because QHT-countermodels are total-closed, this proves the claim for any QHT-countermodel $M = \langle I', J', K' \rangle$, such that $\langle I', K', K' \rangle \not\models \Gamma$.

We continue with the case that $\langle I', K', K' \rangle \models \Gamma$. Then $J' \subset K'$ holds, which means that $M$ is a QHT equivalence-interpretation of $\Gamma$. Therefore, $M \not\models \phi$ for some $\phi \in \Gamma$. Additionally, $M \models \neg\neg\psi$ for all $\psi \in \Gamma$ (recall that $\langle I', K', K' \rangle \models \Gamma$, thus $\langle I', K' \rangle \models \Gamma$). By construction this implies $M \models \Gamma_\phi(M|_\mathcal{L})$. Therefore, $M|_\mathcal{L} \models \Gamma_\phi(M|_\mathcal{L})$, i.e., $M|_\mathcal{L}$ is a QHT equivalence-interpretation of $\Gamma$. Since the restriction is totality preserving, $M|_\mathcal{L}$ is non-total. This proves $M|_\mathcal{L} \not\models \Gamma$. □

Note that this property carries over to QHT-models, i.e., $M|_\mathcal{L} \models \Gamma$ implies $M \models \Gamma$, if $M|_\mathcal{L}$ is the restriction of $M$ to $\mathcal{L}$ and this restriction is totality preserving. Otherwise, by the above result $M \not\models \Gamma$ would imply $M|_\mathcal{L} \not\models \Gamma$. We remark that in [13] it is erroneously stated informally that this property does not hold for QHT-models, however the counter-example given there is flawed (Example 5 in [13]).

### 4.3 Relativized Hyperequivalence for First-Order Theories

In this section we extend the notion of relativized hyperequivalence to first-order theories. For this purpose, we distinguish positive and negative occurrences of predicates in sentences. More precisely, the occurrence of a predicate $p$ in a sentence $\phi$ is called *positive* if $\phi$ is implication free, if $p$ occurs in the consequent of an implication in $\phi$, or if $\phi$ is of the form $(\phi_1 \to \phi_2) \to \phi_3$ and $p$ occurs in $\phi_1$. An occurrence of $p$ is called *negative* if $p$ occurs in the antecedent of an implication. The notion of positive and negative occurrence is again extended to (sub-)sentences in the obvious way.

Let $\Gamma$ be a first-order theory over $\mathcal{L} = \langle \mathcal{F}, L^+ \cup L^- \rangle$, where $L^+$ and $L^-$ are sets of predicate symbols with an associated arity, such that if a predicate symbol $p$ occurs in both $L^+$ and $L^-$, then it is also associated the same arity. We say that $\Gamma$ is an *$L^+$-$L^-$-theory* if its sentences have positive occurrences of predicates from $L^+$, and negative occurrences of predicates from $L^-$, only. As in the propositional case, $\perp$ is allowed to appear positively and negatively, and the same holds for equality in the first-order case. Moreover, an $L^+$-$L^-$-theory is called *extended*, if additionally factual formulas over $L^+$ are permitted.

**Definition 10** *Two first-order theories $\Gamma_1, \Gamma_2$ over $\mathcal{L}$ are called* relativized hyperequivalent wrt. $L^+$ and $L^-$*, symbolically $\Gamma_1 \, {}^{L^+}_{L^-}\!\!\equiv \Gamma_2$, iff for any $L^+$-$L^-$-theory $\Gamma$ over $\mathcal{L}' \supseteq \mathcal{L}$, $\Gamma_1 \cup \Gamma$ and $\Gamma_2 \cup \Gamma$ are answer-set equivalent.*

The properties proven for HT-interpretations and extended $A^+$-$A^-$-theories in the propositional case, carry over to QHT-interpretations and extended $L^+$-$L^-$-theories in a straight forward manner.

**Proposition 8** *Consider an extended first-order $L^+$-$L^-$-theory $\Gamma$, and a QHT-interpretation $\langle I, J, K \rangle$. Then, $\langle I, J, K \rangle \models \Gamma$ implies $\langle I, J', K \rangle \models \Gamma$, for all $J' \subseteq K$ such that $J|_{L^+} \subseteq J'|_{L^+}$ and $J'|_{L^-} \subseteq J|_{L^-}$.*

**Proof**. Consider any $L^+$-$L^-$-sentence $\phi$ in $\Gamma$, i.e., any sentence that has positive occurrences of atoms from $L^+$, and negative occurrences of atoms from $L^-$, only. We show by induction on the formula structure of $\phi$, that for all $J' \subseteq K$ such that $J|_{L^+} \subseteq J'|_{L^+}$ and $J'|_{L^-} \subseteq J|_{L^-}$:

**(a)** $\langle I, J, K \rangle \models \phi$ implies $\langle I, J', K \rangle \models \phi$ if $\phi$ is a positive occurrence; and

**(b)** $\langle I, J, K \rangle \not\models \phi$ implies $\langle I, J', K \rangle \not\models \phi$ if $\phi$ is a negative occurrence.

For the base case, consider any atomic sentence $\phi$, and suppose first that (a) the occurrence of $\phi$ is a positive occurrence. Then, $\langle I, J, K \rangle \models \phi$ implies that $\phi$ is not $\perp$, and thus $\phi$ is either of the form $t_1 = t_2$, or of the form $p(t_1, \ldots, t_n)$ such that $p \in L^+$ and $p(c_{t_1^I}, \ldots, c_{t_n^I}) \in J$. If $\phi$ is of the form $t_1 = t_2$, then $\langle I, J, K \rangle \models \phi$ implies $t_1^I = t_2^I$, hence $\langle I, J', K \rangle \models \phi$, for any $J'$ under consideration. If $\phi$ is of the form

$p(t_1,\ldots,t_n)$, since $J|_{L^+} \subseteq J'|_{L^+}$ for all $J'$ under consideration, we conclude that $p(c_{t_1^I},\ldots,c_{t_n^I}) \in J'$. Hence, $\langle I,J',K\rangle \models \phi$. Suppose (b) $\phi$ is a negative occurrence. If $\langle I,J,K\rangle \not\models \phi$, then either $\phi$ is $\bot$, and $\langle I,J',K\rangle \not\models \phi$ follows trivially. Otherwise, $\phi$ is either of the form $t_1 = t_2$, or of the form $p(t_1,\ldots,t_n)$ such that $p \in L^-$ and $p(c_{t_1^I},\ldots,c_{t_n^I}) \notin J$. If $\phi$ is of the form $t_1 = t_2$, then $\langle I,J,K\rangle \not\models \phi$ implies $t_1^I \neq t_2^I$, hence $\langle I,J',K\rangle \not\models \phi$, for any $J'$ under consideration. If $\phi$ is of the form $p(t_1,\ldots,t_n)$, since $J'|_{L^-} \subseteq J|_{L^-}$ for all $J'$ under consideration, we conclude that $p(c_{t_1^I},\ldots,c_{t_n^I}) \notin J'$, i.e., $\langle I,J',K\rangle \not\models \phi$. This proves (a) and (b) for atomic sentences.

For the induction step, assume that (a) and (b) hold for any $L^+$-$L^-$-sentence of connective nesting depth $n-1$, and let $\phi$ be a sentence of connective nesting depth $n$. Consider the case where $\phi$ is of the form $\phi_1 \wedge \phi_2$, respectively $\phi_1 \vee \phi_2$. If $\phi$ is a positive occurrence (a), then so are $\phi_1$ and $\phi_2$, both of connective nesting depth depth $n-1$. From $\langle I,J,K\rangle \models \phi$ we conclude $\langle I,J,K\rangle \models \phi_1$ and (or) $\langle I,J,K\rangle \models \phi_2$. The induction hypothesis applies, proving $\langle I,J',K\rangle \models \phi_1$ and (or) $\langle I,J',K\rangle \models \phi_2$, for all $J' \subseteq K$ such that $J|_{L^+} \subseteq J'|_{L^+}$ and $J'|_{L^-} \subseteq J|_{L^-}$, i.e., $\langle I,J',K\rangle \models \phi$ for all $J'$ under consideration. In case $\phi$ is a negative occurrence (b), then so are $\phi_1$ and $\phi_2$, both of connective nesting depth $n-1$. Then, $\langle I,J,K\rangle \not\models \phi$ implies $\langle I,J,K\rangle \models \phi_1$ or (and) $\langle I,J,K\rangle \models \phi_2$, and the same holds for any $\langle I,J',K\rangle$ under consideration by induction hypothesis. This proves $\langle I,J,K\rangle \not\models \phi$ implies $\langle I,J',K\rangle \models \phi$.

Next, let $\phi$ be of the form $\phi_1 \rightarrow \phi_2$. Then, independent of whether $\phi$ occurs positively or negatively, $\phi_1$ is a negative occurrence and $\phi_2$ is a positive occurrence, both of connective nesting depth $n-1$. First, suppose that $\phi$ is a positive occurrence (a), as well as that $\langle I,J,K\rangle \models \phi$. Towards a contradiction assume that there exists $J' \subseteq K$ such that $J|_{L^+} \subseteq J'|_{L^+}$, $J'|_{L^-} \subseteq J|_{L^-}$, and $\langle I,J',K\rangle \not\models \phi$. Since $\langle I,J,K\rangle \models \phi$ implies that $\langle I,K\rangle \models \phi$, we conclude that both, $\langle I,J',K\rangle \models \phi_1$ and $\langle I,J',K\rangle \not\models \phi_2$, hold. From the latter, since $\phi_2$ is a positive occurrence of connective nesting depth $n-1$, it follows that $\langle I,J,K\rangle \not\models \phi_2$ (otherwise by induction hypothesis (a) $\langle I,J',K\rangle \models \phi_2$). This implies $\langle I,J,K\rangle \not\models \phi_1$ since $\langle I,J,K\rangle \models \phi$. However, $\phi_1$ is a negative occurrence of connective nesting depth $n-1$, thus by induction hypothesis (b) we conclude that $\langle I,J',K\rangle \not\models \phi_1$, a contradiction. Therefore, $\langle I,J',K\rangle \models \phi$ for all $J'$ under consideration, which proves (a). For (b), let $\phi$ be a negative occurrence and suppose $\langle I,J,K\rangle \not\models \phi$. If $\langle I,K\rangle \not\models \phi$, then also $\langle I,J',K\rangle \not\models \phi$ for all $J'$ under consideration. In case $\langle I,K\rangle \models \phi$, we conclude that $\langle I,J,K\rangle \models \phi_1$ and $\langle I,J,K\rangle \not\models \phi_2$. Since $\phi$ is a negative occurrence, not only $\phi_1$ but also $\phi_2$ is a negative occurrence, both of of connective nesting depth $n-1$.

Therefore, by induction hypothesis (b) we conclude that $\langle I,J',K\rangle \not\models \phi_2$. Moreover, also because $\phi$ is a negative occurrence, $\phi_1$ is a positive occurrence as well. Hence, by induction hypothesis (a) we conclude $\langle I,J',K\rangle \models \phi_1$ from $\langle I,J,K\rangle \models \phi_1$, viz. $\langle I,J',K\rangle \not\models \phi$, for all $J'$ under consideration.

Eventually, consider a quantified sentence $\phi$, i.e., $\phi$ is of the form $\forall x \phi_1(x)$ or $\exists x \phi_1(x)$. If $\phi$ is a positive occurrence (a), then so are the sentences $\phi_1(c_\varepsilon)$, for all $\varepsilon \in \mathcal{U}$, which are of connective nesting depth $n-1$. Then, $\langle I,J,K\rangle \models \phi$ implies $\langle I,J,K\rangle \models \phi_1(c_\varepsilon)$ and $\langle I,K\rangle \models \phi_1(c_\varepsilon)$, for all $\varepsilon \in \mathcal{U}$, respectively $\langle I,J,K\rangle \models \phi_1(c_\varepsilon)$, for some $\varepsilon \in \mathcal{U}$. The induction hypothesis applies, proving for all $J'$ under consideration, that $\langle I,J',K\rangle \models \phi_1(c_\varepsilon)$ and $\langle I,K\rangle \models \phi_1(c_\varepsilon)$, for all $\varepsilon \in \mathcal{U}$, respectively that $\langle I,J',K\rangle \models \phi_1(c_\varepsilon)$, for some $\varepsilon \in \mathcal{U}$. Therefore, $\langle I,J',K\rangle \models \phi$ for all $J' \subseteq K$ such that $J|_{L^+} \subseteq J'|_{L^+}$ and $J'|_{L^-} \subseteq J|_{L^-}$. If $\phi$ is a negative occurrence (b), then so are the sentences $\phi_1(c_\varepsilon)$, for all $\varepsilon \in \mathcal{U}$, which are of connective nesting depth $n-1$. Assume $\langle I,J,K\rangle \not\models \phi$. Then, $\langle I,J,K\rangle \not\models \phi_1(c_\varepsilon)$ or $\langle I,K\rangle \not\models \phi_1(c_\varepsilon)$, for some $\varepsilon \in \mathcal{U}$, respectively $\langle I,J,K\rangle \not\models \phi_1(c_\varepsilon)$, for all $\varepsilon \in \mathcal{U}$. The induction hypothesis applies, proving for all $J'$ under consideration, that $\langle I,J',K\rangle \not\models \phi_1(c_\varepsilon)$ or $\langle I,K\rangle \not\models \phi_1(c_\varepsilon)$, for some $\varepsilon \in \mathcal{U}$, respectively that $\langle I,J',K\rangle \not\models \phi_1(c_\varepsilon)$, for all $\varepsilon \in \mathcal{U}$. Therefore, $\langle I,J',K\rangle \not\models \phi$ for all $J' \subseteq K$ such that $J|_{L^+} \subseteq J'|_{L^+}$ and $J'|_{L^-} \subseteq J|_{L^-}$. This concludes the inductive argument and proves (a) and (b) for $L^+$-$L^-$-sentences of arbitrary connective nesting.

Next, we turn to factual sentences $\psi$ in $\Gamma$, and prove by induction on the formula structure of $\psi$, that

**(c)** $\langle I, J, K\rangle \models \psi$ implies $\langle I, J', K\rangle \models \psi$, for all $J' \subseteq K$ such that $J|_{L^+} \subseteq J'|_{L^+}$ and $J'|_{L^-} \subseteq J|_{L^-}$; and

**(d)** $\langle I, K, K\rangle \not\models \psi$ implies $\langle I, J', K\rangle \not\models \psi$, for all $J' \subseteq K$.

For the base case, consider any atomic sentence $\psi$, and suppose first that (c) $\langle I, J, K\rangle \models \psi$. Then, $\psi$ is not $\perp$, but either of the form $t_1 = t_2$, or of the form $p(t_1, \ldots, t_n)$ such that $p \in L^+$ and $p(c_{t_1^I}, \ldots, c_{t_n^I}) \in J$. If $\psi$ is of the form $t_1 = t_2$, then $\langle I, J, K\rangle \models \psi$ implies $t_1^I = t_2^I$, hence $\langle I, J', K\rangle \models \psi$, for any $J'$ under consideration. If $\psi$ is of the form $p(t_1, \ldots, t_n)$, since $J|_{L^+} \subseteq J'|_{L^+}$ for all $J'$ such that $J|_{L^+} \subseteq J'|_{L^+}$ and $J'|_{L^-} \subseteq J|_{L^-}$, we conclude that $p(c_{t_1^I}, \ldots, c_{t_n^I}) \in J'$. Hence, $\langle I, J', K\rangle \models \psi$. For (d), assume $\langle I, K, K\rangle \not\models \psi$. Then $\psi$ is $\perp$ or $\psi$ is either of the form $t_1 = t_2$, or of the form $p(t_1, \ldots, t_n)$ such that $p \in L^+$ and $p(c_{t_1^I}, \ldots, c_{t_n^I}) \notin J$. In the first case, $\langle I, J', K\rangle \not\models \psi$ follows trivially for all $J' \subseteq K$. If $\psi$ is of the form $t_1 = t_2$, then $\langle I, K, K\rangle \not\models \psi$ implies $t_1^I \neq t_2^I$, hence $\langle I, J', K\rangle \not\models \psi$, for all $J' \subseteq K$. If $\psi$ is of the form $p(t_1, \ldots, t_n)$, then $p(c_{t_1^I}, \ldots, c_{t_n^I})$ also cannot be a member of any $J'$ such that $J' \subseteq K$. Therefore, $\langle I, J', K\rangle \not\models \psi$, for all $J' \subseteq K$. This proves (c) and (d) for atomic sentences.

For the induction step, assume that (c) and (d) hold for any factual sentence of connective nesting depth $n-1$, and let $\psi$ be a factual sentence of connective nesting depth $n$. Consider the case where $\psi$ is of the form $\psi_1 \wedge \psi_2$, respectively $\psi_1 \vee \psi_2$. Since $\psi$ is factual, so are $\psi_1$ and $\psi_2$, both of connective nesting depth depth $n-1$. In case (c), from $\langle I, J, K\rangle \models \psi$ we conclude $\langle I, J, K\rangle \models \psi_1$ and (or) $\langle I, J, K\rangle \models \psi_2$. The induction hypothesis applies, proving $\langle I, J', K\rangle \models \psi_1$ and (or) $\langle I, J', K\rangle \models \psi_2$, for all $J' \subseteq K$ such that $J|_{L^+} \subseteq J'|_{L^+}$ and $J'|_{L^-} \subseteq J|_{L^-}$, i.e., $\langle I, J', K\rangle \models \psi$ for all $J'$ under consideration. Assume (d), i.e., $\langle I, K, K\rangle \not\models \psi$. As a consequence, $\langle I, K, K\rangle \not\models \psi_1$ or (and) $\langle I, K, K\rangle \not\models \psi_2$, hence by induction hypothesis, for all $J' \subseteq K$, it holds that $\langle I, J', K\rangle \not\models \psi_1$ or (and) $\langle I, J', K\rangle \not\models \psi_2$. Therefore, $\langle I, J', K\rangle \not\models \psi$, for all $J' \subseteq K$.

Next, let $\psi$ be of the form $\psi_1 \to \perp$. Then, $\psi_1$ is factual and of connective nesting depth depth $n-1$. In case (c), if $\langle I, J, K\rangle \models \psi$, then $\langle I, K\rangle \models \psi$, hence $\langle I, K\rangle \not\models \psi_1$, i.e., $\langle I, K, K\rangle \not\models \psi_1$ and by induction hypothesis (d), the same holds for any $\langle I, J', K\rangle$ such that $J' \subseteq K$. Thus, in particular for $J' \subseteq K$ such that $J|_{L^+} \subseteq J'|_{L^+}$ and $J'|_{L^-} \subseteq J|_{L^-}$, it follows that $\langle I, J', K\rangle \not\models \psi_1$. Moreover, $\langle I, K\rangle \models \psi$, and therefore $\langle I, J', K\rangle \models \psi \to \perp$, for all $J' \subseteq K$ such that $J|_{L^+} \subseteq J'|_{L^+}$ and $J'|_{L^-} \subseteq J|_{L^-}$. For (d), assume $\langle I, K, K\rangle \not\models \psi$. Consequently $\langle I, K\rangle \not\models \psi$, and this implies $\langle I, J', K\rangle \not\models \psi$, for all $J' \subseteq K$.

Eventually, consider a quantified sentence $\psi$, i.e., $\psi$ is of the form $\forall x \psi_1(x)$ or $\exists x \psi_1(x)$. Since $\psi$ is factual, so are the sentences $\psi_1(c_\varepsilon)$, for all $\varepsilon \in \mathcal{U}$, which are of connective nesting depth $n-1$. Suppose (c) $\langle I, J, K\rangle \models \psi$. Then, $\langle I, J, K\rangle \models \psi_1(c_\varepsilon)$ and $\langle I, K\rangle \models \psi_1(c_\varepsilon)$, for all $\varepsilon \in \mathcal{U}$, respectively $\langle I, J, K\rangle \models \psi_1(c_\varepsilon)$, for some $\varepsilon \in \mathcal{U}$. The induction hypothesis applies, proving for all $J'$ under consideration, that $\langle I, J', K\rangle \models \psi_1(c_\varepsilon)$ and $\langle I, K\rangle \models \psi_1(c_\varepsilon)$, for all $\varepsilon \in \mathcal{U}$, respectively that $\langle I, J', K\rangle \models \phi_1(c_\varepsilon)$, for some $\varepsilon \in \mathcal{U}$. Therefore, $\langle I, J', K\rangle \models \psi$ for all $J' \subseteq K$ such that $J|_{L^+} \subseteq J'|_{L^+}$ and $J'|_{L^-} \subseteq J|_{L^-}$. Assume (d) $\langle I, K, K\rangle \not\models \psi$. Then, $\langle I, K, K\rangle \not\models \psi_1(c_\varepsilon)$ (i.e., $\langle I, K\rangle \not\models \psi_1(c_\varepsilon)$), for some (all) $\varepsilon \in \mathcal{U}$. The induction hypothesis applies, proving for all $J' \subseteq K$, that $\langle I, J', K\rangle \not\models \psi_1(c_\varepsilon)$, for some (all) $\varepsilon \in \mathcal{U}$. Therefore, $\langle I, J', K\rangle \not\models \psi$ for all $J' \subseteq K$. This concludes the inductive argument and proves (c) and (d) for factual sentences over $L^+$ of arbitrary connective nesting.

Concerning the claim of the proposition, since $\langle I, J, K\rangle \models \Gamma$ implies $\langle I, J, K\rangle \models \phi$ and $\langle I, J, K\rangle \models \psi$, for every $L^+$-$L^-$-sentence $\phi$ in $\Gamma$ and every factual sentence $\psi$ in $\Gamma$, we conclude that $\langle I, J', K\rangle \models \phi$ and $\langle I, J', K\rangle \models \psi$, for all $J' \subseteq K$ such that $J|_{L^+} \subseteq J'|_{L^+}$ and $J'|_{L^-} \subseteq J|_{L^-}$. This proves $\langle I, J', K\rangle \models \Gamma$, for all $J'$ under consideration. $\square$

The main differences to the propositional case concern the treatment of equality of terms and that quantification has to be taken into account. The former depends solely on the interpretation part $I$, which is the same for the QHT-interpretations under consideration, and thus has no further influence on the argument. The latter, is a further case to be considered in the inductive argument, however one that reduces easily to the respective induction hypotheses. The remainder simply mirrors the propositional case, with the polarity being considered on the predicate level, rather than for propositional variables. The same holds for the proof of the following result.

**Proposition 9** *Consider an extended first-order $L^+$-$L^-$-theory $\Gamma$, and a total QHT-interpretation $\langle I, K, K\rangle$. Then, $\langle I, K, K\rangle \models \Gamma$ implies $\langle I, J', K\rangle \models \Gamma$, for all $J' \subseteq K$ such that $J'|_{L^+} = K|_{L^+}$.*

**Proof**. Consider any $L^+$-$L^-$-sentence $\phi$ in $\Gamma$, i.e., any sentence that has positive occurrences of atoms from $L^+$, and negative occurrences of atoms from $L^-$, only. We show by induction on the formula structure of $\phi$, that for all $J' \subseteq K$ such that $J'|_{L^+} = K|_{L^+}$:

**(a)** $\langle I, K, K\rangle \models \phi$ implies $\langle I, J', K\rangle \models \phi$ if $\phi$ is a positive occurrence; and

**(b)** $\langle I, K, K\rangle \not\models \phi$ implies $\langle I, J', K\rangle \not\models \phi$ if $\phi$ is a negative occurrence.

For the base case, consider any atomic sentence $\phi$, and suppose first (a) that $\phi$ is a positive occurrence such that $\langle I, K, K\rangle \models \phi$. Then $\phi$ is not $\bot$, and thus it is either of the form $t_1 = t_2$, or of the form $p(t_1, \ldots, t_n)$ such that $p \in L^+$ and $p(c_{t_1^I}, \ldots, c_{t_n^I}) \in K$. If $\phi$ is of the form $t_1 = t_2$, then $\langle I, K, K\rangle \models \phi$ implies $t_1^I = t_2^I$, hence $\langle I, J', K\rangle \models \psi$, for any $J'$ under consideration. If $\phi$ is of the form $p(t_1, \ldots, t_n)$, since $J'|_{L^+} = K|_{L^+}$ for all $J'$ under consideration, we conclude that $p(c_{t_1^I}, \ldots, c_{t_n^I}) \in J'$. Hence, $\langle I, J', K\rangle \models \phi$. Suppose (b) $\phi$ is a negative occurrence. If $\langle I, K, K\rangle \not\models \phi$, then $\phi$ is $\bot$, or of the form $t_1 = t_2$, or of the form $p(t_1, \ldots, t_n)$ such that $p \in L^-$ and $p(c_{t_1^I}, \ldots, c_{t_n^I}) \notin K$. In the first case, $\langle I, J', K\rangle \not\models \phi$ follows trivially for all $J'$ under consideration. If $\phi$ is of the form $t_1 = t_2$, then $\langle I, K, K\rangle \not\models \phi$ implies $t_1^I \neq t_2^I$, hence $\langle I, J', K\rangle \not\models \phi$, for all $J'$ under consideration. If $\phi$ is of the form $p(t_1, \ldots, t_n)$, since $J' \subseteq K$ implies $J'|_{L^-} \subseteq K|_{L^-}$ for all $J'$ under consideration, we conclude that $p(c_{t_1^I}, \ldots, c_{t_n^I}) \notin J'$. Hence, $\langle I, J', K\rangle \not\models \phi$.

For the induction step, assume that (a) and (b) hold for any $L^+$-$L^-$-sentence of connective nesting depth $n-1$, and let $\phi$ be a sentence of connective nesting depth $n$. Consider the case where $\phi$ is of the form $\phi_1 \wedge \phi_2$, respectively $\phi_1 \vee \phi_2$. If $\phi$ is a positive occurrence (a), then so are $\phi_1$ and $\phi_2$, both of connective nesting depth depth $n-1$. From $\langle I, K, K\rangle \models \phi$ we conclude $\langle I, K, K\rangle \models \phi_1$ and (or) $\langle I, K, K\rangle \models \phi_2$. The induction hypothesis applies, proving $\langle I, J', K\rangle \models \phi_1$ and (or) $\langle I, J', K\rangle \models \phi_2$, for all $J' \subseteq K$ such that $J'|_{L^+} = K|_{L^+}$, i.e., $\langle I, J', K\rangle \models \phi$ for all $J'$ under consideration. In case $\phi$ is a negative occurrence (b), then so are $\phi_1$ and $\phi_2$, both of connective nesting depth $n-1$. Then, $\langle I, K, K\rangle \not\models \phi$ implies $\langle I, K, K\rangle \not\models \phi_1$ or (and) $\langle I, K, K\rangle \not\models \phi_2$, and the same holds for any $\langle I, J', K\rangle$ under consideration by induction hypothesis. This proves $\langle I, J', K\rangle \not\models \phi$.

Next, let $\phi$ be of the form $\phi_1 \rightarrow \phi_2$. Then, independent of whether $\phi$ occurs positively or negatively, $\phi_1$ is a negative occurrence and $\phi_2$ is a positive occurrence, both of connective nesting depth $n-1$. First, suppose $\langle I, K, K\rangle \models \phi$. Towards a contradiction assume that there exists $J' \subseteq K$ such that $J'|_{L^+} = K|_{L^+}$ and $\langle I, J', K\rangle \not\models \phi$. Since $\langle I, K, K\rangle \models \phi$ implies that $\langle I, K\rangle \models \phi$, we conclude that both, $\langle I, J', K\rangle \models \phi_1$ and $\langle I, J', K\rangle \not\models \phi_2$, hold. From the latter, since $\phi_2$ is a positive occurrence of connective nesting depth $n-1$, it follows that $\langle I, K, K\rangle \not\models \phi_2$ (otherwise by induction hypothesis (a) $\langle I, J', K\rangle \models \phi_2$). This implies $\langle I, K, K\rangle \not\models \phi_1$ since $\langle I, K, K\rangle \models \phi$. However, $\phi_1$ is a negative occurrence of connective nesting depth $n-1$, thus by induction hypothesis (b) we conclude that $\langle I, J', K\rangle \not\models \phi_1$, a contradiction. Therefore,

$\langle I, J', K\rangle \models \phi$ for all $J'$ under consideration, which proves (a). For (b), let $\phi$ be a negative occurrence and suppose $\langle I, K, K\rangle \not\models \phi$. Then $\langle I, K\rangle \not\models \phi$, hence also $\langle I, J', K\rangle \not\models \phi$ for all $J'$ under consideration.

Eventually, consider a quantified sentence $\phi$, i.e., $\phi$ is of the form $\forall x \phi_1(x)$ or $\exists x \phi_1(x)$. If $\phi$ is a positive occurrence (a), then so are the sentences $\phi_1(c_\varepsilon)$, for all $\varepsilon \in \mathcal{U}$, which are of connective nesting depth $n-1$. Then, $\langle I, K, K\rangle \models \phi$ implies $\langle I, K, K\rangle \models \phi_1(c_\varepsilon)$ (i.e., $\langle I, K\rangle \models \phi_1(c_\varepsilon)$), for all (some) $\varepsilon \in \mathcal{U}$. The induction hypothesis applies, proving for all $J'$ under consideration, that $\langle I, J', K\rangle \models \phi_1(c_\varepsilon)$ and $\langle I, K\rangle \models \phi_1(c_\varepsilon)$, for all $\varepsilon \in \mathcal{U}$, respectively that $\langle I, J', K\rangle \models \phi_1(c_\varepsilon)$, for some $\varepsilon \in \mathcal{U}$. Therefore, $\langle I, J', K\rangle \models \phi$ for all $J' \subseteq K$ such that $J'|_{L^+} = K|_{L^+}$. If $\phi$ is a negative occurrence (b), then so are the sentences $\phi_1(c_\varepsilon)$, for all $\varepsilon \in \mathcal{U}$, which are of connective nesting depth $n-1$. From $\langle I, K, K\rangle \not\models \phi$, we conclude that $\langle I, K, K\rangle \not\models \phi_1(c_\varepsilon)$, for some (all) $\varepsilon \in \mathcal{U}$. (Note that $\langle I, K\rangle \not\models \phi_1(c_\varepsilon)$ also implies $\langle I, K, K\rangle \not\models \phi_1(c_\varepsilon)$.) The induction hypothesis applies, proving for all $J'$ under consideration, that $\langle I, J', K\rangle \not\models \phi_1(c_\varepsilon)$, for some (all) $\varepsilon \in \mathcal{U}$. Therefore, $\langle I, J', K\rangle \not\models \phi$ for all $J' \subseteq K$ such that $J'|_{L^+} = K|_{L^+}$. This concludes the inductive argument and proves (a) and (b) for $L^+$-$L^-$-sentences of arbitrary connective nesting.

Next, we turn to factual sentences $\psi$ in $\Gamma$, and prove by induction on the formula structure of $\psi$, that $\langle I, K, K\rangle \models \psi$ implies $\langle I, J', K\rangle \models \psi$, for all $J' \subseteq K$ such that $J'|_{L^+} = K|_{L^+}$.

For the base case, consider any atomic sentence $\psi$, and suppose that $\langle I, K, K\rangle \models \psi$. Then, $\psi$ is not $\bot$, but either of the form $t_1 = t_2$, or of the form $p(t_1, \ldots, t_n)$ such that $p \in L^+$ and $p(c_{t_1^I}, \ldots, c_{t_n^I}) \in K$. If $\psi$ is of the form $t_1 = t_2$, then $\langle I, K, K\rangle \models \psi$ implies $t_1^I = t_2^I$, hence $\langle I, J', K\rangle \models \psi$, for any $J'$ under consideration. If $\psi$ is of the form $p(t_1, \ldots, t_n)$, since $J'|_{L^+} = K|_{L^+}$ for all $J'$ under consideration, we conclude that $p(c_{t_1^I}, \ldots, c_{t_n^I}) \in J'$. Hence, $\langle I, J', K\rangle \models \psi$, for all $J' \subseteq K$ such that $J'|_{L^+} = K|_{L^+}$.

For the induction step, assume that the claim holds for any factual sentence of connective nesting depth $n-1$, and let $\psi$ be a factual sentence of connective nesting depth $n$. Consider the case where $\psi$ is of the form $\psi_1 \wedge \psi_2$, respectively $\psi_1 \vee \psi_2$. Since $\psi$ is factual, so are $\psi_1$ and $\psi_2$, both of connective nesting depth depth $n-1$. From $\langle I, K, K\rangle \models \psi$ we conclude $\langle I, K, K\rangle \models \psi_1$ and (or) $\langle I, K, K\rangle \models \psi_2$. The induction hypothesis applies, proving $\langle I, J', K\rangle \models \psi_1$ and (or) $\langle I, J', K\rangle \models \psi_2$, for all $J' \subseteq K$ such that $J'|_{L^+} = K|_{L^+}$, i.e., $\langle I, J', K\rangle \models \psi$ for all $J'$ under consideration.

Next, let $\psi$ be of the form $\psi_1 \rightarrow \bot$. Then, $\psi_1$ is factual and of connective nesting depth depth $n-1$. If $\langle I, K, K\rangle \models \psi$, then $\langle I, K\rangle \models \psi$, hence $\langle I, K\rangle \not\models \psi_1$, i.e., $\langle I, K, K\rangle \not\models \psi_1$ and by Case (d) in the proof of Proposition 8, the same holds for any $\langle I, J', K\rangle$ such that $J' \subseteq K$. Thus, in particular for $J' \subseteq K$ such that $J'|_{L^+} = K|_{L^+}$, it follows that $\langle I, J', K\rangle \not\models \psi_1$. Moreover, $\langle I, K\rangle \models \psi$, and therefore $\langle I, J', K\rangle \models \psi \rightarrow \bot$, for all $J' \subseteq K$ such that $J'|_{L^+} = K|_{L^+}$.

Eventually, consider a quantified sentence $\psi$, i.e., $\psi$ is of the form $\forall x \psi_1(x)$ or $\exists x \psi_1(x)$. Since $\psi$ is factual, so are the sentences $\psi_1(c_\varepsilon)$, for all $\varepsilon \in \mathcal{U}$, which are of connective nesting depth $n-1$. Then, $\langle I, K, K\rangle \models \psi$ implies $\langle I, K, K\rangle \models \psi_1(c_\varepsilon)$ (as well as $\langle I, K\rangle \models \psi_1(c_\varepsilon)$), for all (some) $\varepsilon \in \mathcal{U}$. The induction hypothesis applies, proving for all $J'$ under consideration, that $\langle I, J', K\rangle \models \psi_1(c_\varepsilon)$ and $\langle I, K\rangle \models \psi_1(c_\varepsilon)$, for all $\varepsilon \in \mathcal{U}$, respectively that $\langle I, J', K\rangle \models \psi_1(c_\varepsilon)$, for some $\varepsilon \in \mathcal{U}$. Therefore, $\langle I, J', K\rangle \models \psi$ for all $J' \subseteq K$ such that $J'|_{L^+} = K|_{L^+}$. This concludes the inductive argument and proves the claim for factual sentences over $L^+$ of arbitrary connective nesting.

Concerning the claim of the proposition, since $\langle I, K, K\rangle \models \Gamma$ implies $\langle I, K, K\rangle \models \phi$ and $\langle I, K, K\rangle \models \psi$, for every $L^+$-$L^-$-sentence $\phi$ in $\Gamma$ and every factual sentence $\psi$ in $\Gamma$, we conclude that $\langle I, J', K\rangle \models \phi$ and $\langle I, J', K\rangle \models \psi$, for all $J' \subseteq K$ such that $J'|_{L^+} = K|_{L^+}$. This proves $\langle I, J', K\rangle \models \Gamma$, for all $J'$ under consideration. □

Having lifted the essential properties to the case of $L^+$-$L^-$-theories, it comes at no surprise that we end up with respective closure conditions for QHT-equivalence interpretations.

**Definition 11** *Given a first-order theory $\Gamma$ over $\mathcal{L}$, sets of predicate symbols $L^+ \subseteq \mathcal{L}'$, $L^- \subseteq \mathcal{L}'$, $\mathcal{L}' \supseteq \mathcal{L}$, and a QHT-interpretation $M = \langle I, J, K\rangle$, we say that*

- $\langle I, K, K\rangle$ *is* $L^+$-total *iff* $\langle I, K|_{L^+}, K\rangle$ *is closed in* $E_s(\Gamma)$*;*
- $M$ *is* $L^+$-closed *in* $E_s(\Gamma)$ *iff* $\langle I, J', K\rangle \in E_s(\Gamma)$*, for all* $J' \subseteq K$ *such that* $J|_{L^+} \subseteq J'|_{L^+}$ *and* $J'|_{L^-} \subseteq J|_{L^-}$.

Also the characteristic structures for a semantic characterization are defined in straight-forward analogy.

**Definition 12** *A QHT-interpretation $M = \langle I, J, K\rangle$ is a* QHT-hyperequivalence interpretation wrt. $L^+$ and $L^-$ *of a first-order theory $\Gamma$ iff $\langle I, K, K\rangle$ is $L^+$-total and there exists a QHT-interpretation $\langle I, J', K\rangle$ such that $J = J'|_{L^+\cup L^-}$ and $\langle I, J', K\rangle$ is $L^+$-closed in $E_s(\Gamma)$.*
*The set of QHT-hyperequivalence interpretations wrt. $L^+$ and $L^-$ of a first-order theory $\Gamma$ is denoted by $E^{L^+}_{L^-}(\Gamma)$.*

Eventually, we arrive at a characterization of relativized hyperequivalence for general first-order theories under answer-set semantics, where contexts are restricted on the predicate level.

**Theorem 8** *Two first-order theories $\Gamma_1, \Gamma_2$ are relativized hyperequivalent wrt. $L^+$ and $L^-$ if and only if they coincide on their QHT-hyperequivalence interpretations wrt. $L^+$ and $L^-$, symbolically $\Gamma_1 \;{}^{L^+}_{L^-}\!\!\equiv\; \Gamma_2$ iff $E^{L^+}_{L^-}(\Gamma_1) = E^{L^+}_{L^-}(\Gamma_2)$.*

**Proof**. In the following, we will use the following notational simplification: For any set of ground atoms $J$, we write $J_+$ for $J|_{L^+}$, and $J_-$ for $J|_{L^-}$.

For the only-if direction suppose $\Gamma_1 \;{}^{L^+}_{L^-}\!\!\equiv\; \Gamma_2$ and towards a contradiction assume that $E^{L^+}_{L^-}(\Gamma_1) \neq E^{L^+}_{L^-}(\Gamma_2)$. W.l.o.g. let $\langle I, J, K\rangle \in E^{L^+}_{L^-}(\Gamma_1)$ and $\langle I, J, K\rangle \notin E^{L^+}_{L^-}(\Gamma_2)$ (the other case is symmetric). Note that $\langle I, J, K\rangle \in E^{L^+}_{L^-}(\Gamma_1)$ implies that $\langle I, K, K\rangle$ is $L^+$-total, i.e., $\langle I, K_+, K\rangle$ is closed in $E_s(\Gamma_1)$. This implies that $\langle I, K_+, K\rangle$ is in $E^{L^+}_{L^-}(\Gamma_1)$. Suppose $\langle I, K_+, K\rangle$ is not in $E^{L^+}_{L^-}(\Gamma_2)$. Then, either $\langle I, K, K\rangle \not\models \Gamma_2$, or there exists $K_+ \subseteq J' \subset K$ such that $\langle I, J', K\rangle \models \Gamma_2$. Let $\Gamma = K_+$ over $\mathcal{L}' = \langle \mathcal{F} \cup \mathcal{U}_\mathcal{C}, \mathcal{P}\rangle$. and observe that in both cases $\langle I, K\rangle$ is not an answer set of $\Gamma_2 \cup \Gamma$. In the former case because $\langle I, K, K\rangle \not\models \Gamma_2 \cup \Gamma$, in the latter because $J' \subset K$ and $\langle I, J', K\rangle \models \Gamma_2 \cup \Gamma$ (note that $\langle I, J', K\rangle \models \Gamma$ by Proposition 9). However, $\langle I, K\rangle$ is an answer set of $\Gamma_1 \cup \Gamma$. Indeed, $\langle I, K_+, K\rangle$ is closed in $E_s(\Gamma_1)$. And for any $J' \subset K$ such that $K_+ \not\subseteq J'_+$, obviously $\langle I, J', K\rangle$ is a non-total QHT-countermodel of $\Gamma$. Consequently $\langle I, K, K\rangle$ is total-closed in $E_s(\Gamma_1 \cup \Gamma)$. Because $\Gamma$ is an $L^+$-$L^-$-theory, this contradicts $\Gamma_1 \;{}^{L^+}_{L^-}\!\!\equiv\; \Gamma_2$. Thus, we conclude that $\langle I, K_+, K\rangle \in E^{L^+}_{L^-}(\Gamma_2)$. Note that therefore $\langle I, K, K\rangle$ is $L^+$-total for $\Gamma_2$, which implies that $\langle I, K|_A, K\rangle$ is in $E^{L^+}_{L^-}(\Gamma_2)$, hence $J \subset K|_A$ and $J_+ \subset K_+$. Consider the following theory over $\mathcal{L}' = \langle \mathcal{F} \cup \mathcal{U}_\mathcal{C}, \mathcal{P}\rangle$: $\Gamma = J_+ \cup \{\alpha \rightarrow \beta \mid \alpha \in K_- \setminus J_-, \beta \in K_+ \setminus J_+\}$. We show that $\langle I, K\rangle$ is an answer set of $\Gamma_1 \cup \Gamma$. Obviously, $\langle I, K, K\rangle \models \Gamma$ because $J_+ \subset K_+$ and $\beta \in K$ for every $\beta \in K_+ \setminus J_+$. Therefore, $\langle I, K, K\rangle \models \Gamma_1 \cup \Gamma$. Towards a contradiction, assume that there exists $J' \subset K$ such that $\langle I, J', K\rangle \models \Gamma_1 \cup \Gamma$. From $\langle I, J', K\rangle \models \Gamma$, we conclude that either $J'_+ = K_+$, or that $J_+ \subseteq J'_+ \subset K_+$ and $J'_- \subseteq J_-$. In both cases, $\langle I, J', K\rangle \not\models \Gamma_1$. In the former case because $\langle I, K, K\rangle$ is $L^+$-total, i.e., $\langle I, K_+, K\rangle$ is closed in $E_s(\Gamma_1)$. In the latter case it is a consequence of the fact that $\langle I, J, K\rangle \in E^{L^+}_{L^-}(\Gamma_1)$, which implies $\langle I, J', K\rangle \not\models \Gamma_1$ by $L^+$-closure. This contradicts our assumption concerning the existence of $J' \subset K$ such that $\langle I, J', K\rangle \models \Gamma_1 \cup \Gamma$, and proves that $\langle I, K\rangle$ is an answer set of $\Gamma_1 \cup \Gamma$. However, $\langle I, K\rangle$ is not an answer set of $\Gamma_2 \cup \Gamma$. To wit, since $\langle I, J, K\rangle \notin E^{L^+}_{L^-}(\Gamma_2)$, there exists $J' \subset K$ such that $J_+ \subseteq J'_+$,

$J'_- \subseteq J_-$, and $\langle I, J', K\rangle \models \Gamma_2$. Moreover, $\langle I, J', K\rangle$ is a QHT-model of $\Gamma$. Observe that $J'_- \subseteq J_-$ implies that $\langle I, J', K\rangle$ is a QHT-model of every sentence of the form $\alpha \to \beta$ in $\Gamma$. Hence, $\langle I, J', K\rangle \models \Gamma_2 \cup \Gamma$, and since $J' \subset K$, it follows that $\langle I, K\rangle$ is not an answer set of $\Gamma_2 \cup \Gamma$. Note that $\Gamma$ is an $L^+$-$L^-$-theory, which contradicts $\Gamma_1 \,{}^{L^+}_{L^-}\!\!\equiv \Gamma_2$. This proves $E^{L^+}_{L^-}(\Gamma_1) = E^{L^+}_{L^-}(\Gamma_2)$.

For the if direction, suppose $E^{L^+}_{L^-}(\Gamma_1) = E^{L^+}_{L^-}(\Gamma_2)$ and towards a contradiction assume that $\Gamma_1 \,{}^{L^+}_{L^-}\!\!\not\equiv \Gamma_2$. W.l.o.g. let $\langle I, K\rangle$ be an answer set of $\Gamma_1 \cup \Gamma$ for some $L^+$-$L^-$-theory $\Gamma$, such that $\langle I, K\rangle$ is not an answer set of $\Gamma_2 \cup \Gamma$ (the other case is symmetric). Then, $\langle I, K, K\rangle$ is an equivalence interpretation of both, $\Gamma_1$ and $\Gamma$, and $\langle I, K_+, K\rangle$ is closed in $E_s(\Gamma_1 \cup \Gamma)$, which implies (taking Proposition 9 into account) that $\langle I, K, K\rangle$ is $L^+$-total for $\Gamma_1$ and $\langle I, K|_A, K\rangle$ is in $E^{L^+}_{L^-}(\Gamma_1)$. Therefore, $\langle I, K|_A, K\rangle$ is also in $E^{L^+}_{L^-}(\Gamma_2)$, with the consequence that $\langle I, K, K\rangle$ is in $E_s(\Gamma_2)$, and thus $\langle I, K, K\rangle \in E_s(\Gamma_2 \cup \Gamma)$. Since by assumption $\langle I, K\rangle$ is not an answer set of $\Gamma_2 \cup \Gamma$, there exists $J \subset K$ such that $\langle I, J, K\rangle \notin E_s(\Gamma_2 \cup \Gamma)$, i.e., $\langle I, J, K\rangle \models \Gamma_2 \cup \Gamma$. Since $\langle I, K|_A, K\rangle \in E^{L^+}_{L^-}(\Gamma_2)$, it holds that $J|_A \subset K|_A$. Moreover, $J_+ \subset K_+$ due to $L^+$-totality of $\langle I, K, K\rangle$. Clearly, $\langle I, J|_A, K\rangle$ is not in $E^{L^+}_{L^-}(\Gamma_2)$ as witnessed by $\langle I, J, K\rangle \models \Gamma_2$, and thus $\langle I, J|_A, K\rangle \notin E^{L^+}_{L^-}(\Gamma_1)$ since $E^{L^+}_{L^-}(\Gamma_1) = E^{L^+}_{L^-}(\Gamma_2)$. From $\langle I, J|_A, K\rangle \notin E^{L^+}_{L^-}(\Gamma_1)$, we conclude that there exists $J' \subseteq K$, such that $J_+ \subseteq J'_+$, $J'_- \subseteq J_-$, and $\langle I, J', K\rangle \notin E_s(\Gamma_1)$, i.e., $J' \subset K$ and $\langle I, J', K\rangle \models \Gamma_1$. By Proposition 8, $\langle I, J, K\rangle \models \Gamma$ implies $\langle I, J', K\rangle \models \Gamma$. Consequently, $\langle I, J', K\rangle \models \Gamma_1 \cup \Gamma$, and since $J' \subset K$, this contradicts our assumption that $\langle I, K\rangle$ is an answer set of $\Gamma_1 \cup \Gamma$, and proves $\Gamma_1 \,{}^{L^+}_{L^-}\!\!\equiv \Gamma_2$. $\square$

In the same way as for propositional theories, the prominent notions of equivalence are obtained as special cases, and the framework gives rise to relativized notions of strong and uniform equivalence for general first-order theories under answer-set semantics. Also in analogy, the role of factual theories is governed by Proposition 8, yielding the following:

**Corollary 6** *Two first-order theories* $\Gamma_1, \Gamma_2$ *are* relativized hyperequivalent wrt. extended $L^+$-$L^-$-theories *if and only if they coincide on their QHT-hyperequivalence interpretations wrt.* $L^+$ *and* $L^-$.

## 5 Non-ground Logic Programs

In this section we apply the characterizations obtained for first-order theories to non-ground logic programs under various extended semantics—compared to the traditional semantics in terms of Herbrand interpretations. For a proper treatment of these issues, further background is required and introduced (succinctly, but at sufficient detail) below.

In non-ground logic programming, we restrict to a function-free first-order signature $\mathcal{L} = \langle \mathcal{F}, \mathcal{P}\rangle$ (i.e., $\mathcal{F}$ contains object constants only) without equality. A *program* $\Pi$ (over $\mathcal{L}$) is a set of rules (over $\mathcal{L}$) of the form (1). A rule $r$ is *safe* if each variable occurring in $H(r) \cup B^-(r)$ also occurs in $B^+(r)$; a rule $r$ is *ground*, if all atoms occurring in it are ground. A program is safe, respectively ground, if all of its rules enjoy this property.

Given $\Pi$ over $\mathcal{L}$ and a universe $\mathcal{U}$, let $\mathcal{L}^{\mathcal{U}}$ be the extension of $\mathcal{L}$ as before. The *grounding of* $\Pi$ wrt. $\mathcal{U}$ and an interpretation $I|_{\mathcal{L}_\mathcal{F}}$ of $\mathcal{L}_\mathcal{F}$ on $\mathcal{U}$ is defined as the set $grd_\mathcal{U}(\Pi, I|_{\mathcal{L}_\mathcal{F}})$ of ground rules obtained from $r \in \Pi$ by $(i)$ replacing any constant $c$ in $r$ by $c_\varepsilon$ such that $I|_{\mathcal{L}_\mathcal{F}}(c) = \varepsilon$, and $(ii)$ all possible substitutions of elements in $\mathcal{C}_\mathcal{U}$ for the variables in $r$.

Adapted from [16], the *reduct* of a program $\Pi$ with respect to a first-order interpretation $I = \langle I|_{\mathcal{L}_\mathcal{F}}, I|_{\mathcal{C}_\mathcal{U}}\rangle$ on universe $\mathcal{U}$, in symbols $grd_\mathcal{U}(\Pi, I|_{\mathcal{L}_\mathcal{F}})^I$, is given by the set of rules

$$a_1 \vee \cdots \vee a_k \leftarrow b_1, \ldots, b_m,$$

obtained from rules in $grd_{\mathcal{U}}(\Pi, I|_{\mathcal{L}_{\mathcal{F}}})$ of the form (1), such that $I \models a_i$ for all $k < i \leq l$ and $I \not\models b_j$ for all $m < j \leq n$.

A first-order interpretation $I$ satisfies a rule $r$, $I \models r$, iff $I \models \Gamma_r$, where $\Gamma_r = \forall \vec{x}(\beta_r \rightarrow \alpha_r)$, $\vec{x}$ are the free variables in $r$, $\alpha_r$ is the disjunction of $H(r)$, and $\beta_r$ is the conjunction of $B(r)$. It satisfies a program $\Pi$, symbolically $I \models \Pi$, iff it satisfies every $r \in \Pi$, i.e., if $I \models \Gamma_\Pi$, where $\Gamma_\Pi = \bigcup_{r \in \Pi} \Gamma_r$.

A first-order interpretation $I$ is called a *generalized answer set* of $\Pi$ iff it satisfies $grd_{\mathcal{U}}(\Pi, I|_{\mathcal{L}_{\mathcal{F}}})^I$ and it is subset minimal among the interpretations of $\mathcal{L}$ on $\mathcal{U}$ with this property.

Traditionally, only *Herbrand interpretations* are considered as the answer sets of a logic program. The set of all (object) constants occurring in $\Pi$ is called the *Herbrand universe* of $\Pi$, symbolically $\mathcal{H}$. If no constant appears in $\Pi$, then $\mathcal{H} = \{c\}$, for an arbitrary constant $c$. A Herbrand interpretation is any interpretation $I$ of $\mathcal{L}_{\mathcal{H}} = \langle \mathcal{H}, \mathcal{P} \rangle$ on $\mathcal{H}$ interpreting object constants by identity, $id$, i.e., $I(c) = id(c) = c$ for all $c \in \mathcal{H}$. A Herbrand interpretation $I$ is an *ordinary answer set* of $\Pi$ iff it is subset minimal among the interpretations of $\mathcal{L}_{\mathcal{H}}$ on $\mathcal{H}$ satisfying $grd_{\mathcal{H}}(\Pi, id)^I$.

Furthermore, an *extended Herbrand interpretation* is an interpretation of $\mathcal{L}$ on $\mathcal{U} \supseteq \mathcal{F}$ interpreting object constants by identity. An extended Herbrand interpretation $I$ is an *open answer set* [18] of $\Pi$ iff it is subset minimal among the interpretations of $\mathcal{L}$ on $\mathcal{U}$ satisfying $grd_{\mathcal{U}}(\Pi, id)^I$.

Note that since we consider programs without equality, we semantically resort to the logic $\mathbf{QHT^s}$, which results from $\mathbf{QHT^s_=}$ by dropping the axioms for equality. Concerning Kripke models, however, in slight abuse of notation, we reuse QHT-models as defined for the general case. A QHT-interpretation $M = \langle I, J, K \rangle$ is called an (extended) QHT Herbrand interpretation, if $\langle I, K \rangle$ is an (extended) Herbrand interpretation. Given a program $\Pi$, $\langle I, K \rangle$ is a generalized answer set of $\Pi$ iff $\langle I, K, K \rangle$ is a QEL-model of $\Gamma_\Pi$, and $\langle I, K \rangle$ is an open, respectively ordinary, answer set of $\Pi$ iff $\langle I, K, K \rangle$ is an extended Herbrand, respectively Herbrand, QEL-model of $\Gamma_\Pi$. Notice that the static interpretation of constants introduced by Item $(i)$ of the grounding process is essential for this correspondences in terms of $\mathbf{QHT^s}$. In slight abuse of notation, we further on identify $\Pi$ and $\Gamma_\Pi$.

As already mentioned for propositional programs, uniform equivalence is usually understood wrt. sets of *ground facts* (i.e., ground atoms). Obviously, uniform equivalence wrt. factual theories implies uniform equivalence wrt. ground atoms. We show the converse direction (lifting Theorem 2 in [29]).

**Proposition 10** *Given two programs* $\Pi_1, \Pi_2$, *then* $\Pi_1 \equiv_u \Pi_2$ *iff* $(\Pi_1 \cup A) \equiv_a (\Pi_2 \cup A)$, *for any set of ground atoms* $A$.

**Proof**. The only-if direction is trivial since any set of ground atoms constitutes a factual theory.

For the if direction, let $(\Pi_1 \cup A) \equiv_a (\Pi_2 \cup A)$, for any set of ground atoms $A$ and towards a contradiction assume that $\Pi_1 \not\equiv_u \Pi_2$. Then, there exists a factual theory $\Gamma$ and a QHT-interpretation $M = \langle I, K, K \rangle$, such that w.l.o.g., $M$ is in $E_a(\Pi_1 \cup \Gamma)$, but $M \notin E_a(\Pi_2 \cup \Gamma)$. From the hypothesis, we conclude that $M \models \Pi_2$ (otherwise $(\Pi_1 \cup K) \not\equiv_a (\Pi_2 \cup K)$, where $K$ is a set of ground facts over $\mathcal{L}' = \langle \mathcal{F} \cup \mathcal{U}_\mathcal{C}, \mathcal{P} \rangle$). Hence, there exists $M' = \langle I, J, K \rangle$, $J \subset K$, such that $M' \models \Pi_2 \cup \Gamma$ whereas $M' \not\models \Pi_1 \cup \Gamma$. This implies that $M' \models \Gamma$, and thus, $M' \not\models \Pi_1$. Moreover, for any $J \subseteq J' \subset K$, $\langle I, J', K \rangle \models \phi$ for any sentence $\phi \in \Gamma$ by Lemma 5, i.e., $\langle I, J', K \rangle \models \Gamma$. Therefore, by the assumption that $M \in E_a(\Pi_1 \cup \Gamma)$, we conclude that, for any $J \subseteq J' \subset K$, $\langle I, J', K \rangle \not\models \Pi_1$, which implies that $M$ is in $E_a(\Pi_1 \cup J)$. On the other hand, $M \notin E_a(\Pi_2 \cup J)$, since $M' \subset M$ and $M' \models \Pi_2 \cup J$. Note that $J$ is a set of ground facts over $\mathcal{L}' = \langle \mathcal{F} \cup \mathcal{U}_\mathcal{C}, \mathcal{P} \rangle$, and because $(\Pi_1 \cup J) \not\equiv_a (\Pi_2 \cup J)$ follows, we arrive at a contradiction. □

Thus, there is no difference whether we consider uniform equivalence wrt. sets of ground facts or factual theories. Since one can also consider sets of clauses, i.e. disjunctions of atomic formulas and their negations, which is a more suitable representation of facts according to the definition of program rules in this article, we adopt the following terminology. A rule $r$ is called a *fact* if $B(r) = \emptyset$, and a *factual program* is a set of facts. Then, by our result $\Pi_1 \equiv_u \Pi_2$ holds for programs $\Pi_1, \Pi_2$ iff $(\Pi_1 \cup \Pi) \equiv_a (\Pi_2 \cup \Pi)$, for any factual program $\Pi$.

## 5.1 Uniform Equivalence under Herbrand Interpretations

The results in the previous section generalize the notion of uniform equivalence to programs under generalized open answer-set semantics and provide alternative characterizations for other notions of equivalence. They apply to programs under open answer-set semantics and ordinary answer-set semantics, when QHT-interpretations are restricted to extended Herbrand interpretations and Herbrand interpretations, respectively. In order to capture strong and uniform equivalence under ordinary answer-set semantics correctly, interpretations under the Standard Name Assumption (SNA) have to be considered, accounting for the potential extensions. For programs $\Pi_1$ and $\Pi_2$ and $e \in \{c, a, s, u\}$, we use $\Pi_1 \equiv_e^{\mathcal{E}} \Pi_2$ and $\Pi_1 \equiv_e^{\mathcal{H}} \Pi_2$ to denote (classical, answer-set, strong, or uniform) equivalence under open answer-set semantics and ordinary answer-set semantics, respectively.

**Corollary 7** *Given two programs $\Pi_1$ and $\Pi_2$, it holds that*

- $\Pi_1 \equiv_e^{\mathcal{E}} \Pi_2$, $C_e^{\mathcal{E}}(\Pi_1) = C_e^{\mathcal{E}}(\Pi_2)$, *and* $E_e^{\mathcal{E}}(\Pi_1) = E_e^{\mathcal{E}}(\Pi_2)$ *are equivalent; and*
- $\Pi_1 \equiv_e^{\mathcal{H}} \Pi_2$, $C_e^{\mathcal{H}}(\Pi_1) = C_e^{\mathcal{H}}(\Pi_2)$, *and* $E_e^{\mathcal{H}}(\Pi_1) = E_e^{\mathcal{H}}(\Pi_2)$ *are equivalent;*

*where $e \in \{c, a, s, u\}$, superscript $\mathcal{E}$ denotes the restriction to extended Herbrand interpretations, and superscript $\mathcal{H}$ denotes the restriction to Herbrand interpretations for $e \in \{c, a\}$, respectively to SNA interpretations for $e \in \{s, u\}$.*

For safe programs the notion of open answer set and the notion of ordinary answer set coincide [5]. Note that a fact is safe if it is ground. We obtain that uniform equivalence coincides under the two semantics even for programs that are not safe. Intuitively, the potential addition of arbitrary facts accounts for the difference in the semantics since it requires to consider larger domains than the Herbrand universe.[5]

**Theorem 9** *Let $\Pi_1, \Pi_2$ be programs over $\mathcal{L}$. Then, $\Pi_1 \equiv_u^{\mathcal{E}} \Pi_2$ iff $\Pi_1 \equiv_u^{\mathcal{H}} \Pi_2$.*

**Proof**. The only-if direction is trivial. For the if direction, towards a contradiction assume that $\Pi_1 \equiv_u^{\mathcal{H}} \Pi_2$ and $\Pi_1 \not\equiv_u^{\mathcal{E}} \Pi_2$. Let $\Pi$ be a factual program such that $M = \langle id, K, K\rangle$ is an extended Herbrand QHT-interpretation over $\mathcal{L}' \supseteq \mathcal{L}$ on $\mathcal{U}'$, such that $M$ is in $E_a^{\mathcal{E}}(\Pi_1 \cup \Pi)$, but $M \notin E_a^{\mathcal{E}}(\Pi_2 \cup \Pi)$. Consider the signature $\mathcal{L}_{\mathcal{U}'} = \langle \mathcal{U}', \mathcal{L}'_{\mathcal{P}} \cup \{d\}\rangle$, where $\mathcal{L}'_{\mathcal{P}}$ are the predicate symbols of $\mathcal{L}'$, and $d \notin \mathcal{L}'_{\mathcal{P}}$ is a fresh unary predicate symbol. Clearly, $\mathcal{L}_{\mathcal{U}'} \supset \mathcal{L}'$. Furthermore let $\Pi'_1 = \Pi_1 \cup \Pi \cup \{d(X)\}$, $\Pi'_2 = \Pi_2 \cup \Pi \cup \{d(X)\}$, and $K' = K \cup \{d(c) \mid c \in \mathcal{U}'\}$. We show that $M' = \langle id, K', K'\rangle$ is in $E_a^{\mathcal{H}}(\Pi'_1)$, but $M' \notin E_a^{\mathcal{H}}(\Pi'_2)$. Since $M \models \Pi_1 \cup \Pi$ and no sentence in $\Pi_1 \cup \Pi$ involves $d$, we conclude $M' \models \Pi_1 \cup \Pi$. By construction, $M'$ is also a QHT-model of $d(X)$, hence $M' \models \Pi'_1$. Moreover, $\langle id, J, K\rangle \not\models \Pi_1 \cup \Pi$, for every $J \subset K$. Therefore, for every $J' = J \cup \{d(c) \mid c \in \mathcal{U}'\}$ such that $J \subset K$, $\langle id, J', K'\rangle \not\models \Pi'_1$. So let us consider proper subsets $J'$ of $K'$ such that $K \subseteq J$, i.e., $J' \subset \{d(c) \mid c \in \mathcal{U}'\}$. In this case $\langle id, J', K'\rangle \not\models d(X)$, and again $\langle id, J', K'\rangle \not\models \Pi'_1$. This proves that $M'$ is in $E_a^{\mathcal{H}}(\Pi'_1)$. On the other hand, if $M \not\models \Pi_2 \cup \Pi$, then $M \not\models \Pi_2$, and since no sentence in $\Pi_2$ involves $d$, we conclude $M' \not\models \Pi_2$, thus $M' \not\models \Pi'_2$. If

[5]Note that this observation also holds for $\mathbf{QHT}^{\mathbf{s}}_{=}$ with functions and the result could be strengthened accordingly.

$M \models \Pi_2 \cup \Pi$, then $\langle id, J, K\rangle \models \Pi_2 \cup \Pi$ for some $J \subset K$. Consider $J' = J \cup \{d(c) \mid c \in \mathcal{U}'\}$. Since $J \subset K$, it holds that $J' \subset K'$, and since no sentence in $\Pi_2 \cup \Pi$ involves $d$, $\langle id, J', K'\rangle \models \Pi_2 \cup \Pi$. Moroever, $\langle id, J', K'\rangle \models \{d(X)\}$ by construction, hence $\langle id, J', K'\rangle \models \Pi_2'$. This proves $M' \notin E_a^{\mathcal{H}}(\Pi_2')$. By observing that $\Pi \cup \{d(X)\}$ is a factual program, we arrive at a contradiction to $\Pi_1 \equiv_u^{\mathcal{H}} \Pi_2$. □

Finally, we turn to the practically relevant setting of finite, possibly unsafe, programs under Herbrand interpretations, i.e., ordinary (and open) answer-set semantics. For finite programs, uniform equivalence can be characterized by HT-models of the grounding, also for infinite domains. In other words, the problems of "infinite chains" as in Example 1 cannot be generated by the process of grounding. Note that the restriction to finite programs also applies to the programs considered to be potentially added.

**Theorem 10** *Let $\Pi_1, \Pi_2$ be finite programs over $\mathcal{L}$. Then, $\Pi_1 \equiv_u^{\mathcal{H}} \Pi_2$ iff $\Pi_1$ and $\Pi_2$ have the same $(i)$ total and $(ii)$ maximal, non-total extended Herbrand QHT-models.*

**Proof**. The only-if direction is obvious. If $\Pi_1 \equiv_u^{\mathcal{H}} \Pi_2$ then also $\Pi_1 \equiv_u^{\mathcal{E}} \Pi_2$ by Theorem 9. This means that $\Pi_1$ and $\Pi_2$ have $(i)$ the same total extended Herbrand QHT-models, as well as the same sets of closed extended Herbrand QHT equivalence interpretations, and thus $(ii)$ the same maximal, non-total extended Herbrand QHT-models.

For the if direction, assume that $\Pi_1$ and $\Pi_2$ have the same total and the same maximal, non-total extended Herbrand QHT-models but, towards a contradiction, that $\Pi_1 \not\equiv_u^{\mathcal{H}} \Pi_2$. Then, there exists a finite factual program $\Pi$, such that $(\Pi_1 \cup \Pi) \not\equiv_a^{\mathcal{H}} (\Pi_2 \cup \Pi)$. W.l.o.g. let $M = \langle I, K, K\rangle$ over $\mathcal{L}' \supseteq \mathcal{L}$ be in $E_a^{\mathcal{H}}(\Pi_1 \cup \Pi)$ and $M \notin E_a^{\mathcal{H}}(\Pi_2 \cup \Pi)$. Let $\mathcal{H}$ denote the Herbrand universe of $\Pi_1 \cup \Pi$. Since $\Pi_1$ and $\Pi$ are finite, $\mathcal{H}$ is finite and so is $grd_{\mathcal{H}}(\Pi_1 \cup \Pi, id)$. Therefore, by minimality, $K$ is finite as well. Note also, that $M$ is a total extended Herbrand QHT-model of $\Pi_1$. By hypothesis $(i)$, $\Pi_1$ and $\Pi_2$ have the same total extended Herbrand QHT-models. Thus, $M$ is also a total extended Herbrand QHT-model of $\Pi_2$. Moreover, there exists a QHT-interpretation $M' = \langle I, J, K\rangle$, such that $J \subset K$ and $M' \models (\Pi_2 \cup \Pi)$, hence $M' \models \Pi_2$. Since $K$ is finite, we conclude that $\Pi_2$ has a maximal, non-total QHT-model $M'' = \langle I, J'', K\rangle$, such that $J' \subseteq J'' \subset K$. We show that this is not the case for $\Pi_1$. $M' \models (\Pi_2 \cup \Pi)$ implies $M' \models \Pi$. Since $\Pi$ is a factual program, by Lemma 5 we conclude that $M'' \models \Pi$. However $M'' \not\models \Pi_1 \cup \Pi$, because $M \in E_a^{\mathcal{H}}(\Pi_1 \cup \Pi)$. Taken together, $M'' \models \Pi$ and $M'' \not\models \Pi_1 \cup \Pi$ implies $M'' \not\models \Pi_1$. Therefore, $M''$ is not a maximal, non-total QHT-model of $\Pi_1$. Observing that $M''$ is an Herbrand QHT-model over $\mathcal{L}'$ and $\mathcal{L}' \supseteq \mathcal{L}$, we conclude that $M''$ is a maximal non-total extended Herbrand QHT-model of $\Pi_2$, but not of $\Pi_1$. Contradiction. □

## 6 Conclusion

Countermodels in equilibrium logic have recently been used in [2] to show that propositional disjunctive logic programs with negation in the head are strongly equivalent to propositional theories, and in [3] to generate a minimal logic program for a given propositional theory.

By means of Quantified Equilibrium Logic, in [24], the notion of strong equivalence has been extended to first-order theories with equality, under the generalized notion of answer set we have adopted. QEL has also been shown to capture open answer-sets [18] and generalized open answer-sets [17], and is a promising framework to study hybrid knowledge bases providing a unified semantics, since it encompasses classical logic as well as disjunctive logic programs under the answer-set semantics [5].

Our results extend these foundations for the research of semantic properties in these generalized settings. First, they complete the picture concerning the prominent notions of equivalence by making uniform equivalence, which so far has only been dealt with for finite programs under ordinary answer-set semantics, amenable to these generalized settings without any finiteness restrictions, in particular on the domain. In addition, the developed notion of relativized hyperequivalence interpretation provides a means for the study of more specific semantic relationships under generalized answer-set semantics. Thus, a general and uniform model-theoretic framework is achieved for the characterization of various notions of equivalence studied in ASP. We have also shown that for finite programs, i.e., those programs solvers are able to deal with, infinite domains do not cause the problems observed for infinite propositional programs, when dealing with uniform equivalence in terms of HT-models of the grounding.

An intersting theoretical problem for further work in this direction is to consider equivalences and correspondence under projections of answer sets [9, 27, 33, 32]. It is not difficult to apply existing techniques to our characterizations in order to obtain characterizations for projective versions of uniform and strong equivalence, as well as for relativized notions thereof, i.e., as long as the same alphabet is permitted for positive and negative occurrences in the context. However, it is not trivial to characterize projective versions of relativized hyperequivalence in the general case, something which also has not been considered for propositional logic programs so far.

Concerning the application of our results, there is ongoing work in the context of combining ontologies and nonmonotonic rules, which is an important issue in knowledge representation and reasoning for the Semantic Web. The study of equivalences and correspondences under an appropriate (unifying) semantics, such as the generalizations of answer-set semantics characterized by QEL, constitute a highly relevant topic for research in this application domain [14]. Like for Datalog, uniform equivalence may serve investigations on query equivalence and query containment in these hybrid settings, and due to the combination of two formalisms, more specific notions of equivalence are needed to obtain the intended notions of correspondence. While our characterizations serve as a basis for these investigations, in particular the simplified treatment of extended signatures for (equivalence) interpretations is expected to be of avail, when considering separate alphabets.

On the foundational level, our results raise the interesting question whether extensions of intuitionistic logics that allow for a direct characterization of countermodels or equivalence interpretations, would provide a more suitable logical foundation of these structures and a more suitable formal apparatus for the study of (at least notions of uniform) equivalences in ASP.

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