Evaluating QBF Solvers: Quantifier Alternations Matter

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Quantified Boolean Formulas (QBF):
- Existential (∃) / universal (∀) quantification of propositional variables.
- Checking QBF satisfiability: PSPACE-complete.
- QBF encodings: potentially more succinct than propositional logic.

Progress in QBF Reasoning:
- Theory: proof systems (foundations of solver implementations).
- Practice: solving, preprocessing.

Example
Syntax:
- QBF $\psi := \Pi \phi$ in prenex conjunctive normal form (PCNF).
- $\psi = \forall u \exists x. (\overline{u} \lor x) \land (u \lor \overline{x})$.

quantifier prefix propositional CNF


**Example**

**Syntax:**

- QBF $\psi := \hat{Q}.\phi$ in *prenex conjunctive normal form (PCNF)*.
- $\psi = \forall u \exists x. \left( \overline{u} \lor x \right) \land \left( u \lor \overline{x} \right)$.  
  
  **quantifier prefix**  
  **propositional CNF**

**Semantics (recursive):**

- Assign variables in prefix ordering, recurse on simplified formula $\psi[A]$ under current assignment $A$.
- **Base cases:** $\bot$ is unsatisfiable, $\top$ is satisfiable.
- $\forall u.\psi$ is satisfiable iff $\psi[u/\bot]$ and $\psi[u/\top]$ are satisfiable.
- $\exists x.\psi$ is satisfiable iff $\psi[x/\bot]$ or $\psi[x/\top]$ is satisfiable.

PCNF $\psi$ above is satisfiable:

- $\psi[u/\bot] = \exists x. (\overline{x})$ is satisfiable by setting $x$ to $\bot$.
- $\psi[u/\top] = \exists x. (x)$ is satisfiable by setting $x$ to $\top$. 
Example

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Quantifier Alternations in PCNFs:
- A PCNF $Q_1B_1Q_2B_2\ldots Q_nB_n$. $\phi$ has $n \geq 1$ quantifier blocks $Q_iB_i$.
- $Q_iB_i$: sets $B_i$ of variables, quantifiers $Q_i \in \{\forall, \exists\}$ with $Q_i \neq Q_{i+1}$.
- A PCNFs with $n$ quantifier blocks has $n - 1$ quantifier alternations.

Polynomial Hierarchy (PH): cf. [MS72, Sto76, Wra76]
- Framework to describe the complexity of problems beyond NP.
- Satisfiability problem of a given PCNF is located in PH.

Proposition (cf. [BB09, MS72, Sto76, Wra76])
- Let $\psi := Q_1B_1\ldots Q_nB_n$. $\phi$ be a PCNF with $k \geq 0$ alternations.
- $Q_1 = \exists$: satisfiability problem of $\psi$ is $\Sigma_{k+1}^P$-complete.
- $Q_1 = \forall$: satisfiability problem of $\psi$ is $\Pi_{k+1}^P$-complete.
### Introduction (4)

<table>
<thead>
<tr>
<th>Class</th>
<th>Prefix Pattern</th>
<th>Problems (e.g.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Sigma_1^P = NP$</td>
<td>$\exists B_1.\phi$</td>
<td>Checking prop. logic satisfiability</td>
</tr>
<tr>
<td>$\Pi_1^P = co-NP$</td>
<td>$\forall B_1.\phi$</td>
<td>Checking prop. logic validity</td>
</tr>
<tr>
<td>$\Sigma_2^P$</td>
<td>$\exists B_1 \forall B_2.\phi$</td>
<td>MUS membership testing [JS11, Lib05], encodings of conformant planning [Rin07], ASP-related problems [FR05], abstract argumentation [CDG+15]</td>
</tr>
<tr>
<td>$\Pi_2^P$</td>
<td>$\forall B_1 \exists B_2.\phi$</td>
<td></td>
</tr>
<tr>
<td>...</td>
<td></td>
<td></td>
</tr>
<tr>
<td>PSPACE</td>
<td>$Q_1 B_1 \ldots Q_n B_n.\phi$</td>
<td>LTL model checking [SC85], NFA language inclusion, games [Sch78] (n depending on problem instance)</td>
</tr>
</tbody>
</table>
Introduction (5): Solving Paradigms

1. **Expansion** [AB02, Bie04]:
   - RAReQS [JKMSC16], Ijtihad [BBSH+18], Rev-Qfun [Jan18],
   - DynQBF [CW17].

2. **QDPLL (backtracking search)** [CGS98]:
   - GhostQ [JKMSC16, KSGC10].

3. **Nested SAT solving**:
   - QSTS [BJT16a, BJT16b].

4. **Clause selection and clausal abstraction**:
   - QESTO [JM15b], CAQE [RT15, Ten17].

5. **Backtracking search with learning (QCDCL)** [GNT06, Let02, ZM02b]:
   - DepQBF [LE17], Qute [PSS17].

6. **Hybrid approach (expansion, QCDCL)**:
   - Heretic [BBSH+18] (applies Ijtihad and DepQBF).

Theory of (orthogonal) proof systems, e.g.: [BCJ15, JM15a, Ten17].
Progress in QBF Solving — Problems:

- Largely driven by empirical evaluation.
- Practically relevant problems: QBF encodings on low levels in PH.
- Risk of convergence of research to few alternations, cf. [Hoo95].
- Solver rankings by solved instances might not reflect diversity and strength of available paradigms.
Outline and Contributions

Our Contributions:

- Study impact of quantifier alternations on solver performance.
- Performance of paradigms varies wrt. alternations.
- More fine-grained analysis: highlighting diversity of paradigms.
- Motivation for combining orthogonal paradigms (proof complexity).

⇒ Improve QBF solving for encodings at higher levels up to PSPACE.
Example

\[ \psi = \exists x \forall u \exists y. (\bar{x} \vee y) \land (x \vee \bar{y}) \land (\bar{u} \vee y) \land (u \vee \bar{y}) \]

- **Expand** \( u \): copy CNF and replace \( y \) by fresh \( y_d \) in copy of CNF.
- **Expand** \( \forall \): replace all universal variables by Shannon expansion [Sha49].
  
  Replace \( \hat{Q}\forall x. \phi \) by \( \hat{Q}.(\phi[x/\bot] \land \phi[x/\top]) \).

Expansion of \( \forall \)-Variables: cf. [AB02, Bie04]

- Idea: eliminate all universal variables by Shannon expansion [Sha49].
- Replace \( \hat{Q}\forall x. \phi \) by \( \hat{Q}.(\phi[x/\bot] \land \phi[x/\top]) \).
- Duplicate existential variables inner to \( x \) [Bie04, BK07].
- Finally, apply SAT solving to propositional formula.
- Modern: counter example guided abstraction refinement (CEGAR).
Solving Paradigm (2/2): Q-Resolution Calculus

Definition (Q-Resolution Rule)

\[
\frac{C_1 \cup \{p\} \quad C_2 \cup \{\bar{p}\}}{\text{for all } x \in \hat{Q}: \{x, \bar{x}\} \not\subset (C_1 \cup C_2), \quad \bar{p} \not\in C_1, \quad p \not\in C_2, \text{ and } q(p) = \exists}
\]

Example

\[
\psi = \exists x \forall u \exists y. (x) \land (\bar{x} \lor u \lor y) \land (\bar{x} \lor u \lor \bar{y})
\]

- Traditional Q-resolution [BKF95].
- Must resolve on \(\exists\) pivots (cf. variant [VG12]).
- Cf. stronger variants [ZM02a, BJ12].

- PCNF \(\psi\) is unsatisfiable iff empty clause \(\emptyset\) can be derived.
- Resolution-based QBF solvers: inspired by conflict-driven clause learning (CDCL) and DPLL algorithm for SAT solving.
Solving Paradigm (2/2): Q-Resolution Calculus

**Definition (Reduction Rule)**

\[
\begin{align*}
C \cup \{l\} & \quad \text{for all } x \in \hat{Q}: \{x, \bar{x}\} \not\subseteq (C \cup \{l\}), \; q(l) = \forall, \text{ and} \\
& \quad l' < l \text{ for all } l' \in C \text{ with } q(l') = \exists \\
\end{align*}
\]

**Example**

\[
\psi = \exists x \forall u \exists y. (x) \land (\bar{x} \lor u \lor y) \land (\bar{x} \lor u \lor \bar{y})
\]

- Reduction removes “trailing” \(\forall\)-literals.
- Local rule, applied to individual clauses.

- PCNF \(\psi\) is unsatisfiable iff empty clause \(\emptyset\) can be derived.
- Resolution-based QBF solvers: inspired by conflict-driven clause learning (CDCL) and DPLL algorithm for SAT solving.
Experiments (1)

Benchmark Set and Solvers:

- QBFEVAL’17: 523 prenex CNF instances, 1800 CPU sec., 7 GB mem.
- Focus: instances not solved in preprocessing by HQSpre [WRMB17].
- Top-ranked solvers, based on orthogonal paradigms / proof systems.

Goals of Experimental Evaluation:

- Typical solver rankings: by total number of solved instances.
- Theory: numbers of alternations $\approx$ levels in polynomial hierarchy.
- Performance analysis wrt. instances and their numbers of alternations.
- How do different solving paradigms perform wrt. alternations?
- Is there a single best approach that dominates all the others?
Experiments (2): Alternation Bias

Table: Histograms of the benchmark sets illustrating the numbers of formulas (#f) in classes given by the number of qblocks (#q).

<table>
<thead>
<tr>
<th>#q</th>
<th>#f</th>
<th>#q</th>
<th>#f</th>
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<td>4–</td>
<td>97</td>
<td>4–</td>
<td>97</td>
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</table>

(a) 523 original benchmarks.
(b) 312 benchmarks filtered by HQSpree.
(c) 312 benchmarks preprocessed by HQSpree.
Table: Solvers and corresponding paradigms ($P$), solved instances ($S$), unsatisfiable ($\perp$) and satisfiable ones ($\top$), and uniquely solved instances.

<table>
<thead>
<tr>
<th>Solver</th>
<th>$P$</th>
<th>$S$</th>
<th>$\perp$</th>
<th>$\top$</th>
<th>$U$</th>
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<td>16</td>
<td>1</td>
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<td>34</td>
<td>15</td>
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<tr>
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<td>46</td>
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<td>22</td>
<td>9</td>
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<td>0</td>
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<td>Ijtihad</td>
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<td>36</td>
<td>27</td>
<td>9</td>
<td>1</td>
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</table>

(a) Filtered instances.

<table>
<thead>
<tr>
<th>Solver</th>
<th>$P$</th>
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<th>$\perp$</th>
<th>$\top$</th>
<th>$U$</th>
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<tr>
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<td>6</td>
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<td>QESTO</td>
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<td>97</td>
<td>63</td>
<td>34</td>
<td>1</td>
</tr>
<tr>
<td>Rev-Qfun</td>
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<td>28</td>
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<td>24</td>
<td>21</td>
<td>17</td>
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</table>

(b) Preprocessed instances.
Table: Instances solved in classes by numbers of qblocks (\#q) and numbers of formulas in each class (\#f). Only class winners (bold face) are shown, paradigms (P:) are indicated in the first row.

(a) Filtered instances.

(b) Preprocessed instances.
Experiments (5): Class-Based Analysis — Paradigms

Table: Instances solved by solving paradigms 1 to 6 in classes by numbers of qblocks (#q).

<table>
<thead>
<tr>
<th>#q</th>
<th>1</th>
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<td>9</td>
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(a) Filtered instances.

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(b) Preprocessed instances.
**Table:** Instances solved by the virtual best solver (VBS) in classes by number of qblocks (#q), number of formulas (#f) in each class, and relative contribution (%) of each solver to instances solved by the VBS.

<table>
<thead>
<tr>
<th>#q</th>
<th>#f</th>
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<th>Rev-Qfun</th>
<th>CAQE</th>
<th>DepQBF</th>
<th>QSTS</th>
<th>RAReQS</th>
<th>Heretic</th>
<th>Qute</th>
<th>DynQBF</th>
<th>QESTO</th>
<th>Ijtihad</th>
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</table>

(a) Filtered instances.
Experiments (6): Class-Based VBS Analysis — Solvers

Table: Instances solved by the virtual best solver (VBS) in classes by number of qblocks (#q), number of formulas (#f) in each class, and relative contribution (%) of each solver to instances solved by the VBS.

<table>
<thead>
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<th>#q</th>
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<th>RAReqS</th>
<th>QESTO</th>
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<td>2.1</td>
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</table>

(a) Preprocessed instances.
Table: Instances solved by the virtual best solver (VBS) in classes by number of qblocks (#q), number of formulas (#f) in each class, and relative contribution (%) of solving paradigms to instances solved by the VBS.

<table>
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</tbody>
</table>

(a) Preprocessed instances.
Summary

**QBF Solving:**
- Different approaches, empirically-driven development of QBF tools.
- Power of different approaches often not reflected in overall rankings.
- Majority of available QBF benchmarks: problems from low PH levels.

**Our Empirical Results:**
- More fine-grained picture of solver performance.
- Highlighting different strengths in instance classes by alternations.
- VBS: large potential for combining different approaches.

**Future Work and Open Problems:**
- Risk of convergence of research to certain approaches / formulas.
- Proof complexity and alternations, cf. [BHP17, BBH18, Che16].
References
References


References III


References IV


References V


References VII

Towards Generalization in QBF Solving via Machine Learning.

In AAAI. AAAI Press, 2018.

Solving QBF with counterexample guided refinement.

Expansion-based QBF solving versus Q-resolution.

Solving QBF by Clause Selection.
On Deciding MUS Membership with QBF. 

A Non-prenex, Non-clausal QBF Solver with Game-State Learning. 

[LE17] Florian Lonsing and Uwe Egly. 
DepQBF 6.0: A Search-Based QBF Solver Beyond Traditional QCDCL. 
[Let02] Reinhold Letz.
Lemma and Model Caching in Decision Procedures for Quantified Boolean Formulas.

[Lib05] Paolo Liberatore.
Redundancy in logic I: CNF propositional formulae.

The Equivalence Problem for Regular Expressions with Squaring Requires Exponential Space.


[Sch78] Thomas J Schaefer. 
On the Complexity of Some Two-Person Perfect-Information Games. 

[Sha49] Claude Elwood Shannon. 
The Synthesis of Two-Terminal Switching Circuits. 

[Sto76] Larry J. Stockmeyer. 
The Polynomial-Time Hierarchy. 
[Ten17] Leander Tentrup. 
On Expansion and Resolution in CEGAR Based QBF Solving. 

Contributions to the Theory of Practical Quantified Boolean Formula Solving. 

[Wra76] Celia Wrathall. 
Complete Sets and the Polynomial-Time Hierarchy. 
