# Taxonomic and Uncertain Reasoning in Object-Oriented Databases

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**Abstract**

We present a coherent modeling and reasoning methodology to extend object-oriented databases towards taxonomic and uncertain knowledge. Current ISA-hierarchies are enriched by more general t-classes to improve conceptual modeling. The t-classes themselves are then integrated with probabilistic constraints to express uncertainty. The information content of such taxonomic and probabilistic knowledge can be visualized naturally by special Hasse-diagrams. The second contribution concerns efficient taxonomic and probabilistic deduction. Through a careful interaction between taxonomic and probabilistic knowledge much better inference methods can be devised.

## 1 Introduction

The evolution of database technology from the relational model into object-oriented databases (OODBs) and deductive databases has been pushed forward to a state where stable and usable systems are becoming widely available; also an integration of both paradigms into so-called DOOD-systems is taking place. Such systems can deal with various kinds of objects and query mechanisms, however, except for attempts to deal with null-values, only the specification and processing of certain (true/false) information is supported so far. But uncertainty pervades the real world and it seems mandatory for future advanced data models to capture it explicitly and appropriately. Being a topic of interest in AI for quite some period of time, database researchers are picking up increased interest in the field recently. There is e.g. work in the relational context by Barbará et al. [BGMP92] and for deductive databases by Ng and Subramanian [NS92b], Lakshmanan and Sadri [LS94]. Our own previous work comprises a major project with the so-called DUCK-system for reasoning under a conditional probability model (see e.g. [GKT91], [KTG92], [Thö94], [TKG94]).

Before extending current OODBs towards uncertainty, the following aspects must be considered: Since uncertain knowledge comes in a variety of flavors in the real world, which of its facets should we provide? (See [Pea88], [Som90] for a discussion.) Like other database researchers we decided on the probabilistic model. Next consider inheritance along ISA-hierarchies, being characteristic of current OODBs. This allows limited forms of classification into subclass-relationships, but not reaching the expressiveness of taxonomic classification found in terminological systems (see e.g. [Bra91]). On the other hand it is natural to understand taxonomic knowledge as special case of probabilistic knowledge. Consider the sentences "all dogs are domestic animals" and "at least 70% of all domestic animals are dogs" as examples of taxonomic and probabilistic knowledge respectively.

The rest of this paper is now organized as follows. Sec. 2 is concerned about taxonomic knowledge and introduces the central notion of t-classes, being the basis for formulating and querying taxonomic knowledge. Sec. 3 adds uncertainty to a taxonomic knowledge-base by a clean coupling mechanism. Taxonomic and uncertain knowledge can be visualized by special Hasse-diagrams. Thereafter Sec. 4 deals with the problem of efficient deduction of taxonomic knowledge. Similarly, Sec. 5 describes the deduction calculus for uncertain knowledge. In contrast to previous work within the DUCK-approach, taxonomic knowledge can be incorporated into the uncertain deduction process for increased precision. We describe how even new taxonomic knowledge can be derived dynamically from uncertain knowledge. Sec. 6 compares our results to related work and finally Sec. 7 gives a summary and an outlook on ongoing work.

# 2 Taxonomic knowledge

*Taxonomic knowledge* and reasoning is a widely explored field. One of its uses is in terminological reasoning to answer typical questions like "is a class of objects subset of or equal to some other classes of objects". Considering object-oriented databases, it immediately comes to mind that ISA-hierarchies express similar constraints on the extents of classes. However, except for the basic acyclic ISA-graphs supported, more general features to express taxonomic knowledge are missing.

## 2.1 Syntax and semantics of taxonomic knowledge

We start out with the definition of an expressive language for taxonomic classes, called t-classes.

**Definition 2.1 (Syntax of t-class terms)**

a) We consider an alphabet $\mathcal{A} := \{\boldsymbol{\emptyset}, \boldsymbol{\mathcal{O}}, B_1, \ldots, B_k\}$ of constants.

   1) $\boldsymbol{\emptyset}$ is called *empty t-class term*, $\boldsymbol{\mathcal{O}}$ is called *universal t-class term.*

   2) $\mathcal{B}^+ := \{B_1, \ldots, B_k\}$ denotes the set of *positive t-class terms.*

3) $\mathcal{B} := \mathcal{B}^+ \cup \{\overline{B}_1, \ldots, \overline{B}_k\}$ denotes the set of *basic t-class terms.*

b) The set $\mathcal{C}$ of all *conjunctive t-class terms* is the minimal set with

1) $\boldsymbol{\emptyset} \in \mathcal{C}$, $\boldsymbol{\mathcal{O}} \in \mathcal{C}$, $\mathcal{B} \subseteq \mathcal{C}$,
2) $C, D \in \mathcal{C} \Longrightarrow (CD) \in \mathcal{C}$.

c) The set $\mathcal{G}$ of all *general t-class terms* is the minimal set with

1) $\mathcal{A} \subseteq \mathcal{G}$,
2) $C, D \in \mathcal{G} \Longrightarrow (CD), (C \cup D), (\overline{C}) \in \mathcal{G}$. ∎

For convenience we apply the usual preference rules to omit unnecessary parentheses.

**Definition 2.2 (Semantics of t-class terms)**
Let $\mathcal{D}$ be the (not necessarily finite) set of admissable objects of an OODB schema $\mathcal{S}$. An *interpretation* $\mathcal{J} := (\mathcal{O}, J)$ of the set of general t-class terms $\mathcal{G}$ consists of:

a) A finite set of objects $\mathcal{O} \subseteq \mathcal{D}$ of an OODB over $\mathcal{S}$.

b) A mapping $J : \mathcal{B}^+ \longrightarrow 2^{\mathcal{O}}$. A positive t-class $B_i$ is interpreted as a class instance $J(B_i) = \{o_1, \ldots, o_l\}$ for objects $o_i \in \mathcal{O}$ and $l \geq 0$. $J$ is extended to $\mathcal{G}$ by:

1) $J(\boldsymbol{\mathcal{O}}) := \mathcal{O}$ and $J(\boldsymbol{\emptyset}) := \emptyset$,
2) $J(CD) := J(C) \cap J(D)$,
3) $J(C \cup D) := J(C) \cup J(D)$,
4) $J(\overline{C}) := \mathcal{O} \backslash J(C)$.

In the sequel, we identify $\boldsymbol{\mathcal{O}}$ with $\mathcal{O}$, $\boldsymbol{\emptyset}$ with $\emptyset$ and $B_i$ with $J(B_i)$. ∎

General t-class terms allow us to formulate all propositional expressions over the alphabet $\mathcal{A}$. Classification among general t-classes can now be achieved by formulating taxonomic knowledge in the following language.

**Definition 2.3 (Syntax of taxonomic formulas)**
Let $D_1, \ldots, D_m, D, C \in \mathcal{G}$.

a) $C \subseteq D$ is called a *subclass* formula,

b) $C = D$ is called an *equality* formula,

c) $D_1 \parallel \ldots \parallel D_m$ is called a *disjointness* formula,

d) $D_1 \parallel \ldots \parallel D_m = D$ is called a *partition* formula.

The set of all taxonomic formulas is denoted $\mathcal{TF}$. ∎

**Definition 2.4 (Semantics of taxonomic formulas)**
An interpretation $\mathcal{J} = (\mathcal{O}, J)$ of $\mathcal{G}$ is extended to an interpretation of $\mathcal{TF}$ into $\{true, false\}$ as follows:

a) $J(C \subseteq D)$, iff $J(C) \subseteq J(D)$,

b) $J(C = D)$, iff $J(C \subseteq D)$ and $J(D \subseteq C)$,

c) $J(D_1 \parallel \ldots \parallel D_m)$, iff $J(D_i) \cap J(D_j) = \emptyset$ for all $i, j \in [1 : m]$ with $i < j$,

d) $J(D_1 \parallel \ldots \parallel D_m = D)$, iff $J(D_1 \parallel \ldots \parallel D_m)$ and $J(D_1 \cup \ldots \cup D_m = D)$. ■

**Definition 2.5 (Models, satisfiability, logical consequence)**
Let $\mathcal{F}_1, \mathcal{F}_2 \subseteq \mathcal{TF}$.

a) An interpretation $(\mathcal{O}, J)$ of $\mathcal{G}$ is a *model* of $\mathcal{F}_1$, iff $J(F)$ for all $F \in \mathcal{F}_1$.

b) $\mathcal{F}_1$ is *satisfiable*, iff there exists a model of $\mathcal{F}_1$.

c) $\mathcal{F}_2$ is a *logical consequence* of $\mathcal{F}_1$ (denoted $\mathcal{F}_1 \models \mathcal{F}_2$), iff every model of $\mathcal{F}_1$ is a model of $\mathcal{F}_2$.

We abbreviate $\emptyset \models \mathcal{F}_1$ by $\models \mathcal{F}_1$. ■

**Definition 2.6 (Taxonomic knowledge-base and queries)**

a) A *taxonomic knowledge-base* consists of a set of taxonomic formulas.

b) Let $V$ be an infinite set of variables, let $F \in \mathcal{TF}$ and $A, B \in \mathcal{C} \cup V$. The expressions $?F$, $?A \subseteq B$ and $?AB = \emptyset$ are called *taxonomic queries*. ■

## 2.2 Integrating taxonomic knowledge and object-oriented databases

From the various OODB-features (see e.g. [ABD+89]), our interest here focuses on ISA-hierarchies, which specify super-/subclass relationships with inheritance. We can easily verify that the relationship $B_i$ *isa* $B_j$ for two classes $B_i$ and $B_j$ of an OODB schema implies a subclass formula $B_i \subseteq B_j$ for two positive t-classes $B_i$ and $B_j$. On the other hand, many general t-classes and associated taxonomic formulas have no counterpart in existing OODBs, for instance the negated t-class $\overline{B}_i$.

To make full taxonomic reasoning available as an extension of existing OODB technology, we propose the following two-step procedure:

*Step 1:* ISA-hierarchies of a conventional OODB schema are translated into $\mathcal{TF}$ according to Fig. 1. The diagrams are given in OMT-notation (see [RBP+91]),

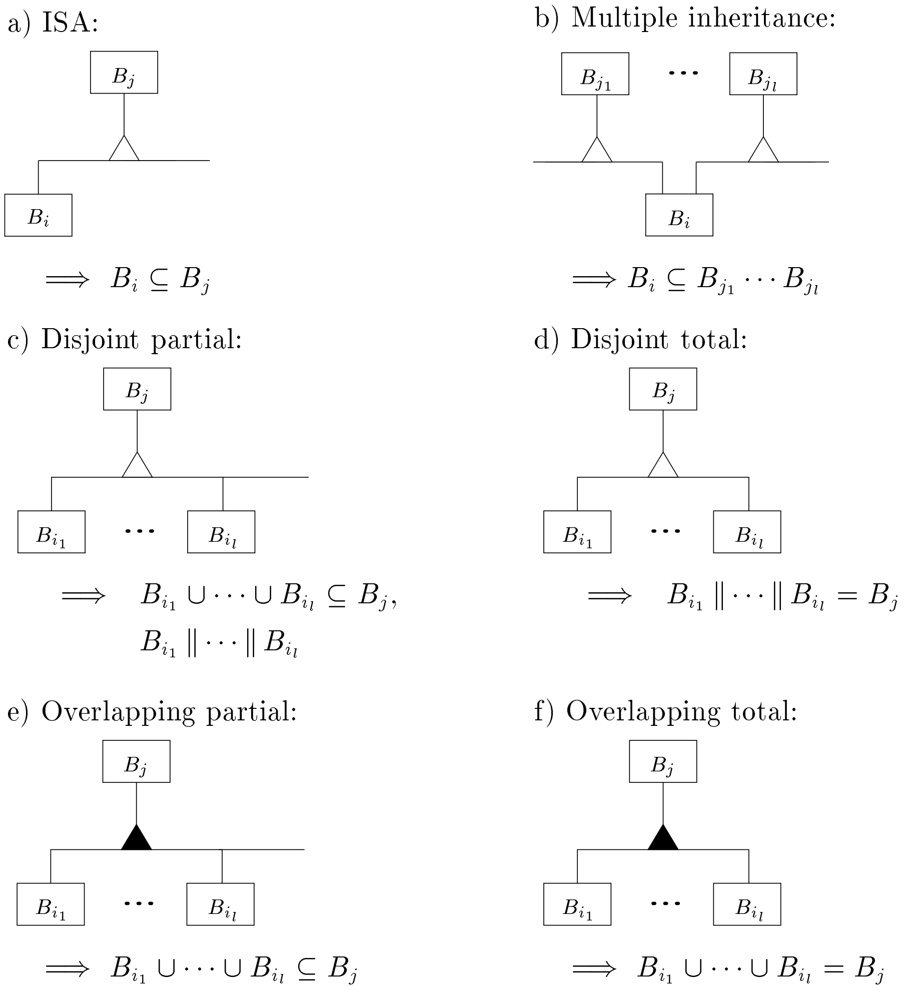


Figure 1: Some OMT-constructs translated into $\mathcal{TF}$-formulas

neglecting attributes and methods. The translation into $\mathcal{TF}$ provides a natural extensional semantics of the OMT-notation, which originally just deals with inheritance relationships without considering the extensions of classes at all.

*Step 2 (optional):* Full $\mathcal{TF}$ is made available as a means to specify *additional* taxonomic knowledge. This has no impact on the inheritance schema fixed before, it solely affects class extensions.

**Example 2.7**

*Step 1:* Let's consider an airline application and assume that the persons relevant for an air carrier are passengers and employees. These groups may not be disjoint, i.e. employees can be passengers as well. Employees are distinguished in pilots, ground_staff, flight_attendants and the remaining personnel. Subgroups of the ground_staff are doctors and nurses. They are not necessarily disjoint.

After designing relevant attributes and methods, assume that the inheritance hierarchy is fixed (see Fig. 2, left side).

*Step 2:* Additionally it is required that the group of pilots should be disjoint to all other groups of personnel. This is modeled by extra taxonomic knowledge (see Fig. 2, right side).

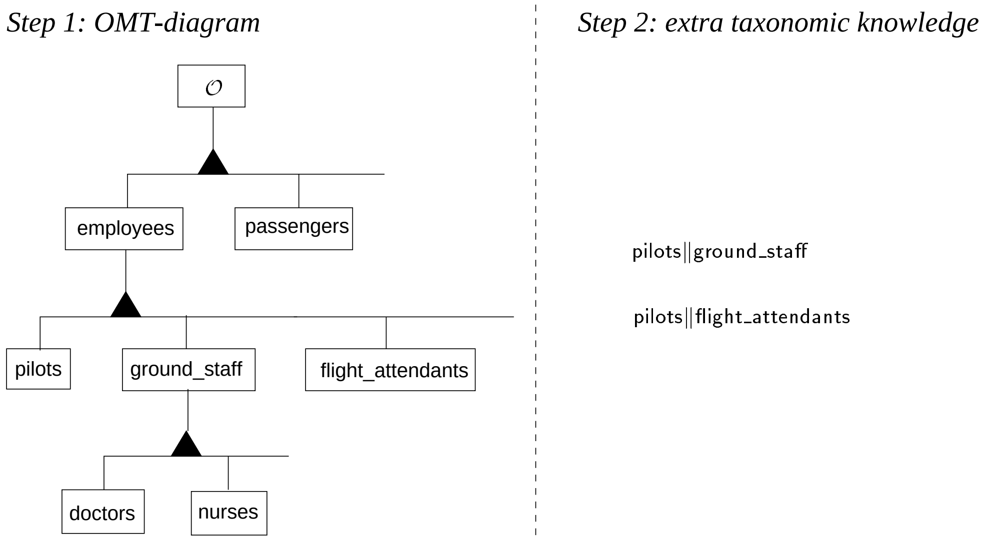


Figure 2: Modeling of an airline application

It is crucial to observe that these two disjointness formulas could have been implemented by changing the original OMT-diagram radically by introducing an "artificial" superclass for flight_attendants and ground_staff. Obviously this procedure becomes very clumsy for larger numbers of subclasses and would render the conceptual model hard to follow up. Moreover, the introduction of those superclasses does not possess any advantages w.r.t. software reuse, since by assumption in step 1 all cases of useful inheritance had been fixed before. Our approach avoids such problems by decoupling inheritance and extra extensional taxonomic knowledge carefully.
The conceptual modeling process, i.e. the design of the OMT-diagram and the formulation of additional taxonomic knowledge, should be supported interactively by the system. For instance one has to be careful about disjoint specialization in combination with multiple inheritance. Therefore we use taxonomic deduction to check the consistency of the model (i.e. checking whether a class is forced to be empty) and to allow queries about the consequences of postulated taxonomic knowledge. For instance the user may wonder whether the groups of pilots and nurses are disjoint and pose the following taxonomic query:

$$?\mathsf{nurses} \parallel \mathsf{pilots},$$

yielding the answer *Yes.* □

**Example 2.8**
*Step 1:* Let's assume an OODB capturing the knowledge that all dogs and cats are domestic animals, that no dog is a cat and that we want to talk about

barking animals (see Fig. 3, left side).

*Step 2:* The additional knowledge that all cats do not bark can be expressed by the taxonomic formula $\mathsf{cats} \subseteq \overline{\mathsf{bark}}$ (see Fig. 3, right side). This leaves the class hierarchy unchanged.[1]

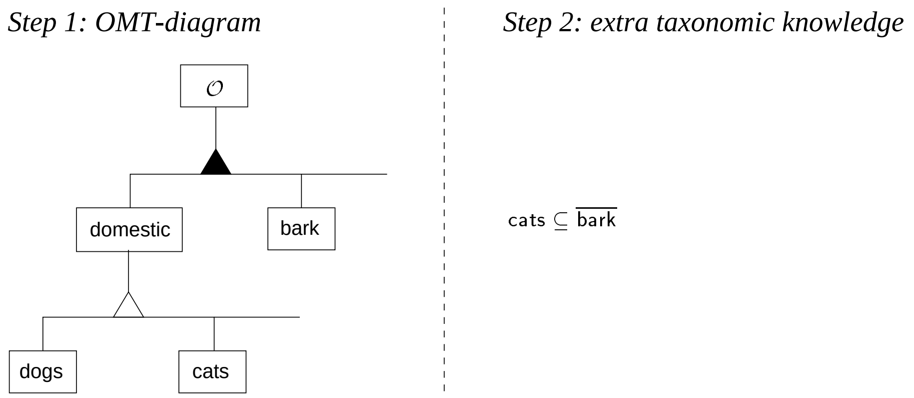


Figure 3: Modeling of an animal world □

**Example 2.9**
For the rest of this paper, we want to consider a slightly different example. Let the alphabet $\mathcal{A}$ be defined by $\mathcal{A} = \{\emptyset, \mathcal{O}, \mathsf{dogs}, \mathsf{cats}, \mathsf{domestic}, \mathsf{bark}\}$ and let the taxonomic knowledge-bases $\mathcal{T}_0$ and $\mathcal{T}_1$ be given by:

$$\begin{aligned}\mathcal{T}_0 &= \{\mathsf{dogs} \cup \mathsf{cats} \subseteq \mathsf{domestic}, \mathsf{cats} \parallel \mathsf{dogs}\},\\ \mathcal{T}_1 &= \{\mathsf{dogs} \cup \mathsf{cats} \subseteq \mathsf{domestic}, \mathsf{cats} \parallel \mathsf{dogs}, \mathsf{bark} \subseteq \mathsf{dogs}\}.\end{aligned}$$ □

## 2.3 Equivalent t-classes and Hasse-diagrams

All taxonomic knowledge introduced so far has a natural visualization as *Hasse-diagrams*. Such Hasse-diagrams shall provide a compact and redundancy-free encoding of all (explicitly defined or deducible) taxonomic knowledge.

There are different general t-class terms with the same interpretation due to the axioms of set theory (e.g. $AA = A$, $A\overline{A} = A\,\emptyset = \emptyset$, $\overline{\overline{A}} = A$, $\ldots$) or because of taxonomic formulas (e.g. $A \subseteq B \Longrightarrow AB = A$). We want to identify general t-class terms with an identical interpretation. For this purpose we define an equivalence relation that takes into account all set-theoretic laws and taxonomic knowledge.

**Definition 2.10 (Equivalent t-classes)**
Let $\mathcal{T}$ be a satisfiable taxonomic knowledge-base.

[1] Note that the original OMT-diagram must be changed radically, if one attempts to express $\mathsf{cats} \subseteq \overline{\mathsf{bark}}$ directly in OMT.

a) The equivalence relation $\sim_{\mathcal{T}}$ on $\mathcal{G}$ is defined by:

$$G_1 \sim_{\mathcal{T}} G_2 \quad :\Leftrightarrow \quad \mathcal{T} \models G_1 = G_2.$$

b) Let $[G]_{\sim_{\mathcal{T}}}$ be abbreviated by $G_{\mathcal{T}}$, then:

$\mathcal{B}_{\mathcal{T}} := \{B_{\mathcal{T}} \mid B \in \mathcal{B}\}$, $\mathcal{C}_{\mathcal{T}} := \{C_{\mathcal{T}} \mid C \in \mathcal{C}\}$, $\mathcal{G}_{\mathcal{T}} := \{G_{\mathcal{T}} \mid G \in \mathcal{G}\}$.

c) The partial order $\subseteq$ is canonically extended to $\mathcal{G}_{\mathcal{T}}$ by:

$$A_{\mathcal{T}} \subseteq_{\mathcal{T}} B_{\mathcal{T}} \quad :\Leftrightarrow \quad (AB)_{\mathcal{T}} = A_{\mathcal{T}}.$$

d) The operations $\cap$, $\cup$ and $^{-}$ are canonically extended to $\mathcal{G}_{\mathcal{T}}$ by:

$$\begin{array}{lcl} A_{\mathcal{T}} B_{\mathcal{T}} & := & (AB)_{\mathcal{T}}, \\ A_{\mathcal{T}} \cup_{\mathcal{T}} B_{\mathcal{T}} & := & (A \cup B)_{\mathcal{T}}, \\ \mathrm{comp}_{\mathcal{T}}(A_{\mathcal{T}}) & := & (\overline{A})_{\mathcal{T}}. \end{array}$$

■

**Lemma 2.11**
*The predicate $\subseteq_{\mathcal{T}}$ and the operations $\cap_{\mathcal{T}}$, $\cup_{\mathcal{T}}$ and $\mathrm{comp}_{\mathcal{T}}$ are well-defined.*

**Proof:** Let $A, B, C, D \in \mathcal{C}$ with $A \sim_{\mathcal{T}} C$ and $B \sim_{\mathcal{T}} D$. Since $\mathcal{T} \models A = C$ and $\mathcal{T} \models B = D$, we get $\mathcal{T} \models AB = CD$, $\mathcal{T} \models A \cup B = C \cup D$ and $\mathcal{T} \models \overline{A} = \overline{C}$. Thus the operations $\cap_{\mathcal{T}}$, $\cup_{\mathcal{T}}$ and $\mathrm{comp}_{\mathcal{T}}$ are well-defined. Since $\cap_{\mathcal{T}}$ is well-defined, also the predicate $\subseteq_{\mathcal{T}}$ is well-defined. □

The finite partially ordered set $(\mathcal{C}_{\mathcal{T}}, \subseteq_{\mathcal{T}})$[2] completely represents the taxonomic knowledge $\mathcal{T}$.[3] It can graphically be represented by a Hasse-diagram:
Let $R$ be the minimal subset of $\subseteq_{\mathcal{T}}$ for which the transitive and reflexive closure is equal to $\subseteq_{\mathcal{T}}$. Hasse-diagrams are acyclic directed graphs that take all the elements of $\mathcal{C}_{\mathcal{T}}$ as nodes and the elements of $R$ as directed edges drawn "arrow up". We use the elements of $\mathcal{B} \cup \{\emptyset, \mathcal{O}\}$ as labels to the corresponding nodes in the Hasse-diagrams. In this way we can get a representative for all non-labeled nodes: nodes corresponding to $\mathcal{C}_{\mathcal{T}}$ contain the conjunction of all basic t-class terms attached to upper nodes.[4]

[2] Note that $(\mathcal{C}_{\mathcal{T}}, \subseteq_{\mathcal{T}})$ is a complete lattice with $A_{\mathcal{T}} \wedge B_{\mathcal{T}} = A_{\mathcal{T}} B_{\mathcal{T}}$ and $A_{\mathcal{T}} \vee B_{\mathcal{T}} \supseteq_{\mathcal{T}} A_{\mathcal{T}} \cup_{\mathcal{T}} B_{\mathcal{T}}$ for all $A_{\mathcal{T}}, B_{\mathcal{T}} \in \mathcal{C}_{\mathcal{T}}$. Note that $A_{\mathcal{T}} \vee B_{\mathcal{T}} = A_{\mathcal{T}} \cup_{\mathcal{T}} B_{\mathcal{T}}$ generally does not hold (see Ex. 2.12: $\mathsf{dogs}_{\mathcal{T}_1} \vee \mathsf{cats}_{\mathcal{T}_1} = \mathsf{domestic}_{\mathcal{T}_1} \neq \mathsf{dogs}_{\mathcal{T}_1} \cup_{\mathcal{T}_1} \mathsf{cats}_{\mathcal{T}_1}$).

[3] Note that we can represent $\mathcal{T}$ in conjunctive normal form. Each disjunctive minterm is now equivalent to a subclass formula over $\mathcal{C}$.

[4] The Hasse-diagrams in the following examples are constructed according to Sec. 4.

**Example 2.12 (Ex. 2.9 continued)**
Fig. 4 shows the Hasse-diagram of $\mathcal{C}_{\mathcal{T}_1}$ of Ex. 2.9. Each node represents an element of $\mathcal{C}_{\mathcal{T}_1}$, e.g. the node labeled $\mathsf{domestic}$ represents $\mathsf{domestic}_{\mathcal{T}_1} \in \mathcal{C}_{\mathcal{T}_1}$. The node directly lying under the nodes labeled $\mathsf{domestic}$ and $\overline{\mathsf{cats}}$ represents $(\mathsf{domestic}\,\overline{\mathsf{cats}})_{\mathcal{T}_1} \in \mathcal{C}_{\mathcal{T}_1}$. The taxonomic knowledge is represented by all the edges between the nodes: since the node labeled $\mathsf{dogs}$ is subnode of the node labeled $\mathsf{domestic}$, we have $\mathsf{dogs}_{\mathcal{T}_1} \subseteq_{\mathcal{T}_1} \mathsf{domestic}_{\mathcal{T}_1}$ and thus $\mathcal{T}_1 \models \mathsf{dogs} \subseteq \mathsf{domestic}$. Since the intersection of the two nodes labeled $\mathsf{dogs}$ and $\mathsf{cats}$ is $\emptyset$, we have $(\mathsf{dogs}\,\mathsf{cats})_{\mathcal{T}_1} = \emptyset_{\mathcal{T}_1}$ and thus $\mathcal{T}_1 \models \mathsf{dogs} \parallel \mathsf{cats}$.
The taxonomic query $?\mathsf{domestic} = \mathsf{cats} \parallel \mathsf{dogs}$ yields the answer *No*, since there may be other domestic animals than dogs or cats (in Fig. 4 we have $\overline{\mathsf{domestic}}_{\mathcal{T}_1} \neq (\overline{\mathsf{cats}}\,\overline{\mathsf{dogs}})_{\mathcal{T}_1}$). The query $?\mathsf{cats} \subseteq \overline{\mathsf{bark}}$ yields the answer *Yes*, since all barking animals are dogs and no dog is a cat (in Fig. 4 we have $\mathsf{cats}_{\mathcal{T}_1} \subseteq_{\mathcal{T}_1} \overline{\mathsf{bark}}_{\mathcal{T}_1}$). □

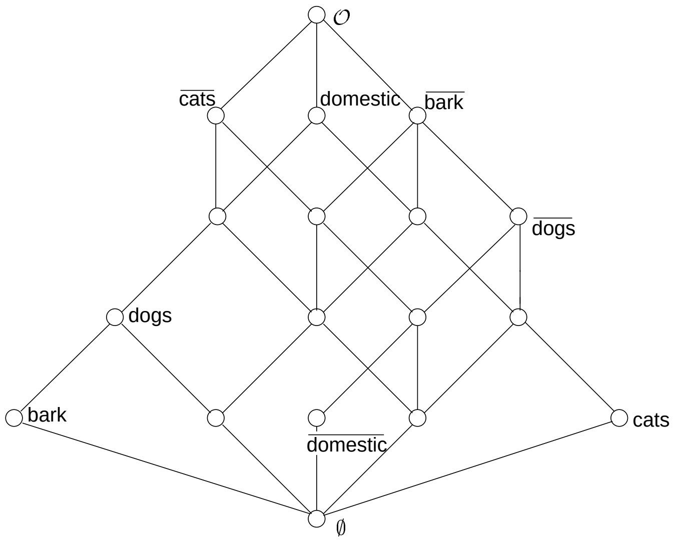


Figure 4: Hasse-diagram for the taxonomic knowledge of Ex. 2.9

# 3 Probabilistic knowledge

As announced, we combine object-orientation and taxonomies with probability for modeling uncertainty. To achieve this we assume that we already have defined a taxonomic knowledge-base as described by the two-step procedure before. In a third step we now add uncertainty capabilities by a probabilistic model.

## 3.1 Syntax and semantics of uncertain knowledge

Probabilistic reasoning is a tremendously complex task with many intractability results known. Therefore many researchers a priori restrict their attention to tractable subclasses. For the purposes of this paper we present a very wide class, which continues our previous work with the DUCK-approach ([GKT91], [KTG92], [TKG94]).

**Definition 3.1 (Syntax of probabilistic formulas)**
Let $A, B, C \in \mathcal{C}$, $x_1, x_2, y_1, y_2 \in [0,1]$, $x_1 \le x_2$, $y_1 \le y_2$, $k \in \boldsymbol{R}^+$ and $\theta \in \{\le, \ge\}$.

a) $pos(A)$ is called a *positive probability statement*,

b) $A \xrightarrow{x_1,x_2} B$ is called an *uncertain rule*,

c) $A \underset{y_1,y_2}{\overset{x_1,x_2}{\longleftrightarrow}} B$ is called a *correlation rule*,

d) $A \xrightarrow{\theta k[B]} C$ is called a *comparative rule*,

e) $I(A, B, C)$ is called a *conditional independence*.

We abbreviate $A \xrightarrow{x,x} B$ by $A \xrightarrow{x} B$. The set of all probabilistic formulas is denoted $\mathcal{PF}$. ■

**Lemma 3.2**
*$J(\mathcal{G})$ is a $\sigma$-algebra.*

**Proof:** For $M, N \in J(\mathcal{G})$ there exists elements $G, H \in \mathcal{G}$ with $M = J(G)$ and $N = J(H)$. Since $G \cup H \in \mathcal{G}$ and $\overline{G} \in \mathcal{G}$, we get $J(G \cup H) \in J(\mathcal{G})$ and $J(\overline{G}) \in J(\mathcal{G})$, hence $J(G) \cup J(H) \in J(\mathcal{G})$ and $\mathcal{O} \backslash J(G) \in J(\mathcal{G})$. We do not have to consider infinite unions, since the number of elements in $J(\mathcal{G})$ is finite. □

**Definition 3.3 (Semantics of probabilistic formulas)**
An interpretation $(\mathcal{O}, J, P)$ of $\mathcal{PF}$ consists of an interpretation $(\mathcal{O}, J)$ of $\mathcal{G}$ and a probability measure $P : J(\mathcal{G}) \longrightarrow [0,1]$ for the measure space $(\mathcal{O}, J(\mathcal{G}))$.

a) $J(pos(A))$, iff $P(J(A)) > 0$,

b) $J(A \xrightarrow{x_1,x_2} B)$, iff $P(J(A)) = 0$ or $x_1 \le P(J(B)|J(A)) \le x_2$,[5]

c) $J(A \underset{y_1,y_2}{\overset{x_1,x_2}{\longleftrightarrow}} B)$, iff $J(A \xrightarrow{x_1,x_2} B)$ and $J(B \xrightarrow{y_1,y_2} A)$,

d) $J(A \xrightarrow{\theta k[B]} C)$, iff $P(J(A)) = 0$ or $P(J(C)|J(A)) \ \theta \ k \cdot P(J(B)|J(A))$,

e) $J(I(A, B, C))$, iff $P(J(AB)) = 0$ or $P(J(C)|J(AB)) = P(J(C)|J(B))$.

We disallow the user to formulate an uncertain rule $A \xrightarrow{0,1} B$ or a correlation rule $A \underset{0,1}{\overset{0,1}{\longleftrightarrow}} B$, since they do not carry any probabilistic information. ■

The notions of models, satisfiability and logical consequence are defined in the usual way.

[5] $P(J(B)|J(A))$ is the conditional probability of $J(B)$ under $J(A)$.

**Definition 3.4 (Probabilistic knowledge-base and queries)**

a) A taxonomic knowledge-base, extended by a non-empty set of probabilistic formulas, is called a *probabilistic knowledge-base*.

b) Let $V$ be an infinite set of variables. For $A, B \in \mathcal{C} \cup V$, $x_1, x_2 \in [0,1] \cup V$, the expression $?A \xrightarrow{x_1, x_2} B$ is called a *probabilistic rule query*.[6] ■

Note that in Def. 3.3 we do not commit ourselves to a specific interpretation of probability. Probabilistic formulas may be derived by statistical means or may express subjective beliefs of an agent. Furthermore they can express user-defined (application-dependent) uncertain constraints for the frame of the "real world" to be modeled in an OODB. The latter usage is illustrated by the following example.

**Example 3.5 (Ex. 2.7 continued)**
*Step 3:* In the OMT-diagram of Fig. 2 employees and passengers are not modeled as disjoint classes, i.e. passengers may be employees. However, fictitiously assume because of economical reasons their number should be restricted to 10 per cent. By company policy the number of pilots, ground_staff and flight_attendants is restricted to maximally 5, 30 and 20 per cent of the number of employees, resp., and the number of doctors and nurses is restricted to maximally 2 and 5 per cent of the number of ground_staff. Additionally assume that by legal regulations there must be at least 1 percent of doctors and 1 percent of nurses working in the ground_staff of an airline.
We model these restrictions by the following probabilistic formulas:

$$\begin{array}{ll} \mathsf{passengers} \xrightarrow{0,0.1} \mathsf{employees}, & \mathsf{employees.} \xrightarrow{0,0.05} \mathsf{pilots}, \\ \mathsf{employees} \xrightarrow{0,0.3} \mathsf{ground\_staff}, & \mathsf{employees} \xrightarrow{0,0.2} \mathsf{flight\_attendants}, \\ \mathsf{ground\_staff} \xrightarrow{0.01,0.02} \mathsf{doctors}, & \mathsf{ground\_staff} \xrightarrow{0.01,0.05} \mathsf{nurses}. \end{array}$$ □

## 3.2 Integrating uncertain and taxonomic knowledge

In Sec. 2 we have introduced taxonomic knowledge-bases as a natural extension of ISA-hierarchies of OODBs. Probabilistic knowledge-bases enable us to represent *uncertain constraints* over object-oriented databases. However, we must be careful to smoothly integrate all pieces of knowledge to come up with a coherent knowledge representation schema. This is achieved by the following observation.

For practical purpose it is sufficient and desirable to restrict the interpretation $(\mathcal{O}, J, P)$ of $\mathcal{PF}$ as follows:

[6] Queries for correlation rules, comparative rules and conditional independences can be treated similarly.

**Assumption 3.6**
*We just consider interpretations* $(\mathcal{O}, J, P)$ *of* $\mathcal{PF}$ *with*
$P(J(A)) = 0 \Longleftrightarrow J(A) = \emptyset$ *for all* $A \in \mathcal{C}$.

Under this realistic assumption the following correspondence between taxonomic and probabilistic knowledge arises:

**Lemma 3.7**
*Let* $A, B \in \mathcal{C}$.

$$
\begin{array}{lccc}
1) & A \xrightarrow{1} B & \Longleftrightarrow & A \subseteq B \\
2) & A \xrightarrow{0} B & \Longleftrightarrow & A \parallel B
\end{array}
$$

An important implication of this simple lemma is that on the one hand taxonomic knowledge can be incorporated in the probabilistic deduction process and on the other hand that probabilistic information can have a feedback on taxonomic knowledge.

**Assumption 3.8**
*The extensions of all positive t-classes of an OODB are assumed to be different from* $\emptyset$ *and* $\mathcal{O}$.

This assumption enables us to state another simple, but important observation. Since $P(J(B)) = 0$ entails $J(B) = \emptyset$ and $J(\overline{B}) = \mathcal{O}$, we get:[7]

**Lemma 3.9**
*For all* $B \in \mathcal{B}$ *we have* $\mathrm{pos}(B)$.

There are more positive probability statements that can be derived from a probabilistic knowledge-base (see calculus for uncertain reasoning in Sec. 5).

## 3.3 Visualization of probabilistic knowledge

The Hasse-diagrams of $\mathcal{C}_{\mathcal{T}}$ can be extended by labeled edges and marks to the nodes to represent also uncertain rules and positive probability statements. Since

$$A \xrightarrow{x_1,x_2} B \Longleftrightarrow A \xrightarrow{x_1,x_2} AB \text{ and } (AB)_{\mathcal{T}} \subseteq_{\mathcal{T}} A_{\mathcal{T}},$$

we can visualize $A \xrightarrow{x_1,x_2} B$ by adding a (dashed, downward) arc from node $A_{\mathcal{T}}$ to $(AB)_{\mathcal{T}}$. A positive probability statement $pos(A)$ can be visualized by a node $A_{\mathcal{T}}$ filled black.

[7]This lemma enables us to state a lot of positive probability statements for Ex. 3.5 ($pos$(employees), $pos$(passengers),...).

**Example 3.10 (Ex. 2.9 continued)**
Assume we gain some new knowledge about domestic animals, dogs and cats and their barking capability:

1) at least 70% of all domestic animals are dogs,
2) at least 30% of all domestic animals are cats,
3) 90% of all dogs bark,
4) between 75% and 85% of all non-barking domestic animals are cats.

We can add the following uncertain rules to our taxonomic knowledge-base $\mathcal{T}_1$:

1) $\mathsf{domestic} \xrightarrow{0.7,1} \mathsf{dogs}$,
2) $\mathsf{domestic} \xrightarrow{0.3,1} \mathsf{cats}$,
3) $\mathsf{dogs} \xrightarrow{0.9} \mathsf{bark}$,
4) $\mathsf{domestic}\ \overline{\mathsf{bark}} \xrightarrow{0.75,0.85} \mathsf{cats}$.

The extended Hasse-diagram of $\mathcal{C}_{\mathcal{T}_1}$ is given by Fig. 5 (compare Fig. 4). □

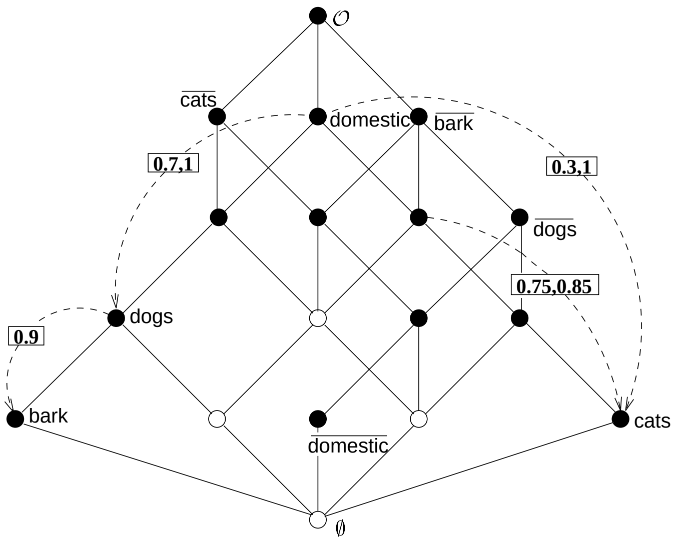


Figure 5: Visualization of the uncertain knowledge of Ex. 3.10

The extended Hasse-diagrams contain all positive probability statements that can be derived from the probabilistic knowledge-base. Note that there may be elements $A_{\mathcal{T}} \in \mathcal{C}_{\mathcal{T}}$ for which it is unknown (due to incomplete knowledge) if the positive probability statement $pos(A)$ or the taxonomic formula $A = \emptyset$ holds. These elements of $\mathcal{C}_{\mathcal{T}}$ are represented by all the white nodes being different from $\emptyset$.

This finishes the conceptual modeling part and we now switch our attention to the technical issues of deduction.

# 4 Deduction of taxonomic knowledge

Efficient processing of taxonomic queries is of course of great concern and poses a major challenge. A straightforward way would be to represent $\mathcal{TF}$ e.g. in disjunctive normal form and to apply some theorem prover for propositional reasoning.

Here we choose an approach that can efficiently be incorporated in the uncertain deduction process. Our aim is to find a representation for the equivalence classes in $\mathcal{C}_\mathcal{T}$ and an efficient technique for the evaluation of taxonomic conditions to be used while processing uncertain inference rules. For this purpose the deduction of taxonomic knowledge is based on a normalized representation for the set of equivalent conjunctive t-classes $\mathcal{C}_\mathcal{T}$. The theoretical background from the field of *formal concept analysis* (see e.g. [Wil82]) makes it easy to find such a normalized representation. The main idea is to take conjunctive t-class terms that are mapped to pairwise disjoint and non-empty subsets of $\mathcal{O}$ and to represent each element of $\mathcal{C}_\mathcal{T}$ as union of some of these conjunctive t-class terms.

## 4.1 Results from formal concept analysis

For technical reasons we have to recapture some notions from formal concept analysis.

**Definition 4.1 (Formal context, [Wil82])**

a) A *formal context* $K = (G, M, \kappa)$ is defined by:

1) a finite set $G(K) := G$ of *objects*,
2) a finite set $M(K) := M$ of *attributes*,
3) an *incidence relation* $\kappa \subseteq G \times M$.

b) For each formal context $(G, M, \kappa)$ the operators $\pi : 2^G \longrightarrow 2^M$ and $\sigma : 2^M \longrightarrow 2^G$ are defined by

$$\begin{aligned} X \longmapsto X\pi &:= \{m \in M \mid \forall g \in X : \ g\,\kappa\, m\}, \\ Y \longmapsto Y\sigma &:= \{g \in G \mid \forall m \in Y : \ g\,\kappa\, m\} \end{aligned}$$

for all $X \subseteq G$ and $Y \subseteq M$. ■

In other words, $X\pi$ are the attributes shared by all objects in $X$, whereas $Y\sigma$ are the objects that share all attributes in $Y$.[8]

[8]Note that the notions of "attributes" and "objects" in this section must not be confused with their usual meaning in object-orientation.

**Example 4.2**
Let's take the objects $G = \{\textsf{Tom, Jerry, Sylvester, Tweety, Snoopy}\}$ and the attributes $M = \{\textsf{dogs, birds, mice, cats, domestic}\}$ denoting that an object is a dog, bird, mouse or a cat and that an object is domestic. Let $\kappa$ be given by the following table:

| | dogs | birds | mice | cats | domestic |
|---|---|---|---|---|---|
| Tom | | | | × | × |
| Jerry | | | × | | |
| Sylvester | | | | × | × |
| Tweety | | × | | | × |
| Snoopy | × | | | | × |

We get e.g.: $\{\textsf{Tom, Snoopy}\}\pi = \{\textsf{domestic}\}$,
$\{\textsf{domestic}\}\sigma = \{\textsf{Tom, Sylvester, Tweety, Snoopy}\}$. □

**Definition 4.3 (Concepts, concept order, [Wil82])**
Let $K = (G, M, \kappa)$ be a formal context.

a) $(A, B) \subseteq G \times M$ is called a *concept* of $K$, iff $A\pi = B$ and $A = B\sigma$.
$A$ is called the *extent* and $B$ the *intent* of $(A, B)$.

b) Let $\mathcal{CL}(K)$ be the set of all concepts of $K$. The partial order $\sqsubseteq$ on $\mathcal{CL}(K)$ is defined by

$$(A_1, B_1) \sqsubseteq (A_2, B_2) \quad :\Leftrightarrow \quad A_1 \subseteq A_2 \quad (\Leftrightarrow B_2 \subseteq B_1).$$

We call $(A_1, B_1)$ a subconcept of $(A_2, B_2)$. ■

**Example 4.4 (Ex. 4.2 continued)**
In Ex. 4.2 we can state:

a) $(\{\textsf{Tom, Sylvester, Tweety, Snoopy}\}, \{\textsf{domestic}\})$ is a concept.
$(\{\textsf{Tweety}\}, \{\textsf{birds}\})$ is no concept, since $\{\textsf{Tweety}\}\pi = \{\textsf{birds, domestic}\}$.

b) $(\{\textsf{Tom, Sylvester}\}, \{\textsf{cats, domestic}\})$ and $(\{\textsf{Snoopy}\}, \{\textsf{dogs, domestic}\})$ are subconcepts of $(\{\textsf{Tom, Sylvester, Tweety, Snoopy}\}, \{\textsf{domestic}\})$. □

**Theorem 4.5 (Main theorem for concept lattices, [Wil82])**
*$(\mathcal{CL}(K), \sqsubseteq)$ is a complete lattice. It holds*

$$\bigvee_{i\in I} (A_i, B_i) = \left(\left(\bigcup_{i\in I} A_i\right)\pi\sigma, \bigcap_{i\in I} B_i\right),$$

$$\bigwedge_{i\in I} (A_i, B_i) = \left(\bigcap_{i\in I} A_i, \left(\bigcup_{i\in I} B_i\right)\sigma\pi\right)$$

*for each family $(A_i, B_i)_{i\in I}$ of concepts.*

$(\mathcal{CL}(K), \sqsubseteq)$ is called the *concept lattice of $K$*.

**Example 4.6 (Ex. 4.2 continued)**
In Ex. 4.2 we have:

$$
\begin{aligned}
&(\{\mathsf{Tom},\ \mathsf{Sylvester}\},\ \{\mathsf{cats},\ \mathsf{domestic}\}) \vee (\{\mathsf{Snoopy}\},\ \{\mathsf{dogs},\ \mathsf{domestic}\})\\
&= (\{\mathsf{Tom},\ \mathsf{Sylvester},\ \mathsf{Tweety},\ \mathsf{Snoopy}\},\ \{\mathsf{domestic}\}),\\
&(\{\mathsf{Tom},\ \mathsf{Sylvester}\},\ \{\mathsf{cats},\ \mathsf{domestic}\}) \wedge (\{\mathsf{Snoopy}\},\ \{\mathsf{dogs},\ \mathsf{domestic}\})\\
&= (\emptyset,\ \{\mathsf{dogs},\ \mathsf{birds},\ \mathsf{mice},\ \mathsf{cats},\ \mathsf{domestic}\}).
\end{aligned}
$$
□

## 4.2 Representation theorem for equivalent t-classes

We recaptured these results from formal concept analysis to find a way to deduce and represent taxonomic knowledge. Now we represent the taxonomic knowledge $\mathcal{T}$ by a formal context, containing all non-empty minimal intersections of basic t-classes from $\mathcal{B}$ as objects. These minimal intersections of basic t-classes can be computed by an incremental "generate and test" algorithm with exponential time-complexity.

**Definition 4.7 (Formal context with respect to $\mathcal{T}$)**
Let $\mathcal{B}^+ = \{B_1, \ldots, B_k\}$ be the set of all positive basic t-classes.

a) The set $\mathcal{CC}$ of all *minimal* conjunctive t-class terms is defined by

$$\mathcal{CC} := \{C_1 C_2 \ldots C_k \mid C_i = B_i \text{ or } C_i = \overline{B_i} \text{ for all } i \in [1:k]\}.$$

b) The formal context $\mathcal{FC_T}$ is defined by:

$$
\begin{aligned}
G(\mathcal{FC_T}) &:= \{C \in \mathcal{CC} \mid \mathcal{T} \not\models C = \emptyset\},\\
M(\mathcal{FC_T}) &:= \mathcal{B},\\
C\ \kappa\ B &:\Leftrightarrow\ \models C \subseteq B.
\end{aligned}
$$

c) $\mathcal{CL_T} := \mathcal{CL}(\mathcal{FC_T})$. ■

**Example 4.8 (Ex. 2.9 continued)**
The formal context $\mathcal{FC}_{\mathcal{T}_1}$ for the taxonomic knowledge $\mathcal{T}_1$ of Ex. 2.9 is given by:

| | bark | $\overline{\text{bark}}$ | domestic | $\overline{\text{domestic}}$ | dogs | $\overline{\text{dogs}}$ | cats | $\overline{\text{cats}}$ |
|---|---|---|---|---|---|---|---|---|
| bark domestic dogs $\overline{\text{cats}}$ | × | | × | | × | | | × |
| $\overline{\text{bark}}$ domestic dogs $\overline{\text{cats}}$ | | × | × | | × | | | × |
| $\overline{\text{bark}}$ $\overline{\text{domestic}}$ $\overline{\text{dogs}}$ $\overline{\text{cats}}$ | | × | | × | | × | | × |
| $\overline{\text{bark}}$ domestic $\overline{\text{dogs}}$ $\overline{\text{cats}}$ | | × | × | | | × | | × |
| $\overline{\text{bark}}$ domestic $\overline{\text{dogs}}$ cats | | × | × | | | × | × | |

□

The formal context $\mathcal{FC_T}$ and the concept lattice $\mathcal{CL_T}$ are complete representations of the taxonomic knowledge $\mathcal{T}$. The next theorem shows that $\mathcal{CL_T}$ is isomorphic to $\mathcal{C_T}$. Therefore the Hasse-diagram of $\mathcal{C_T}$ can be constructed from $\mathcal{CL_T}$.

**Theorem 4.9 (Representation theorem)**
*Let the concept lattice $(\mathcal{CL}_{\mathcal{T}}, \sqsubseteq)$ be defined according to Def. 4.7 and 4.3, let $\mathcal{C}_{\mathcal{T}}$ and $\subseteq_{\mathcal{T}}$ be defined according to Def. 2.10.*

$$(\mathcal{CL}_{\mathcal{T}}, \sqsubseteq) \quad \sim \quad (\mathcal{C}_{\mathcal{T}}, \subseteq_{\mathcal{T}|_{\mathcal{C}_{\mathcal{T}} \times \mathcal{C}_{\mathcal{T}}}})$$

**Proof:** Let the mapping $\beta : \mathcal{C}_{\mathcal{T}} \longrightarrow \mathcal{CL}_{\mathcal{T}}$ be defined by

$$\beta(A_{\mathcal{T}}) = ((\{B \in \mathcal{B} \mid A_{\mathcal{T}} \subseteq_{\mathcal{T}} B_{\mathcal{T}}\})\sigma, \{B \in \mathcal{B} \mid A_{\mathcal{T}} \subseteq_{\mathcal{T}} B_{\mathcal{T}}\})$$

for all $A_{\mathcal{T}} \in \mathcal{C}_{\mathcal{T}}$. Since $\beta((B_{i_1} \cap \ldots \cap B_{i_l})_{\mathcal{T}}) = ((\{B_{i_1}, \ldots, B_{i_l}\})\sigma, \{B_{i_1}, \ldots, B_{i_l}\})$ for all intents $\{B_{i_1}, \ldots, B_{i_l}\} \subseteq M(\mathcal{FC}_{\mathcal{T}})$, the mapping $\beta$ is surjective. $\beta$ is injective, since for each $A_{\mathcal{T}} \in \mathcal{C}_{\mathcal{T}}$ the intersection of all attributes in the intent of its image is a representative for $A_{\mathcal{T}}$. The mapping $\beta$ is obviously order-preserving. $\square$

It can be observed that the elements of $G(\mathcal{FC}_{\mathcal{T}})$ (i.e. the non-empty minimal intersections of basic t-classes) correspond to those nodes of $\mathcal{C}_{\mathcal{T}}$ that belong to the lowest level above $\emptyset$.

## 4.3 Implementation issues and query evaluation

In this section we show that the concept lattice $\mathcal{CL}_{\mathcal{T}}$ provides efficient datastructures to represent $\mathcal{C}_{\mathcal{T}}$. In more detail we elaborate how the elements of $\mathcal{C}_{\mathcal{T}}$ can be represented by the extents and intents of $\mathcal{CL}_{\mathcal{T}}$ and how to perform the $\vee$- and $\wedge$-computation within the corresponding representation by referring to the formal context $\mathcal{FC}_{\mathcal{T}}$. Furthermore we indicate how user-defined conjunctive t-class terms can efficiently be translated into elements of the selected representation.

First we have a look at the representation of $\mathcal{C}_{\mathcal{T}}$ by the extents of $\mathcal{CL}_{\mathcal{T}}$. Within this representation, we can realize the $\vee$- and $\wedge$-computation in linear time and we are able to evaluate all taxonomic queries by bitstring-operations with complexity O(1). The advantages are illustrated by some examples.

**Definition 4.10**
For a set of taxonomic formulas $\mathcal{T}$ and the operations $\pi$ and $\sigma$ of the corresponding formal context $\mathcal{FC}_{\mathcal{T}}$ let *columns*$(\mathcal{FC}_{\mathcal{T}})$ be defined by:

$$\mathit{columns}(\mathcal{FC}_{\mathcal{T}}) \quad := \quad \{(B, \{B\}\sigma) \mid B \in M(\mathcal{FC}_{\mathcal{T}})\}. \qquad \blacksquare$$

**Lemma 4.11 (Extent representation of conjunctive t-classes)**
*Let all conjunctive t-class terms be represented by a subset of $\mathcal{B}$. The operator* trans *translating conjunctive t-class terms into extents of $\mathcal{CL}_{\mathcal{T}}$ and the operations $\vee$ and $\wedge$ computing the extents for the supremum and infimum of the corresponding elements of $\mathcal{CL}_{\mathcal{T}}$ are given by*

$$\begin{aligned}
\mathit{trans}(E) &= \bigcap\{A \mid \exists_{B \in E} : (B, A) \in \mathit{columns}(\mathcal{FC}_{\mathcal{T}})\}, \\
C \vee D &= \bigcap\{A \mid \exists_{B} : (B, A) \in \mathit{columns}(\mathcal{FC}_{\mathcal{T}}), C \cup D \subseteq A\}, \\
C \wedge D &= C \cap D
\end{aligned}$$

*for all $E \subseteq \mathcal{B}$ and all extents $C$ and $D$ of $\mathcal{CL}_\mathcal{T}$. Note that the intersection of an empty family of sets is defined to be $G(\mathcal{FC}_\mathcal{T})$.*

**Proof:** We just prove the correctness of the $\vee$-computation: the claim follows directly from theorem 4.5: let $C$ and $D$ be the extents of two elements of $\mathcal{CL}_\mathcal{T}$. The set $\{A \mid \exists_B : (B, A) \in \mathit{columns}(\mathcal{FC}_\mathcal{T}), C \cup D \subseteq A\}$ contains exactly for each attribute of $(C \cup D)\pi$ one column of the context table. Thus $\bigcap\{A \mid \exists_B : (B, A) \in \mathit{columns}(\mathcal{FC}_\mathcal{T}), C \cup D \subseteq A\}$ contains exactly all the objects which share these attributes and is equal to $(C \cup D)\sigma\pi$. $\square$

**Example 4.12 (Ex. 2.9 continued)**
Let us take the taxonomic knowledge $\mathcal{T}_1$ of Ex. 2.9 to illustrate these definitions. We take a bitstring representation of the extents of $\mathcal{CL}_{\mathcal{T}_1}$, assuming the following order on the elements of $G(\mathcal{FC}_{\mathcal{T}_1})$:

$\#1$ $\mathsf{bark}$ $\mathsf{domestic}$ $\mathsf{dogs}$ $\overline{\mathsf{cats}}$, $\#2$ $\overline{\mathsf{bark}}$ $\mathsf{domestic}$ $\mathsf{dogs}$ $\overline{\mathsf{cats}}$,
$\#3$ $\overline{\mathsf{bark}}$ $\overline{\mathsf{domestic}}$ $\overline{\mathsf{dogs}}$ $\overline{\mathsf{cats}}$, $\#4$ $\overline{\mathsf{bark}}$ $\mathsf{domestic}$ $\overline{\mathsf{dogs}}$ $\overline{\mathsf{cats}}$,
$\#5$ $\overline{\mathsf{bark}}$ $\mathsf{domestic}$ $\overline{\mathsf{dogs}}$ $\mathsf{cats}$.

The bitstring-operations *bitwise and*, *bitwise or*, *bitwise implication* and *bitwise equivalence* are denoted by $\cap$, $\cup$, $\subseteq$ and $=$, respectively. We get:

$$\begin{aligned} \mathit{columns}(\mathcal{FC}_\mathcal{T}) \quad := \quad & \{(\mathsf{domestic}, 11011), (\overline{\mathsf{domestic}}, 00100), (\mathsf{dogs}, 11000), \\ & (\overline{\mathsf{dogs}}, 00111), (\mathsf{cats}, 00001), (\overline{\mathsf{cats}}, 11110), (\mathsf{bark}, 10000), \\ & (\overline{\mathsf{bark}}, 01111)\}. \end{aligned}$$

The bitstring-length is 5 ($\leq 2^4$). The conjunctive t-class $\overline{\mathsf{dogs}}$ $\overline{\mathsf{cats}}$ is translated into the bitstring-representation as follows:

$$\mathit{trans}(\{\overline{\mathsf{dogs}}, \overline{\mathsf{cats}}\}) \quad = \quad 00111 \cap 11110 \quad = \quad 00110.$$

The taxonomic query $?\mathsf{cats} \subseteq \overline{\mathsf{bark}}$ can be evaluated by simply performing the following bitstring-operations, yielding the answer *Yes*:

$$\mathsf{cats} \subseteq \overline{\mathsf{bark}} \quad \equiv \quad 00001 \subseteq 01111 \quad \equiv \quad 11111.$$

The query $?\mathsf{domestic} = \mathsf{cats} \parallel \mathsf{dogs}$ evaluates to *No*:

$$\begin{aligned} \mathsf{domestic} = \mathsf{cats} \parallel \mathsf{dogs} \quad \equiv \quad & (11011 = (00001 \cup 11000)) \\ & \cap((00001 \cap 11000) = 00000) \\ \equiv \quad & (11011 = 11001) \cap (00000 = 00000) \\ \equiv \quad & 00000 \cap 11111 \\ \equiv \quad & 00000. \end{aligned}$$

$\square$

**Efficiency:** In the worst case the length of the bitstring-representation and the number of evaluations of propositional formulas for the construction of the formal context $\mathcal{FC}_\mathcal{T}$ is less or equal to $2^{\frac{|\mathcal{B}|}{2}}$. In practical cases with many taxonomic formulas this length is very small. The complexity of the translation of the user-defined conjunctive t-class terms into the bitstring-representation and the complexity of $\vee$-computation is $\mathrm{O}(|\mathcal{B}|)$. The complexity of $\wedge$-computation and the complexity of query evaluation is $\mathrm{O}(1)$.

Even though the previously presented representation can be implemented efficiently, we should also have a look at the following representation of $\mathcal{C}_\mathcal{T}$ by the intents of $\mathcal{CL}_\mathcal{T}$, which is useful for the development of explanation tools.

**Definition 4.13**
For a set of taxonomic formulas $\mathcal{T}$ and the operations $\pi$ and $\sigma$ of the corresponding formal context $\mathcal{FC}_\mathcal{T}$ let $rows(\mathcal{FC}_\mathcal{T})$ be defined by:

$$rows(\mathcal{FC}_\mathcal{T}) \quad := \quad \{\{C\}\pi \mid C \in G(\mathcal{FC}_\mathcal{T})\}. \qquad \blacksquare$$

**Lemma 4.14 (Intent representation of conjunctive t-classes)**
*Let all conjunctive t-class terms be represented by a subset of $\mathcal{B}$. The operator* trans *translating conjunctive t-class terms into intents of $\mathcal{CL}_\mathcal{T}$ and the operations $\vee$ and $\wedge$ computing the intents for the supremum and infimum of the corresponding elements of $\mathcal{CL}_\mathcal{T}$ are given by*

$$\begin{aligned} trans(E) \quad &= \quad \bigcap\{A \mid A \in rows(\mathcal{FC}_\mathcal{T}), E \subseteq A\}, \\ C \wedge D \quad &= \quad \bigcap\{A \mid A \in rows(\mathcal{FC}_\mathcal{T}), C \cup D \subseteq A\}, \\ C \vee D \quad &= \quad C \cap D \end{aligned}$$

*for all $E \subseteq \mathcal{B}$ and all intents $C$ and $D$ of $\mathcal{CL}_\mathcal{T}$. Note that the intersection of an empty family of sets is defined to be $M(\mathcal{FC}_\mathcal{T})$.*

**Proof:** We just prove the correctness of the $\wedge$-computation: the claim follows directly from theorem 4.5: let $C$ and $D$ be the intents of two elements of $\mathcal{CL}_\mathcal{T}$. The set $\{A \mid A \in rows(\mathcal{FC}_\mathcal{T}), C \cup D \subseteq A\}$ contains exactly for each object of $(C \cup D)\sigma$ one row of the context table. Thus $\bigcap\{A \mid A \in rows(\mathcal{FC}_\mathcal{T}), C \cup D \subseteq A\}$ contains exactly all the attributes that all these objects share and is equal to $(C \cup D)\sigma\pi$. $\square$

**Example 4.15 (Ex. 2.9 continued)**
For $\mathcal{T}_1$ of Ex. 2.9 we get:

$$\begin{aligned} rows(\mathcal{FC}_\mathcal{T}) \quad := \quad &\{\{\mathsf{bark}, \mathsf{domestic}, \mathsf{dogs}, \overline{\mathsf{cats}}\}, \{\overline{\mathsf{bark}}, \mathsf{domestic}, \mathsf{dogs}, \overline{\mathsf{cats}}\}, \\ &\{\overline{\mathsf{bark}}, \overline{\mathsf{domestic}}, \overline{\mathsf{dogs}}, \overline{\mathsf{cats}}\}, \{\overline{\mathsf{bark}}, \mathsf{domestic}, \overline{\mathsf{dogs}}, \overline{\mathsf{cats}}\}, \\ &\{\overline{\mathsf{bark}}, \mathsf{domestic}, \overline{\mathsf{dogs}}, \mathsf{cats}\}\}. \end{aligned}$$

The conjunctive t-class term $\overline{\mathsf{dogs}}\,\overline{\mathsf{cats}}$ is translated into the intent representation by:

$$\begin{aligned}
trans(\{\overline{\mathsf{dogs}}, \overline{\mathsf{cats}}\}) &= \bigcap\{A \mid A \in rows(\mathcal{FC}_{\mathcal{T}}), \{\overline{\mathsf{dogs}}, \overline{\mathsf{cats}}\} \subseteq A\} \\
&= \{\overline{\mathsf{bark}}, \overline{\mathsf{domestic}}, \overline{\mathsf{dogs}}, \overline{\mathsf{cats}}\} \cap \{\overline{\mathsf{bark}}, \mathsf{domestic}, \overline{\mathsf{dogs}}, \overline{\mathsf{cats}}\} \\
&= \{\overline{\mathsf{bark}}, \overline{\mathsf{dogs}}, \overline{\mathsf{cats}}\}.
\end{aligned}$$

The number of elements in $M(\mathcal{FC}_{\mathcal{T}})$ is 8 ($= 2 \cdot 4$). □

### 4.4 Further improvements

In the previous sections, we suggested to represent the elements of $\mathcal{C}_{\mathcal{T}}$ by subsets of $G(\mathcal{FC}_{\mathcal{T}})$. Unfortunately the number of elements in $G(\mathcal{FC}_{\mathcal{T}})$ grows exponentially with the number of positive t-class terms. Thus in this section, we are presenting another representation that generally needs much less space. The main idea is to divide the formal context $\mathcal{FC}_{\mathcal{T}}$ into smaller subcontexts that cover the same information. A comparison of all possible representations is given in the appendix A.2.

**Definition 4.16**
Let $\mathcal{R} \subseteq \mathcal{B}$ with $\mathcal{R} = \{R_1, \overline{R}_1, \ldots, R_l, \overline{R}_l\}$.

a) The set $\mathcal{CC}_{\mathcal{R}}$ of all $\mathcal{R}$-*minimal* conjunctive t-class terms is defined by

$$\mathcal{CC}_{\mathcal{R}} := \{C_1C_2 \ldots C_l \mid C_i = R_i \text{ or } C_i = \overline{R_i} \text{ for all } i \in [1:l]\}.$$

b) The formal context $\mathcal{FC}_{\mathcal{T},\mathcal{R}}$ is defined by:

$$\begin{aligned}
G(\mathcal{FC}_{\mathcal{T},\mathcal{R}}) &:= \{C \in \mathcal{CC}_{\mathcal{R}} \mid \mathcal{T} \not\models C = \emptyset\}, \\
M(\mathcal{FC}_{\mathcal{T},\mathcal{R}}) &:= \mathcal{R}, \\
C \; \kappa \; R &:\Leftrightarrow \; \models C \subseteq R.
\end{aligned}$$ ■

**Definition 4.17**
Let $\mathcal{R}_1 \dot\cup \mathcal{R}_2 \dot\cup \cdots \dot\cup \mathcal{R}_n = \mathcal{B}$ with

$$|G(\mathcal{FC}_{\mathcal{T},\mathcal{R}_1})| \cdot |G(\mathcal{FC}_{\mathcal{T},\mathcal{R}_2})| \cdot \ldots \cdot |G(\mathcal{FC}_{\mathcal{T},\mathcal{R}_n})| = |G(\mathcal{FC}_{\mathcal{T}})|.$$

a) Let $\mathcal{CL}_{\mathcal{T},\mathcal{R}_i} := \mathcal{CL}(\mathcal{FC}_{\mathcal{T},\mathcal{R}_i})$ for all $i \in [1:n]$.

b) Let $\emptyset_i$ denote the minimum of $\mathcal{CL}_{\mathcal{T},\mathcal{R}_i}$ and $\sqsubseteq_i$ the partial order defined on $\mathcal{CL}_{\mathcal{T},\mathcal{R}_i}$ for all $i \in [1:n]$.

c) Let $\mathcal{CL}_{\mathcal{T},\mathcal{R}_1,\ldots,\mathcal{R}_n} := \times_{i\in[1:n]}(\mathcal{CL}_{\mathcal{T},\mathcal{R}_i} \backslash \emptyset_i) \cup \{(\emptyset_1, \ldots, \emptyset_n)\}$.

d) Let the partial order $\sqsubseteq_{\mathcal{R}_1,\ldots,\mathcal{R}_n}$ on $\mathcal{CL}_{\mathcal{T},\mathcal{R}_1,\ldots,\mathcal{R}_n}$ be defined by

$$(A_1, \ldots, A_n) \sqsubseteq_{\mathcal{R}_1,\ldots,\mathcal{R}_n} (B_1, \ldots, B_n) \;:\Leftrightarrow\; A_i \sqsubseteq_i B_i \text{ for all } i \in [1:n]$$ ■

A partition $\mathcal{R}_1 \dot\cup \mathcal{R}_2 \dot\cup \cdots \dot\cup \mathcal{R}_n$ of $\mathcal{B}$ can be user-defined or automatically computed from the taxonomic knowledge-base.

**Example 4.18**
Let us have a look at the taxonomic knowledge-base $\mathcal{T}_0$ of Ex. 2.9.
With $\mathcal{R}_1 = \{\mathsf{bark}, \overline{\mathsf{bark}}\}$ and $\mathcal{R}_2 = \{\mathsf{dogs}, \overline{\mathsf{dogs}}, \mathsf{cats}, \overline{\mathsf{cats}}, \mathsf{domestic}, \overline{\mathsf{domestic}}\}$ we get $|G(\mathcal{FC}_{\mathcal{T}_0,\mathcal{R}_1})| = 2$, $|G(\mathcal{FC}_{\mathcal{T}_0,\mathcal{R}_2})| = 4$ and $|G(\mathcal{FC}_{\mathcal{T}_0})| = 8$. □

**Theorem 4.19**
*With the propositions of Def. 4.17 we get the following isomorphism:*

$$(\mathcal{CL}_{\mathcal{T},\mathcal{R}_1,\ldots,\mathcal{R}_n}, \sqsubseteq_{\mathcal{R}_1,\ldots,\mathcal{R}_n}) \quad \sim \quad (\mathcal{C}_{\mathcal{T}}, \subseteq_{\mathcal{T}|\mathcal{C}_{\mathcal{T}}\times\mathcal{C}_{\mathcal{T}}}).$$

**Proof:** Let $(G_i, \mathcal{R}_i, \kappa_i) := \mathcal{FC}_{\mathcal{T},\mathcal{R}_i}$ for all $i \in [1:n]$. Since the attribute-sets $\mathcal{R}_i$ are pairwise taxonomically independent from each other, we have

$$B\sigma\pi \quad = \quad (B \cap \mathcal{R}_1)\sigma\pi \,\dot{\cup}\, \ldots \,\dot{\cup}\, (B \cap \mathcal{R}_n)\sigma\pi$$

for all $B \subseteq \mathcal{B}$. Let the mapping $\delta : \mathcal{CL}_{\mathcal{T}} \longrightarrow \mathcal{CL}_{\mathcal{T},\mathcal{R}_1,\ldots,\mathcal{R}_n}$ be defined by

$$\delta((A,B)) \quad = \quad (((B \cap \mathcal{R}_1)\sigma, B \cap \mathcal{R}_1), \ldots, ((B \cap \mathcal{R}_n)\sigma, B \cap \mathcal{R}_n))$$

for all $(A,B) \in \mathcal{CL}_{\mathcal{T}}$. The mapping $\delta$ is obviously surjective. Since

$$B_1 = B_2 \quad \Longleftrightarrow \quad B_1 \cap \mathcal{R}_1 = B_2 \cap \mathcal{R}_1, \ldots, B_1 \cap \mathcal{R}_n = B_2 \cap \mathcal{R}_n$$

for all intents $B_1 \neq \mathcal{B}$, $B_2 \neq \mathcal{B}$ of $\mathcal{FC}_{\mathcal{T}}$, the mapping is injective. The mapping is order-preserving according to the definition of $\sqsubseteq_{\mathcal{R}_1,\ldots,\mathcal{R}_n}$. □

The representation of $\mathcal{C}_{\mathcal{T}}$ by all $\mathcal{CL}_{\mathcal{T},\mathcal{R}_i}$ with $i \in [1:n]$ generally needs much less space than the representation of $\mathcal{C}_{\mathcal{T}}$ by $\mathcal{CL}_{\mathcal{T}}$. Unfortunately the evaluation of taxonomic conditions is more difficult, since generally just a proper subset of $\mathcal{G}_{\mathcal{T}}$ is representable within this new representation.

The taxonomic knowledge given in $\mathcal{T}$ can be completely visualized by *Hasse-diagrams* of all $\mathcal{CL}_{\mathcal{T},\mathcal{R}_i}$ with $i \in [1:n]$. We can also represent some positive probability statements and some uncertain rules in these concept lattices.

**Example 4.20**
Let us take the taxonomic knowledge-base $\mathcal{T}_0$ of Ex. 2.9 and the probabilistic formulas 1) to 3) of Ex. 3.10. This taxonomic and probabilistic knowledge can be visualized by the two extended Hasse-diagrams in Fig. 6. □

# 5 Deduction of uncertain knowledge

The scope of this paper prevents us to explore the complex optimization problem for probabilistic queries in full depth. Here we want to highlight the advantages gained by the careful interaction between taxonomic and probabilistic knowledge.

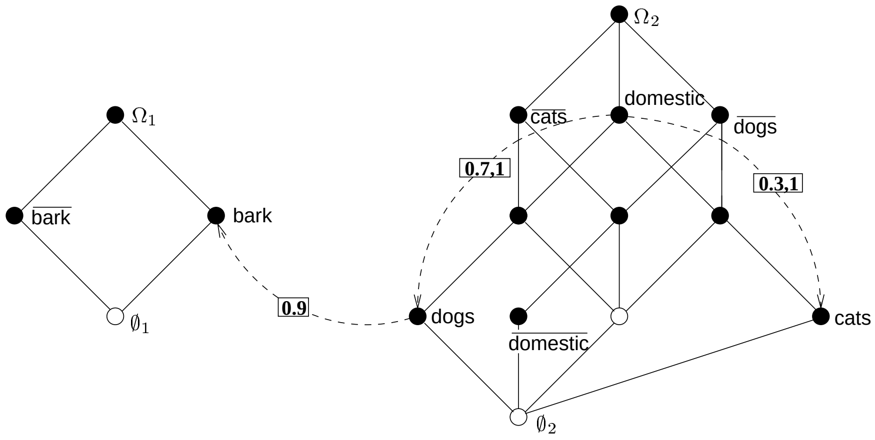


Figure 6: Visualization of the taxonomic and probabilistic knowledge of Ex. 4.20

### 5.1 Internal representation

All conjunctive t-class terms are assumed to be represented by their corresponding elements of $\mathcal{C}_\mathcal{T}$. This presumes that taxonomic and probabilistic formulas are defined on $\mathcal{G}_\mathcal{T}$ and $\mathcal{C}_\mathcal{T}$, respectively. Taxonomic formulas are extended to $\mathcal{G}_\mathcal{T}$ by Def. 2.10, probabilistic formulas can similarly be extended to $\mathcal{C}_\mathcal{T}$. The translation of the user-defined probabilistic formulas over $\mathcal{C}$ into the corresponding probabilistic formulas over $\mathcal{C}_\mathcal{T}$ is done as follows:

**(IR)** Internal Representation:

(a) $\{ \text{pos}(A) \} \vdash \text{pos}(A_\mathcal{T})$

(b) $\{ A \xrightarrow{x_1,x_2} B \} \vdash A_\mathcal{T} \xrightarrow{x_1,x_2} (AB)_\mathcal{T}$

(c) $\{ A \underset{y_1,y_2}{\overset{x_1,x_2}{\longleftrightarrow}} B \} \vdash A_\mathcal{T} \underset{y_1,y_2}{\overset{x_1,x_2}{\longleftrightarrow}} B_\mathcal{T}$

(d) $\{ I(A, B, C) \} \vdash I(A_\mathcal{T}, B_\mathcal{T}, C_\mathcal{T})$

### 5.2 Inference rules

Consider an uncertain rule query $?A \xrightarrow{x_1,x_2} B$. If $A \subseteq B$ or $A \parallel B$ hold, the answer $x_1 = x_2 = 1$ or $x_1 = x_2 = 0$, resp., can be returned by taxonomic deduction without engaging in probabilistic deduction. Otherwise probabilistic query evaluation has to be initiated, employing probabilistic inference rules.

The previous transformation into an internal representation over $\mathcal{C}_\mathcal{T}$ immediately pays off for probabilistic query evaluation, because the search space is enormously reduced (as demonstrated by Ex. 5.3 subsequently). The decisive difference between the inference rules for probabilistic knowledge stated below and previous inference rules (comp. [GKT91], [TGK92]) is that here taxonomic formulas (to be evaluated by the taxonomic reasoning part) appear in the premise of the inference rules. As will be shown later (see Ex. 5.4), this will produce more precise conclusions in many cases.[9]

In the inference rules (C) and (CRN) we are computing the expressions $AC_1 \cup AC_2$ and $AC \cap \overline{AC_1}$, respectively. Since all our computations are performed within $\mathcal{C}_\mathcal{T}$, we must carefully check that these expressions represent conjunctive t-classes. We make sure that all deduced pobabilistic formulas do not contain any elements of $\mathcal{G}_\mathcal{T} \backslash \mathcal{C}_\mathcal{T}$.

Let $A, B, C, C_1, C_2, D \in \mathcal{C}_\mathcal{T}$ and $F \in \mathcal{B}_\mathcal{T}$.

**(CO)** Correlation:

(a) $\{ A \xrightarrow{x_1,x_2} AB,\ B \xrightarrow{y_1,y_2} AB \} \vdash A \xleftrightarrow[y_1,y_2]{x_1,x_2} B$

(b) $\{ A \xrightarrow{x_1,x_2} AB \} \vdash A \xleftrightarrow[0,1]{x_1,x_2} B$

(c) $\{ A \xleftrightarrow[y_1,y_2]{x_1,x_2} B \} \vdash B \xleftrightarrow[x_1,x_2]{y_1,y_2} A$

(d) $\{ A \xleftrightarrow[y_1,y_2]{x_1,x_2} B \} \vdash A \xrightarrow{x_1,x_2} AB$

**(PRC1)** Precise rule chaining 1:[10]

$$\{ A \xleftrightarrow[v_1,v_2]{u_1,u_2} B,\ \mathrm{pos}(A),\ B \xleftrightarrow[y_1,y_2]{x_1,x_2} C \} \vdash A \xrightarrow{z_1,z_2} AC,$$

$$z_1 = \max(\mathrm{PRC1}_{lb}(A \xleftrightarrow[v_1,v_2]{u_1,u_2} B, B \xleftrightarrow[y_1,y_2]{x_1,x_2} C))$$

$$z_2 = \min(\mathrm{PRC1}_{ub}(A \xleftrightarrow[v_1,v_2]{u_1,u_2} B, B \xleftrightarrow[y_1,y_2]{x_1,x_2} C))$$

**(PRC2)** Precise Rule Chaining 2:

(a) $\{A \xrightarrow{u_1,u_2} AF, AF \xrightarrow{x_1,x_2} AFC, A\overline{F} \xrightarrow{y_1,y_2} A\overline{F}C,\ \mathrm{pos}(AF),\ \mathrm{pos}(A\overline{F}) \}$
$\vdash A \xrightarrow{z_1,z_2} AC$

$$z_1 = \begin{cases} u_1 \cdot x_1 + (1-u_1) \cdot y_1 & \text{if } x_1 > y_1 \\ u_2 \cdot x_1 + (1-u_2) \cdot y_1 & \text{otherwise} \end{cases}$$

$$z_2 = \begin{cases} u_2 \cdot x_2 + (1-u_2) \cdot y_2 & \text{if } x_2 > y_2 \\ u_1 \cdot x_2 + (1-u_1) \cdot y_2 & \text{otherwise} \end{cases}$$

(b) $\{A \xrightarrow{u_1,u_2} AF, AF \xrightarrow{x_1,x_2} AFC, A\overline{F} \xrightarrow{y_1,y_2} A\overline{F}C,\ \mathrm{pos}(AF)\,\mathrm{pos}(A\overline{F}),$
$\mathrm{pos}(AC) \} \vdash AC \xrightarrow{z_1,z_2} AFC,$

$$z_1 = \begin{cases} 1 & \text{if } u_1 = 0, y_2 = 0 \\ \frac{u_1 \cdot x_1}{u_1 \cdot x_1 + (1-u_1) \cdot y_2} & \text{otherwise} \end{cases}$$

[9] For special subclasses [Thö94] describes highly efficient and complete inference rules. For the large probabilistic class considered here, no completeness results are known so far.

[10] The definitions of $\mathrm{PRC1}_{lb}$ and $\mathrm{PRC1}_{ub}$ are given in the appendix.

$$z_2 = \begin{cases} 0 & \text{if } u_2 = 1, x_2 = 0 \\ \frac{u_2 \cdot x_2}{u_2 \cdot x_2 + (1-u_2)\cdot y_1} & \text{otherwise} \end{cases}$$

**(C)** Chaining:

$$\{\, A \xrightarrow{x_1,x_2} AC_1,\ \mathrm{pos}(A),\ A \xrightarrow{y_1,y_2} AC_2,\ AC_1 \parallel AC_2 \,\} \vdash A \xrightarrow{z_1,z_2} AC,$$

$$AC = AC_1 \cup AC_2,\ z_1 = x_1 + y_1,\ z_2 = \min(1, x_2 + y_2)$$

**(CRN)** Conjunction Right with Negation:

$$\{\, A \xrightarrow{x_1,x_2} AC,\ \mathrm{pos}(A),\ A \xrightarrow{y_1,y_2} AC_1,\ AC_1 \subseteq AC \,\} \vdash A \xrightarrow{z_1,z_2} AC_2,$$

$$AC_2 = AC \cap \overline{AC_1},\ z_1 = \max(0, x_1 - y_2),\ z_2 = x_2 - y_1$$

**(WCRN)** Weak Conjunction Right with Negation:[11]

$$\{\, A \xleftrightarrow[v_1,v_2]{u_1,u_2} F,\ \mathrm{pos}(AF),\ \ F \xleftrightarrow[y_1,y_2]{x_1,x_2} C,\ \mathrm{pos}(CF) \,\} \vdash A \xrightarrow{z_1,z_2} A\overline{F}C,$$

$$z_1 = \max(\mathrm{WCRN}_{lb}(A \xleftrightarrow[v_1,v_2]{u_1,u_2} F, F \xleftrightarrow[y_1,y_2]{x_1,x_2} C))$$

$$z_2 = \min(\mathrm{WCRN}_{ub}(A \xleftrightarrow[v_1,v_2]{u_1,u_2} F, F \xleftrightarrow[y_1,y_2]{x_1,x_2} C))$$

**(S)** Sharpening:

$$\{\, A \xrightarrow{x_1,x_2} AB,\ \mathrm{pos}(A),\ A \xrightarrow{y_1,y_2} AB \,\} \vdash A \xrightarrow{z_1,z_2} AB,$$

$$z_1 = \max(x_1, y_1),\ z_2 = \min(x_2, y_2)$$

**(CP)** Comparative Probabilities:

(a) $\{\, A \xrightarrow{x_1,x_2} AB,\ \mathrm{pos}(A),\ A \xrightarrow{\geq k[B]} C \,\} \vdash A \xrightarrow{z_1,1} AC,$
$z_1 = \min(1, k \cdot x_1)$

(b) $\{\, A \xrightarrow{x_1,x_2} AB,\ \mathrm{pos}(A),\ A \xrightarrow{\leq k[B]} C \,\} \vdash A \xrightarrow{0,z_2} AC,$
$z_2 = \min(1, k \cdot x_2)$

**(I)** Invariance:

(a) $\{\, B \xrightarrow{x_1,x_2} BC,\ \mathrm{pos}(AB),\ I(A,B,C) \,\} \vdash AB \xrightarrow{x_1,x_2} ABC$

(b) $\{\, AB \xrightarrow{x_1,x_2} ABC,\ \mathrm{pos}(AB),\ I(A,B,C) \,\} \vdash B \xrightarrow{x_1,x_2} BC$

**(SYM)** Symmetry:

$$\{\, I(A,B,C) \,\} \vdash I(C,B,A)$$

**(DEC)** Decomposition:

$$\{\, I(FB,C,D),\ I(\overline{F}B,C,D) \,\} \vdash I(B,C,D)$$

[11] The definitions of $\mathrm{WCRN}_{lb}$ and $\mathrm{WCRN}_{ub}$ are given in the appendix.

**(WU)** Weak Union:

$$\{ I(FB,C,D),\, I(\overline{F}B,C,D) \} \vdash I(F,BC,D)$$

**(CON)** Contraction:

$$\{ I(A,BC,D),\, I(B,C,D) \} \vdash I(AB,C,D)$$

**(PP)** Positive Probability:

(a) $\vdash \mathrm{pos}(F)$

(b) $\{ A \xrightarrow{x_1,x_2} AB,\ \mathrm{pos}(A),\ x_1 > 0 \} \vdash \mathrm{pos}(AB)$

(c) $\{ \mathrm{pos}(A),\ A \subseteq B \} \vdash \mathrm{pos}(B)$

**(TK)** Taxonomic Knowledge:

(a) $\vdash A \xrightarrow{1} A$

(b) $\vdash A \xrightarrow{0} \emptyset$

**(C)** Contradiction:

(a) $\{ A \xrightarrow{x_1,x_2} AB,\ \mathrm{pos}(A),\ x_2 < x_1 \} \vdash \mathit{contradiction}()$

(b) $\{ \mathrm{pos}(\emptyset) \} \vdash \mathit{contradiction}()$

### 5.3 Subsumption order

We define the following subsumption order $\sqsubseteq_S$ on uncertain rules (see [KKG93], [KKTG94]):

**Definition 5.1**

$$A \xrightarrow{x_1,x_2} B \sqsubseteq_S A \xrightarrow{y_1,y_2} B :\Leftrightarrow x_1 \leq y_1 \wedge y_2 \leq x_2. \qquad \blacksquare$$

### 5.4 From probabilistic to taxonomic knowledge

Very importantly, new taxonomic knowledge may be generated dynamically during uncertain deduction from probabilistic knowledge, which in turn may further reduce the amount of different t-classes. We can use the following rules to deduce taxonomic knowledge from probabilistic knowledge (see lemma 3.7).

**(TK)** Taxonomic Knowledge:

(c) $\{ A \xrightarrow{1} AB \} \vdash A \subseteq AB$

(d) $\{ A \xrightarrow{0} AB \} \vdash AB = \emptyset$

**Example 5.2 (Ex. 2.9 continued)**
The probabilistic knowledge-base of Ex. 3.10 is transformed into the internal representation by:[12]

1) $\{\mathsf{domestic} \xrightarrow{0.3,1} \mathsf{cats}\} \underset{(IR)(b)}{\vdash} \mathsf{domestic} \xrightarrow{0.3,1} \mathsf{domestic\,cats}$,

2) $\{\mathsf{domestic} \xrightarrow{0.7,1} \mathsf{dogs}\} \underset{(IR)(b)}{\vdash} \mathsf{domestic} \xrightarrow{0.7,1} \mathsf{domestic\,dogs}\,\overline{\mathsf{cats}}$.

Applying the inference rules of Sec. 5.2, we get:

3) $\underset{(TK)(a)}{\vdash} \{\mathsf{domestic} \xrightarrow{1} \mathsf{domestic},\ \mathsf{domestic}\,\overline{\mathsf{cats}} \xrightarrow{1} \mathsf{domestic}\,\overline{\mathsf{cats}},$
$\mathsf{domestic\,dogs}\,\overline{\mathsf{cats}} \xrightarrow{1} \mathsf{domestic\,dogs}\,\overline{\mathsf{cats}}\,\}$

4) $\underset{(PP)(a)}{\vdash} \{\,\mathrm{pos}(\mathsf{domestic}),\ \mathrm{pos}(\mathsf{domestic\,dogs}\,\overline{\mathsf{cats}})\,\}$

5) $\{\mathsf{domestic} \xrightarrow{1} \mathsf{domestic},\ \mathrm{pos}(\mathsf{domestic}),\ \mathsf{domestic} \xrightarrow{0.3,1} \mathsf{domestic\,cats},$
$\mathsf{domestic\,cats} \subseteq \mathsf{domestic}\,\} \underset{(CRN)}{\vdash} \mathsf{domestic} \xrightarrow{0,0.7} \mathsf{domestic}\,\overline{\mathsf{cats}}$

6) $\{\mathsf{domestic} \xrightarrow{0,0.7} \mathsf{domestic}\,\overline{\mathsf{cats}},\ \mathsf{domestic}\,\overline{\mathsf{cats}} \xrightarrow{1} \mathsf{domestic}\,\overline{\mathsf{cats}}\,\}$
$\underset{(CO)(a)}{\vdash} \mathsf{domestic} \xleftrightarrow[1]{0,0.7} \mathsf{domestic}\,\overline{\mathsf{cats}}$

7) $\{\mathsf{domestic} \xrightarrow{0.7,1} \mathsf{domestic\,dogs}\,\overline{\mathsf{cats}},\ \mathsf{domestic\,dogs}\,\overline{\mathsf{cats}} \xrightarrow{1} \mathsf{domestic\,dogs}\,\overline{\mathsf{cats}}\,\}$
$\underset{(CO)(a)}{\vdash} \mathsf{domestic} \xleftrightarrow[1]{0.7,1} \mathsf{domestic\,dogs}\,\overline{\mathsf{cats}}$

8) $\{\,\mathrm{pos}(\mathsf{domestic\,dogs}\,\overline{\mathsf{cats}})\} \underset{(PP)(c)}{\vdash} \mathrm{pos}(\mathsf{domestic}\,\overline{\mathsf{cats}})$

9) $\{\mathsf{domestic}\,\overline{\mathsf{cats}} \xleftrightarrow[0,0.7]{1} \mathsf{domestic},\ \mathrm{pos}(\mathsf{domestic}\,\overline{\mathsf{cats}}),$
$\mathsf{domestic} \xleftrightarrow[1]{0.7,1} \mathsf{domestic\,dogs}\,\overline{\mathsf{cats}},\ \mathsf{domestic\,dogs}\,\overline{\mathsf{cats}} \subseteq \mathsf{domestic}\,\overline{\mathsf{cats}}\,\}$
$\underset{(PRC1)}{\vdash} \mathsf{domestic}\,\overline{\mathsf{cats}} \xrightarrow{1} \mathsf{domestic\,dogs}\,\overline{\mathsf{cats}}$

10) $\{\mathsf{domestic}\,\overline{\mathsf{cats}} \xrightarrow{1} \mathsf{domestic\,dogs}\,\overline{\mathsf{cats}}\}$
$\underset{(TK)(c)}{\vdash} \mathsf{domestic}\,\overline{\mathsf{cats}} \subseteq \mathsf{domestic\,dogs}\,\overline{\mathsf{cats}}$

Since we have $\models \mathsf{domestic}\,\overline{\mathsf{cats}} \supseteq \mathsf{domestic\,dogs}\,\overline{\mathsf{cats}}$, the taxonomic formula $\mathsf{domestic}\,\overline{\mathsf{cats}} \subseteq \mathsf{domestic\,dogs}\,\overline{\mathsf{cats}}$ yields a further reduction of the concept lattice. Let $\mathcal{T}_2 = \mathcal{T}_1 \cup \{\mathsf{domestic}\,\overline{\mathsf{cats}} \subseteq \mathsf{domestic\,dogs}\,\overline{\mathsf{cats}}\}$. The Hasse-diagram of $\mathcal{C}_{\mathcal{T}_2}$ is given by Fig. 7 (uncertain rules are omitted, compare Fig. 5).
In Fig. 7 we can easily verify that $\mathcal{T}_2 \models \mathsf{domestic}\,\overline{\mathsf{cats}} = \mathsf{dogs}$ holds. □

[12] For explanation purposes we represent $\mathcal{C}_{\mathcal{T}}$ by the intents of $\mathcal{CL}_{\mathcal{T}}$.

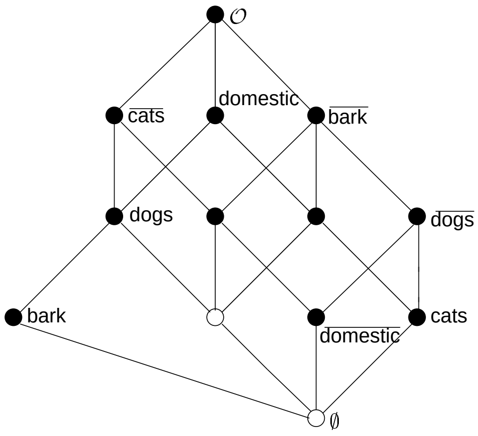


Figure 7: Dynamically reduced Hasse-diagram for Ex. 5.2

### 5.5 Performance considerations

The internal representation over $\mathcal{C}_{\mathcal{T}}$ yields an enormous search space reduction that can be illustrated by the following example.

**Example 5.3 (Ex. 2.9 continued)**
The following table gives a comparison of the number of elements in $\mathcal{C}_{\mathcal{T}}$, the number of uncertain and correlation rules over $\mathcal{C}_{\mathcal{T}}$ occuring w.r.t. $\mathcal{T} = \emptyset$, $\mathcal{T} = \mathcal{T}_1$ and $\mathcal{T} = \mathcal{T}_2$.[13]

| | $\mathcal{T} = \emptyset$ | $\mathcal{T} = \mathcal{T}_1$ | $\mathcal{T} = \mathcal{T}_2$ |
|---|---|---|---|
| number of elements in $\mathcal{C}_{\mathcal{T}}$ | 81 | 18 | 13 |
| number of uncertain rules over $\mathcal{C}_{\mathcal{T}}$ | 625 | 107 | 61 |
| number of correlation rules over $\mathcal{C}_{\mathcal{T}}$ | 6561 | 324 | 169 |

$\square$

The evaluation of taxonomic conditions enables us to produce more precise conclusions in many cases. The following example illustrates this advantage.

**Example 5.4**
Let $\{\, \mathsf{A} \xleftrightarrow[0.5,0.6]{0.2,0.3} \mathsf{B},\ \text{pos}(\mathsf{A}),\ \text{pos}(\mathsf{B}),\ \mathsf{B} \xleftrightarrow[0.2,0.3]{0.2,0.3} \mathsf{C},\ \mathsf{ABC} = \emptyset \,\}$ be a probabilistic knowledge-base. By (PRC1) the query $?A \xrightarrow{x_1,x_2} C$ is evaluated to $x_1 = 0$, $x_2 = 0.706$. In contrast, our previous inference calculus just yields $x_1 = 0$, $x_2 = 0.882$, since it could not incorporate taxonomic knowledge for a precise conclusion. $\square$

[13] The numbers for $\mathcal{T} = \emptyset$ are computed combinatorically, the numbers for $\mathcal{T} = \mathcal{T}_1$ and $\mathcal{T} = \mathcal{T}_2$ are taken from Fig. 5 and Fig. 7, respectively.

## 6 Related work

The combination of taxonomic and uncertain knowledge was also examined by Ng and Subrahmanian [NS92a], von Rimscha [vR90], Heinsohn [Hei91].

• Ng and Subrahmanian present an approach to integrate empirical probabilities in deductive databases. An empirical program consists of two parts, true/false knowledge about classes of individuals (or single individuals) and empirical clauses representing statistical knowledge about "generic" individuals. In a compilation step this knowledge base is enriched by adding logically entailed first order and empirical clauses. Queries about individuals are either answered by deduction or by induction on the enriched knowledge base. In contrast we do not necessarily assume a relative frequency interpretation of probability. Moreover, we apply more general and more precise inference rules on the uncertain rule knowledge allowing a broader range of hypothetical reasoning. Here we do not consider queries considering individuals, but our approach can be extended to uncertain facts as well ([KTG92], [TKG94]).

• Von Rimscha uses taxonomic knowledge to determine the variables of a linear optimization problem. Uncertain knowledge is then transformed to linear inequalities. However, if there is little taxonomic knowledge the number of variables increases exponentially. Although the variables of the optimization problem correspond to the probabilities of the elements on the lowest level of our concept lattice, we can avoid this increase by splitting the concept lattice as described in Sec. 4.4. Conditional independencies cannot be dealt with linear programming, though.

• Heinsohn integrates uncertain knowledge into a terminological language but does not take advantage of it in the deduction process. We show how the search space can be reduced efficiently and how the chaining of correlation rules can be made more precise by the exploitation of taxonomic knowledge. Furthermore we demonstrate that taxonomic knowledge can be deduced from probabilistic knowledge under suitable circumstances.

There is a lot of literature on representing and reasoning about taxonomic knowledge in the area of terminological systems (see i.e. [Bra91]). The query answering mechanism we apply is especially appropiate for the use in uncertain deduction, too.
We are not aware of other work about the integration of object-oriented database systems with taxonomic *and* uncertain knowledge.

## 7 Conclusion and Outlook

We have presented a coherent and evolutionary approach to extend conventional OODB technology by taxonomic and uncertain modeling and reasoning capabilities. For the complex field of taxonomic and uncertain deduction we showed novel techniques and algorithms improving prior results. As an important feature we demonstrated how taxonomic knowledge can be inferred from uncertain knowledge, giving a strong counterargument to critics that deduction of uncertain knowledge (especially in the form of probability intervals) just yields too weak uncertain knowledge again ("garbage in, garbage out"). A project to realize a prototype is under way as follows. Applications are specified conventionally through OMT diagrams, which can be mapped semi-automatically into available OODBs like $O_2$. OMT diagrams can be ameliorated by taxonomic and uncertain knowledge as described. The implementation of uncertain deduction can adapt our prior experiences with the DUCK-system, which will be realized in Datalog_S ([KG94], [KKTG94]) on top of CORAL ([RSS92]).

# Appendix A (Taxonomic deduction)

## A.1 Proof for Ex. 4.8

$\mathsf{bark}\ \mathsf{domestic}\ \mathsf{dogs}\ \mathsf{cats} = \emptyset$, since $\mathsf{cats} \parallel \mathsf{dogs}$
$\mathsf{bark}\ \mathsf{domestic}\ \overline{\mathsf{dogs}}\ \mathsf{cats} = \emptyset$, since $\mathsf{bark} \subseteq \mathsf{dogs}$
$\mathsf{bark}\ \mathsf{domestic}\ \overline{\mathsf{dogs}}\ \overline{\mathsf{cats}} = \emptyset$, since $\mathsf{bark} \subseteq \mathsf{dogs}$
$\mathsf{bark}\ \overline{\mathsf{domestic}}\ \mathsf{dogs}\ \mathsf{cats} = \emptyset$, since $\mathsf{cats} \parallel \mathsf{dogs}$
$\mathsf{bark}\ \overline{\mathsf{domestic}}\ \mathsf{dogs}\ \overline{\mathsf{cats}} = \emptyset$, since $\mathsf{dogs} \subseteq \mathsf{domestic}$
$\mathsf{bark}\ \overline{\mathsf{domestic}}\ \overline{\mathsf{dogs}}\ \mathsf{cats} = \emptyset$, since $\mathsf{cats} \subseteq \mathsf{domestic}$
$\mathsf{bark}\ \overline{\mathsf{domestic}}\ \overline{\mathsf{dogs}}\ \overline{\mathsf{cats}} = \emptyset$, since $\mathsf{bark} \subseteq \mathsf{dogs}$
$\overline{\mathsf{bark}}\ \mathsf{domestic}\ \mathsf{dogs}\ \mathsf{cats} = \emptyset$, since $\mathsf{cats} \parallel \mathsf{dogs}$
$\overline{\mathsf{bark}}\ \overline{\mathsf{domestic}}\ \mathsf{dogs}\ \mathsf{cats} = \emptyset$, since $\mathsf{cats} \parallel \mathsf{dogs}$
$\overline{\mathsf{bark}}\ \overline{\mathsf{domestic}}\ \mathsf{dogs}\ \overline{\mathsf{cats}} = \emptyset$, since $\mathsf{dogs} \subseteq \mathsf{domestic}$
$\overline{\mathsf{bark}}\ \overline{\mathsf{domestic}}\ \overline{\mathsf{dogs}}\ \mathsf{cats} = \emptyset$, since $\mathsf{cats} \subseteq \mathsf{domestic}$

## A.2 Complexity of taxonomic deduction

According to Theorem 4.9 and 4.19 there are four canonic representations for $\mathcal{C}_\mathcal{T}$. We can represent $\mathcal{C}_\mathcal{T}$ by the extents or intents of $\mathcal{CL}_\mathcal{T}$ or all $\mathcal{CL}_{\mathcal{T},\mathcal{R}_i}$ with $i \in [1:n]$. The following table shows a comparison of these representations, assuming that $|G(\mathcal{FC}_{\mathcal{T},\mathcal{R}_i})| = 2$ for all $i \in [1:l]$ and $|G(\mathcal{FC}_{\mathcal{T},\mathcal{R}_i})| > 2$ for all $i \in [l+1,n]$. The set $\mathcal{B}$ is defined according to Def. 2.1. Note that we take bitstrings to represent the extents or intents and sets of bitstrings to represent the underlying contexts. Our computations are based on the bitstring-operations *bitwise and, bitwise or, bitwise complement* and *bitwise implication.* We compare the four different representations with respect to

1) the length of the bitstring-representation,

2) the number of evaluations of propositional formulas for the construction of the formal context $\mathcal{FC}_\mathcal{T}$,

3) the complexity of the translation of the user-defined conjunctive t-class terms into the bitstring-representation,

4) the complexity of $\wedge$-computation,

5) the complexity of $\vee$-computation and

6) the number of elements in $\mathcal{G}_\mathcal{T} \backslash \mathcal{C}_\mathcal{T}$ with a bitstring-representation.

| | extents of | | intents of | |
|---|---|---|---|---|
| | $\mathcal{CL}_\mathcal{T}$ | $\mathcal{CL}_{\mathcal{T},\mathcal{R}_i}$ | $\mathcal{CL}_\mathcal{T}$ | $\mathcal{CL}_{\mathcal{T},\mathcal{R}_i}$ |
| 1) | $\lvert G(\mathcal{FC}_\mathcal{T})\rvert \leq 2^{\frac{\lvert\mathcal{B}\rvert}{2}}$ | $\Sigma_{i\in[1:n]}\lvert G(\mathcal{FC}_{\mathcal{T},\mathcal{R}_i})\rvert \leq \Sigma_{i\in[1:n]}2^{\frac{\lvert\mathcal{R}_i\rvert}{2}}$ | $\lvert\mathcal{B}\rvert$ | $\lvert\mathcal{B}\rvert$ |
| 2) | $2^{\frac{\lvert\mathcal{B}\rvert}{2}}$ | $\Sigma_{i\in[l+1:n]}2^{\frac{\lvert\mathcal{R}_i\rvert}{2}}$ | $2^{\frac{\lvert\mathcal{B}\rvert}{2}}$ | $\Sigma_{i\in[l+1:n]}2^{\frac{\lvert\mathcal{R}_i\rvert}{2}}$ |
| 3) | $\mathrm{O}(\lvert\mathcal{B}\rvert)$ | $\mathrm{O}(\max(\lvert\mathcal{B}\rvert-2l, n-l, 1))$ | $\mathrm{O}(2^{\frac{\lvert\mathcal{B}\rvert}{2}})$ | $\mathrm{O}(\max(2^{\frac{\max(\lvert\mathcal{R}_{l+1}\rvert,\ldots,\lvert\mathcal{R}_n\rvert,0)}{2}}, n-l))$ |
| 4) | $\mathrm{O}(1)$ | $\mathrm{O}(n-l)$ | $\mathrm{O}(2^{\frac{\lvert\mathcal{B}\rvert}{2}})$ | $\mathrm{O}(\max(2^{\frac{\max(\lvert\mathcal{R}_{l+1}\rvert,\ldots,\lvert\mathcal{R}_n\rvert,0)}{2}}, n-l))$ |
| 5) | $\mathrm{O}(\lvert\mathcal{B}\rvert)$ | $\mathrm{O}(\max(\lvert\mathcal{B}\rvert-2l, 1))$ | $\mathrm{O}(1)$ | $\mathrm{O}(1)$ |
| 6) | all | some | none | none |

The complexity of $\wedge$- and $\vee$-computation in the different representations varies according to theorem 4.5. Since in the calculus for uncertain reasoning we are mostly concerned with computing the infimum of two concepts, we may select the extents of $\mathcal{CL}_\mathcal{T}$ or all $\mathcal{CL}_{\mathcal{T},\mathcal{R}_i}$ with $i \in [1:n]$ for representing $\mathcal{C}_\mathcal{T}$. Unfortunately the number of bits for the bitstring-representation of the first representation can grow exponentially and the second representation does not allow us to evaluate all taxonomic conditions with elements in $\mathcal{G}_\mathcal{T}\backslash\mathcal{C}_\mathcal{T}$. Another solution could be a combined representation with the intents of $\mathcal{CL}_\mathcal{T}$ for the use in positive probability statements, uncertain rules, independencies, comparative probabilities and correlation rules, and the extents of $\mathcal{CL}_\mathcal{T}$ for the use in the evaluation of taxonomic conditions. The bijection between these two combined representations could be realized by computing it once for each concept and keeping it stored for later use.

# Appendix B (Probabilistic deduction)

## B.1 The inference rules (PRC1) and (WCRN)

**Definition B.1**
a) The set $\mathrm{PRC1}_{lb}(A \xleftrightarrow[v_1,v_2]{u_1,u_2} B, B \xleftrightarrow[y_1,y_2]{x_1,x_2} C)$ of real numbers is defined to be the smallest set which contains:

$$
\begin{array}{ll}
0, & \\
u_1(1-\frac{1}{v_1}(1-x_1)) & \text{if} \quad v_1 > 0, \\
u_1 & \text{if} \quad AB \subseteq C, \\
\frac{u_1 x_1}{v_2 y_2} & \text{if} \quad C \subseteq A,\ v_2 > 0,\ y_2 > 0, \\
\frac{u_1}{y_2} & \text{if} \quad C \subseteq A,\ AB \subseteq C,\ y_2 > 0, \\
u_1(1-\frac{x_1}{v_2}(1-\frac{1}{y_2})) & \text{if} \quad AB \subseteq C,\ (\overline{B}C \subseteq A \text{ or } \overline{A}C \subseteq B),\ v_2 > 0,\ y_2 > 0, \\
1-u_2 & \text{if} \quad (A\overline{B} \subseteq C \text{ or } A\overline{C} \subseteq B), \\
1-u_2(1-\frac{x_1}{v_2}) & \text{if} \quad BC \subseteq A,\ (A\overline{B} \subseteq C \text{ or } A\overline{C} \subseteq B),\ v_2 > x_1, \\
\frac{x_1}{y_2(v_2-x_1)+x_1} & \text{if} \quad C \subseteq A,\ (A\overline{B} \subseteq C \text{ or } A\overline{C} \subseteq B),\ v_2 > x_1,\ y_2 > 0, \\
1 & \text{if} \quad A \subseteq C.
\end{array}
$$

b) The set $\mathrm{PRC1}_{ub}(A \xleftrightarrow[v_1,v_2]{u_1,u_2} B, B \xleftrightarrow[y_1,y_2]{x_1,x_2} C)$ of real numbers is defined to be the smallest set which contains:

$$
\begin{array}{ll}
1, & \\
1-u_1(1-\frac{x_2}{v_1}) & \text{if} \quad v_1 > x_2, \\
\frac{u_2 x_2}{v_1 y_1} & \text{if} \quad v_1 > 0,\ y_1 > 0, \\
\frac{x_2}{y_1(v_1-x_2)+x_2} & \text{if} \quad v_1 > x_2,\ y_1 > 0, \\
u_2(1-\frac{x_2}{v_1}(1-\frac{1}{y_1})) & \text{if} \quad \mathrm{pos}(B),\ x_2 > v_1,\ v_1 > 0,\ y_1 > 0, \\
u_2 & \text{if} \quad AC \subseteq B, \\
\frac{u_2 x_2}{v_1} & \text{if} \quad AC \subseteq B,\ v_1 > x_2, \\
u_2(1-\frac{1}{v_2}(1-x_2)) & \text{if} \quad AC \subseteq B,\ (\overline{A}B \subseteq C \text{ or } B\overline{C} \subseteq A), \\
 & \phantom{\text{if}} \quad \mathrm{pos}(B),\ v_2 > 0, \\
\frac{u_2 x_2}{y_1(1-x_2)+x_2} & \text{if} \quad AC \subseteq B,\ (\overline{A}B \subseteq C \text{ or } B\overline{C} \subseteq A), \\
 & \phantom{\text{if}} \quad \mathrm{pos}(B),\ v_1 > 0,\ y_1 > 0, \\
\frac{1-u_1}{1-y_2} & \text{if} \quad (\overline{A}C \subseteq B \text{ or } \overline{B}C \subseteq A),\ \mathrm{pos}(C),\ y_2 < 1, \\
\frac{u_2}{y_1} & \text{if} \quad C \subseteq A,\ AB \subseteq C,\ y_1 > 0, \\
1-u_1 & \text{if} \quad ABC = \emptyset, \\
(1-y_1)x_2 \cdot \min(\frac{u_2}{v_1 y_1}, \frac{1}{y_1(v_1-x_2)+x_2}) & \text{if} \quad ABC = \emptyset,\ \mathrm{pos}(B),\ v_1 > 0,\ y_1 > 0, \\
0 & \text{if} \quad AC = \emptyset.
\end{array}
$$

c) The set $\mathrm{WCRN}_{lb}(A \underset{v_1,v_2}{\overset{u_1,u_2}{\longleftrightarrow}} F, F \underset{y_1,y_2}{\overset{x_1,x_2}{\longleftrightarrow}} C)$ of real numbers is defined to be the smallest set which contains:

$$
\begin{array}{ll}
0, & \\
1-u_2 & \text{if} \quad A\overline{F} \subseteq C, \\
1-y_2 & \text{if} \quad A = C, \\
(1-y_2)\frac{u_1}{y_2} & \text{if} \quad AF \subseteq C,\ C \subseteq A,\ y_2 > 0, \\
(1-y_2)\frac{u_1 x_1}{v_2 y_2} & \text{if} \quad C\overline{F} \subseteq A,\ v_2 > 0,\ y_2 > 0, \\
(1-y_2)\frac{x_1}{y_2(v_2-x_1)+x_1} & \text{if} \quad A\overline{F} \subseteq C,\ C\overline{F} \subseteq A,\ v_2 > 0,\ y_2 > 0.
\end{array}
$$

d) The set $\mathrm{WCRN}_{ub}(A \underset{v_1,v_2}{\overset{u_1,u_2}{\longleftrightarrow}} F, F \underset{y_1,y_2}{\overset{x_1,x_2}{\longleftrightarrow}} C)$ of real numbers is defined to be the smallest set which contains:

$$
\begin{array}{ll}
\min(1, 1-u_1), & \\
1-y_1 & \text{if} \quad A = C, \\
(1-y_1)\frac{u_2}{y_1} & \text{if} \quad AF \subseteq C,\ CF \subseteq A,\ y_1 > 0, \\
(1-y_1)x_2 \cdot \min(\frac{u_2}{v_1 y_1}, \frac{1}{y_1(v_1-x_2)+x_2}) & \text{if} \quad v_1 > 0,\ y_1 > 0, \\
0 & \text{if} \quad A\overline{F}C = \emptyset.
\end{array}
$$

■

**Theorem B.2**
*The inference rules (PRC1) and (WCRN) are correct.*

**Proof:** Correctness of (PRC1): Generally we have the following propositions:

$$
\begin{array}{rclcll}
v_1 P(B) & \leq & P(AB) & \leq & v_2 P(B), & v_1, v_2 \in [0,1], \\
x_1 P(B) & \leq & P(BC) & \leq & x_2 P(B), & x_1, x_2 \in [0,1], \\
y_1 P(C) & \leq & P(BC) & \leq & y_2 P(C), & y_1, y_2 \in [0,1],
\end{array}
$$

$$
u = \frac{P(AB)}{P(A)}, \quad u \in [u_1, u_2], \quad u_1, u_2 \in [0,1].
$$

With $P(B) > 0$, we additionally get

$$
v = \frac{P(AB)}{P(B)}, \quad v \in [v_1, v_2] \text{ and}
$$

$$
x = \frac{P(BC)}{P(B)}, \quad x \in [x_1, x_2].
$$

a) Correctness of the lower bound:

1) Case $v_1 > 0$, $P(B) > 0$: since

$$
\begin{aligned}
u(1 - \tfrac{1}{v}(1-x)) &= \frac{P(AB)}{P(A)}\left(1 - \frac{P(B)P(B\overline{C})}{P(AB)P(B)}\right) \\
&= \frac{P(ABC) - P(\overline{A}B\overline{C})}{P(A)} \\
&\leq \frac{P(AC)}{P(A)},
\end{aligned}
$$

we can conclude

$$u_1(1 - \tfrac{1}{v_1}(1 - x_1)) \quad \leq \quad \frac{P(AC)}{P(A)}.$$

Case $v_1 > 0$, $P(B) = 0$: we have $u_1 = 0$, hence $u_1(1 - \frac{1}{v_1}(1 - x_1)) = 0$.

2) Case $AB \subseteq C$: since $P(AB\overline{C}) = 0$, we get

$$u_1 \quad \leq \quad u \quad = \quad \frac{P(AB)}{P(A)} \quad = \quad \frac{P(ABC)}{P(A)} \quad \leq \quad \frac{P(AC)}{P(A)}.$$

3) Case $C \subseteq A$, $v_2 > 0$, $y_2 > 0$: since $P(\overline{A}C) = 0$, we get

$$\begin{aligned} x_1 u_1 \quad &\leq \quad x_1 u \quad = \quad x_1 \frac{P(AB)}{P(A)} \quad \leq \quad x_1 v_2 \frac{P(B)}{P(A)} \\ &\leq \quad v_2 \frac{P(BC)}{P(A)} \quad \leq \quad v_2 y_2 \frac{P(C)}{P(A)} \quad = \quad v_2 y_2 \frac{P(AC)}{P(A)}. \end{aligned}$$

4) Case $C \subseteq A$, $AB \subseteq C$, $y_2 > 0$: since $P(\overline{A}C) = 0$ and $P(AB\overline{C}) = 0$, we get

$$\begin{aligned} u_1 \quad \leq \quad u \quad &= \quad \frac{P(AB)}{P(A)} \quad = \quad \frac{P(ABC)}{P(A)} \quad = \quad \frac{P(BC)}{P(A)} \\ &\leq \quad y_2 \frac{P(C)}{P(A)} \quad = \quad y_2 \frac{P(AC)}{P(A)}. \end{aligned}$$

5) Case $AB \subseteq C$, $\overline{B}C \subseteq A$, $v_2 > 0$, $y_2 > 0$, $P(B) > 0$: (Note that $\overline{B}C \subseteq A \Leftrightarrow \overline{A}C \subseteq B$.) Since $P(AB\overline{C}) = 0$ and $P(\overline{A}\,\overline{B}C) = 0$, we get

$$\begin{aligned} xu \quad &= \quad \frac{P(BC)}{P(B)} \frac{P(AB)}{P(A)} \quad = \quad v \frac{P(BC)}{P(A)} \quad \text{and} \\ xu \quad &\leq \quad v y_2 \frac{P(C)}{P(A)} \quad = \quad v y_2 \frac{P(BC) + P(A\overline{B}C)}{P(A)}. \end{aligned}$$

Therefore we can conclude

$$x_1 u_1(\tfrac{1}{y_2} - 1) \quad \leq \quad xu(\tfrac{1}{y_2} - 1) \quad \leq \quad v \frac{P(A\overline{B}C)}{P(A)} \quad \leq \quad v_2 \frac{P(A\overline{B}C)}{P(A)}.$$

Since

$$u_1 \quad \leq \quad u \quad \leq \quad \frac{P(AB)}{P(A)} \quad = \quad \frac{P(ABC)}{P(A)},$$

we get

$$u_1 + \tfrac{x_1 u_1}{v_2}(\tfrac{1}{y_2} - 1) \quad \leq \quad u + \tfrac{xu}{v_2}(\tfrac{1}{y_2} - 1) \quad \leq \quad \frac{P(AC)}{P(A)}.$$

Case $AB \subseteq C$, $\overline{B}C \subseteq A$, $v_2 > 0$, $y_2 > 0$, $P(B) = 0$: We have $u_1 = 0$, hence $u_1 + \frac{x_1 u_1}{v_2}(\frac{1}{y_2} - 1) = 0$.

6) Case $A\overline{B} \subseteq C$: since $P(A\overline{B}\,\overline{C}) = 0$, we get

$$1 - u_2 \;\leq\; 1 - u \;=\; \frac{P(A\overline{B})}{P(A)} \;=\; \frac{P(A\overline{B}C)}{P(A)} \;\leq\; \frac{P(AC)}{P(A)}.$$

7) Case $BC \subseteq A$, $A\overline{B} \subseteq C$, $v_2 \geq x_1$: since $P(\overline{A}BC) = 0$ and $P(A\overline{B}\,\overline{C}) = 0$, we get

$$\begin{aligned}
x_1 u \;&=\; x_1\frac{P(AB)}{P(A)} \;\leq\; x_1 v_2\frac{P(B)}{P(A)} \;\leq\; v_2\frac{P(BC)}{P(A)} \;=\; v_2\frac{P(ABC)}{P(A)},\\
-u \;&=\; -\frac{P(AB)}{P(A)} \text{ and}\\
1 \;&=\; \frac{P(A)}{P(A)} \;=\; \frac{P(AC)+P(AB\overline{C})}{P(A)}.
\end{aligned}$$

Thus we can conclude

$$1 - u_2(1 - \tfrac{x_1}{v_2}) \;\leq\; 1 - u(1 - \tfrac{x_1}{v_2}) \;\leq\; \frac{P(AC)}{P(A)}.$$

8) Case $C \subseteq A$, $A\overline{B} \subseteq C$, $v_2 > x_1$, $y_2 > 0$: in 3) and 7) we already proved:

$$\begin{aligned}
\tfrac{x_1 u}{v_2 y_2} \;&\leq\; \frac{P(AC)}{P(A)} \text{ and}\\
1 - u(1 - \tfrac{x_1}{v_2}) \;&\leq\; \frac{P(AC)}{P(A)}.
\end{aligned}$$

By local linear optimization we get

$$\tfrac{x_1}{y_2(v_2 - x_1) + x_1} \;\leq\; \frac{P(AC)}{P(A)}.$$

9) Case $A \subseteq C$: the claim follows directly from $P(AC) = P(A)$.

b) Correctness of the upper bound:

1) Case $v_1 > x_2$: Since

$$x_2 u \;=\; x_2\frac{P(AB)}{P(A)} \;\geq\; x_2 v_1\frac{P(B)}{P(A)} \;\geq\; v_1\frac{P(BC)}{P(A)},$$

we get

$$\begin{aligned}
-u \;&=\; -\frac{P(ABC)+P(AB\overline{C})}{P(A)},\\
\tfrac{u x_2}{v_1} \;&\geq\; \frac{P(ABC)+P(\overline{A}BC)}{P(A)} \text{ and}\\
1 \;&=\; \frac{P(ABC)+P(A\overline{B}C)+P(AB\overline{C})+P(A\overline{B}\,\overline{C})}{P(A)}.
\end{aligned}$$

Thus we can conclude

$$\begin{aligned} 1-u_1(1-\tfrac{x_2}{v_1}) &\geq 1-u(1-\tfrac{x_2}{v_1}) \\ &\geq \frac{P(BC)+P(A\overline{B})}{P(A)} \\ &\geq \frac{P(AC)}{P(A)}. \end{aligned}$$

2) Case $v_1>0$, $y_1>0$: we get

$$\begin{aligned} x_2u_2 &\geq x_2u &= x_2\frac{P(AB)}{P(A)} &\geq x_2v_1\frac{P(B)}{P(A)} \\ &\geq v_1\frac{P(BC)}{P(A)} &\geq v_1y_1\frac{P(C)}{P(A)} &\geq v_1y_1\frac{P(AC)}{P(A)}. \end{aligned}$$

3) Case $v_1>x_2$, $y_1>0$: in 1) and 2) we already proved:

$$\begin{aligned} 1-u(1-\tfrac{x_2}{v_1}) &\geq \frac{P(AC)}{P(A)} \quad \text{and} \\ \tfrac{x_2u}{v_1y_1} &\geq \frac{P(AC)}{P(A)}. \end{aligned}$$

By local linear optimization we get

$$\tfrac{x_2}{y_1(v_1-x_2)+x_2} \geq \frac{P(AC)}{P(A)}.$$

4) Case pos$(B)$, $x_2>v_1$, $v_1>0$, $y_1>0$: (Note that the proof does not require $x_2>v_1$. For $x_2\leq v_1$ we have $\frac{u_2x_2}{v_1y_1}\leq u_2(1-\frac{x_2}{v_1}(1-\frac{1}{y_1}))$.)

We get

$$\begin{aligned} u &= \frac{P(ABC)+P(AB\overline{C})}{P(A)}, \\ -\tfrac{ux}{v} &= -\frac{P(ABC)+P(\overline{A}BC)}{P(A)} \quad \text{and} \\ \tfrac{ux}{vy_1} &\geq \frac{P(ABC)+P(\overline{A}BC)+P(A\overline{B}C)+P(\overline{A}\,\overline{B}C)}{P(A)}. \end{aligned}$$

Thus we can conclude

$$\begin{aligned} u_2(1-\tfrac{x_2}{v_1}(1-\tfrac{1}{y_1})) &\geq u(1-\tfrac{x}{v}(1-\tfrac{1}{y_1})) \\ &\geq \frac{P(AB)+P(\overline{B}C)}{P(A)} \\ &\geq \frac{P(AC)}{P(A)}. \end{aligned}$$

5) Case $AC \subseteq B$: since $P(A\overline{B}C) = 0$, we get

$$u_2 \quad \geq \quad u \quad = \quad \frac{P(AB)}{P(A)} \quad \geq \quad \frac{P(ABC)}{P(A)} \quad = \quad \frac{P(AC)}{P(A)}.$$

6) Case $AC \subseteq B$, $v_1 > x_2$: since

$$x_2 u_2 \quad \geq \quad x_2 \frac{P(AB)}{P(A)} \quad \geq \quad x_2 v_1 \frac{P(B)}{P(A)} \quad \geq \quad v_1 \frac{P(BC)}{P(A)}$$

and $AC \subseteq B$ we get

$$\tfrac{x_2 u_2}{v_1} \quad \geq \quad \frac{P(BC)}{P(A)} \quad = \quad \frac{P(ABC)+P(\overline{A}BC)}{P(A)} \quad \geq \quad \frac{P(AC)}{P(A)}.$$

7) Case $AC \subseteq B$, $\overline{A}B \subseteq C$, pos$(B)$, $v_2 > 0$: since $P(A\overline{B}C) = 0$, $P(\overline{A}B\overline{C}) = 0$ and

$$\begin{aligned} -u &= -v \frac{P(B)}{P(A)} \quad \text{and} \\ ux &= v \frac{P(BC)}{P(A)}, \end{aligned}$$

we get

$$u(x-1) \quad = \quad -v \frac{P(AB\overline{C})}{P(A)} \quad \geq \quad -v_2 \frac{P(AB\overline{C})}{P(A)}.$$

Since $u = \dfrac{P(AB)}{P(A)}$, we can conclude

$$u(1 - \tfrac{1}{v_2}(1-x)) \quad \geq \quad \frac{P(ABC)}{P(A)} \quad = \quad \frac{P(AC)}{P(A)}.$$

Thus we get

$$u_2(1 - \tfrac{1}{v_2}(1-x_2)) \quad \geq \quad \frac{P(AC)}{P(A)}.$$

8) Case $AC \subseteq B$, $\overline{A}B \subseteq C$, pos$(B)$, $v_1 > 0$, $y_1 > 0$: analogous to the proofs in 2) and 7), we can show:

$$\begin{aligned} \tfrac{u_2 x_2}{v y_1} &\geq \frac{P(AC)}{P(A)} \quad \text{and} \\ u_2(1 - \tfrac{1}{v}(1-x_2)) &\geq \frac{P(AC)}{P(A)} \quad \text{(note that } v_1 > 0). \end{aligned}$$

By local rational optimization we get

$$\tfrac{u_2 x_2}{y_1(1-x_2)+x_2} \quad \geq \quad \frac{P(AC)}{P(A)}.$$

9) Case $\overline{A}C \subseteq B$, pos$(C)$, $y_2 < 1$: since $P(\overline{A}\,\overline{B}C) = 0$ and

$$\begin{aligned} 1-u &= \frac{P(A\overline{B})}{P(A)} \quad \text{and} \\ \tfrac{1}{1-y_2} &\geq \frac{P(C)}{P(\overline{B}C)}, \end{aligned}$$

we get

$$\tfrac{1-u_1}{1-y_2} \geq \tfrac{1-u}{1-y_2} \geq \frac{P(C)P(A\overline{B})}{P(A)P(\overline{B}C)} = \frac{P(C)P(A\overline{B})}{P(A)P(A\overline{B}C)} \geq \frac{P(AC)}{P(A)}.$$

10) Case $C \subseteq A$, $AB \subseteq C$, $y_1 > 0$: since $P(\overline{A}C) = 0$ and $P(AB\overline{C}) = 0$, we get

$$\begin{aligned} u_2 \geq u &= \frac{P(AB)}{P(A)} = \frac{P(ABC)}{P(A)} = \frac{P(BC)}{P(A)} \\ &\geq y_1 \frac{P(C)}{P(A)} = y_1 \frac{P(AC)}{P(A)}. \end{aligned}$$

11) Case $ABC = \emptyset$: since $P(ABC) = 0$, we get

$$\begin{aligned} 1-u_1 \geq 1-u &= \frac{P(A\overline{B})}{P(A)} \\ \geq \frac{P(A\overline{B}C)}{P(A)} &= \frac{P(AC)}{P(A)}. \end{aligned}$$

12) Case $ABC = \emptyset$, pos$(B)$, $v_1 > 0$, $y_1 > 0$: since

$$\begin{aligned} -\tfrac{ux}{v} &= -\frac{P(ABC)+P(\overline{A}BC)}{P(A)} \quad \text{and} \\ \tfrac{ux}{vy_1} &\geq \frac{P(ABC)+P(\overline{A}BC)+P(A\overline{B}C)+P(\overline{A}\,\overline{B}C)}{P(A)} \end{aligned}$$

we get

$$\begin{aligned} \tfrac{u_2x_2}{v_1y_1} - \tfrac{u_2x_2}{v_1} &\geq \tfrac{ux_2}{v_1y_1} - \tfrac{ux_2}{v_1} \geq \tfrac{ux}{vy_1} - \tfrac{ux}{v} \\ &\geq \frac{P(A\overline{B}C)+P(\overline{A}\,\overline{B}C)}{P(A)} = \frac{P(AC)+P(\overline{A}\,\overline{B}C)}{P(A)} \\ &\geq \frac{P(AC)}{P(A)}. \end{aligned}$$

In 11) we already proved $1-u \geq \frac{P(AC)}{P(A)}$. By local linear optimization we get $\frac{(1-y_1)x_2}{y_1(v_1-x_2)+x_2} \geq \frac{P(AC)}{P(A)}$.

13) Case $AC=\emptyset$: the claim follows directly from $P(AC)=0$.

Correctness of (WCRN): Generally we have the following propositions:

$$\begin{aligned}
u &= \frac{P(AF)}{P(A)}, \quad u\in[u_1,u_2], \quad u_1,u_2\in[0,1],\\
v &= \frac{P(AF)}{P(F)}, \quad v\in[v_1,v_2], \quad v_1,v_2\in[0,1],\\
x &= \frac{P(FC)}{P(F)}, \quad x\in[x_1,x_2], \quad x_1,x_2\in[0,1],\\
y &= \frac{P(FC)}{P(C)}, \quad y\in[y_1,y_2], \quad y_1,y_2\in[0,1].
\end{aligned}$$

a) Correctness of the lower bound:

1) Case $A\overline{F}\subseteq C$: since $P(A\overline{F}\,\overline{C})=0$, we get

$$1-u_2 \;\le\; 1-u \;=\; \frac{P(A\overline{F})}{P(A)} \;=\; \frac{P(A\overline{F}C)}{P(A)}.$$

2) Case $A=C$: we get

$$1-y_2 \;\le\; 1-y \;=\; \frac{P(\overline{F}C)}{P(C)} \;=\; \frac{P(A\overline{F})}{P(A)} \;=\; \frac{P(A\overline{F}C)}{P(A)}.$$

3) Case $AF\subseteq C$, $C\subseteq A$, $y_2>0$: since $P(\overline{A}C)=0$ and $P(AF\overline{C})=0$, we get

$$\begin{aligned}
(1-y_2)u_1 \;&\le\; (1-y)u && = \frac{P(\overline{F}C)P(AF)}{P(C)P(A)}\\
&= \frac{P(A\overline{F}C)P(FC)}{P(C)P(A)} && = y\frac{P(A\overline{F}C)}{P(A)}\\
&\le y_2\frac{P(A\overline{F}C)}{P(A)}.
\end{aligned}$$

4) Case $C\overline{F}\subseteq A$, $v_2>0$, $y_2>0$: since $P(\overline{A}\,\overline{F}C)=0$, we get

$$\begin{aligned}
(1-y_2)u_1x_1 \;&\le\; (1-y_2)ux_1 && \le\; (1-y)ux\\
&= \frac{P(\overline{F}C)P(AF)P(FC)}{P(C)P(A)P(F)} && = vy\frac{P(\overline{F}C)}{P(A)}\\
&= vy\frac{P(A\overline{F}C)}{P(A)} && \le\; v_2y_2\frac{P(A\overline{F}C)}{P(A)}.
\end{aligned}$$

5) Case $A\overline{F} \subseteq C$, $C\overline{F} \subseteq A$, $v_2 > 0$, $y_2 > 0$: in 1) and 4) we already proved:

$$\begin{array}{rcl} 1-u & \leq & \frac{P(A\overline{F}C)}{P(A)} \text{ and} \\ \frac{(1-y_2)ux_1}{v_2 y_2} & \leq & \frac{P(A\overline{F}C)}{P(A)}. \end{array}$$

The claim can now be proved by linear optimization.

b) Correctness of the upper bound:

1) The expression $1-u_1$ is an upper bound, since $\frac{P(A\overline{F}C)}{P(A)} \leq \frac{P(AF)}{P(A)} \leq 1-u_1$.

2) Case $A = C$: analogous to the proof for the correctness of the lower bound for this case.

3) Case $AF \subseteq C$, $CF \subseteq A$, $y_1 > 0$: since $P(AF\overline{C}) = 0$ and $P(\overline{A}FC) = 0$, we get

$$\begin{array}{rcccl} (1-y_1)u_2 & \geq & (1-y)u & = & \frac{P(\overline{F}C)P(AF)}{P(C)P(A)} \\ & \geq & \frac{P(A\overline{F}C)P(FC)}{P(C)P(A)} & = & y\frac{P(A\overline{F}C)}{P(A)} \\ & \geq & y_1\frac{P(A\overline{F}C)}{P(A)}. & & \end{array}$$

4) Case $v_1 > 0$, $y_1 > 0$: already proved in previous papers.

5) Case $A\overline{F}C = \emptyset$: the claim follows directly from $P(A\overline{F}C) = 0$. □

## B.2 The previous calculus

There are a some rules, occuring in the calculus of previous papers, which are not considered anymore in this paper. Our new approach and the refinement of (PRC1) assure that the rules given up are subsumed by the rules of the new calculus. Before proving this claim, let us have a look at the rules given up, modified according to the internal representation of uncertain rules.
Let $A, B, C \in \mathcal{C}$.

**(C)** Chaining:

(b) $\{A \xrightarrow{x_1,x_2} ABC,\ \text{pos}(A)\} \vdash A \xrightarrow{x_1,1} AC$

(c) $\{A \xrightarrow{x_1,x_2} ABC,\ \text{pos}(A),\ C \xrightarrow{1} BC,\ \text{pos}(C)\} \vdash A \xrightarrow{x_1,x_2} AC$

(d) $\{A \xrightarrow{x_1,x_2} ABC,\ \text{pos}(A),\ A \xrightarrow{1} AB\} \vdash A \xrightarrow{x_1,x_2} AC$

**(CL)** Conjunction Left:

(a) $\{A \xrightarrow{u_1,u_2} AB,\ \mathrm{pos}(AB),\ A \xrightarrow{x_1,x_2} ABC\} \vdash AB \xrightarrow{z_1,z_2} ABC,$

$$z_1 = \tfrac{x_1}{u_2},$$
$$z_2 = \begin{cases} 0 & \text{if } x_2 = 0 \\ \min(1, \frac{x_2}{u_1}) & \text{if } x_2 > 0, u_1 > 0 \\ 1 & \text{otherwise} \end{cases}$$

(b) $\{AB \xrightarrow{x_1,x_2} ABC,\ \mathrm{pos}(AB),\ A \xrightarrow{y} AC,\ y = 0 \vee y = 1\} \vdash$

$AB \xrightarrow{y} ABC$

**(WCL)** Weak Conjunction Left:

$\{B \xrightarrow{v_1,v_2} AB,\ \mathrm{pos}(AB),\ B \xrightarrow{x_1,x_2} BC\} \vdash AB \xrightarrow{z_1,z_2} ABC,$

$$z_1 = \begin{cases} 1 & \text{if } x_1 = 1 \\ \max(0, 1 - \frac{1-x_1}{v_1}) & \text{if } x_1 < 1, v_1 > 0 \\ 0 & \text{otherwise} \end{cases}$$
$$z_2 = \begin{cases} 0 & \text{if } x_2 = 0 \\ \min(1, \frac{x_2}{v_1}) & \text{if } x_2 > 0, v_1 > 0 \\ 1 & \text{otherwise} \end{cases}$$

**(CR)** Conjunction Right:

(a) $\{A \xrightarrow{x_1,x_2} AB,\ AB \xrightarrow{y_1,y_2} ABC,\ \mathrm{pos}(AB)\} \vdash A \xrightarrow{z_1,z_2} ABC,$

$z_1 = x_1 \cdot y_1,\ z_2 = x_2 \cdot y_2$

(b) $\{A \xrightarrow{x_1,x_2} AB,\ \mathrm{pos}(A),\ B \xrightarrow{y} BC,\ \mathrm{pos}(B)\} \vdash A \xrightarrow{z_1,z_2} ABC,$

$$z_1 = \begin{cases} 0 & \text{if } y = 0 \\ x_1 & \text{if } y = 1 \end{cases},\ z_2 = \begin{cases} 0 & \text{if } y = 0 \\ x_2 & \text{if } y = 1 \end{cases}$$

**(WCR)** Weak Conjunction Right:

$\{A \xrightarrow{x_1,x_2} AB,\ \mathrm{pos}(A),\ A \xrightarrow{y_1,y_2} AC\} \vdash A \xrightarrow{0,z_2} ABC,$

$z_2 = \min(x_2, y_2)$

**(N)** Negation:

$\{A \xrightarrow{x_1,x_2} AF,\ \mathrm{pos}(A)\} \vdash A \xrightarrow{z_1,z_2} A\overline{F},$

$z_1 = 1 - x_2,\ z_2 = 1 - x_1$

The rules (C)(c), (C)(d), (CL)(b) and (CR)(b) can obviously be given up, since uncertain reasoning according to our new approach is based on $\mathcal{C}_{\mathcal{T}}$ instead of $\mathcal{C}$:

1) (C)(c), (C)(d): since $C \subseteq B$ or $A \subseteq B$, we have

$$A_{\mathcal{T}} \xrightarrow{x_1,x_2} (ABC)_{\mathcal{T}} \quad = \quad A_{\mathcal{T}} \xrightarrow{x_1,x_2} (AC)_{\mathcal{T}}.$$

2) (CL)(b), case $y = 1$: since $A \subseteq C$, we have

$$(AB)_{\mathcal{T}} \xrightarrow{x_1,x_2} (ABC)_{\mathcal{T}} \quad = \quad (ABC)_{\mathcal{T}} \xrightarrow{x_1,x_2} (ABC)_{\mathcal{T}}.$$

$(ABC)_{\mathcal{T}} \xrightarrow{1} (ABC)_{\mathcal{T}}$ is then derived by (TK)(a) and (S).

3) (CL)(b), case $y = 0$: since $AC = \emptyset$, we have

$$(AB)_{\mathcal{T}} \xrightarrow{x_1,x_2} (ABC)_{\mathcal{T}} \quad = \quad (ABC)_{\mathcal{T}} \xrightarrow{x_1,x_2} \emptyset_{\mathcal{T}}.$$

$(ABC)_{\mathcal{T}} \xrightarrow{0} \emptyset_{\mathcal{T}}$ is then derived by (TK)(b) and (S).

4) (CR)(b): analogous to (CL)(b).

All the other rules are special cases of (PRC1) or (CRN):

1) (C)(b) is a special case of (PRC1), because of (TK)(a) and (CO)(b).

2) The rule (CL)(a) can be modified to
$\{A' \xleftrightarrow[u_1,u_2]{1} B',\ \mathrm{pos}(A'),\ B' \xleftrightarrow[1]{x_1,x_2} C'\} \vdash A' \xrightarrow{z_1,z_2} C'$.
Taking into account the additional conditions $\mathrm{pos}(B')$, $A' \subseteq B'$, $C' \subseteq B'$ and $C' \subseteq A'$, it is easy to verify, that the bounds computed according to (PRC1) are at least as precise as the bounds computed according to (CL)(a).

3) Analogous to 2), we can prove that also (WCL) is a special case of (PRC1).

4) The rule (CR)(a) can be modified to
$\{A' \xleftrightarrow[1]{x_1,x_2} B',\ \mathrm{pos}(A'),\ B' \xleftrightarrow[1]{y_1,y_2} C'\} \vdash A' \xrightarrow{z_1,z_2} C'$.
Taking into account the additional conditions $\mathrm{pos}(B')$, $B' \subseteq A'$, $C' \subseteq B'$ and $C' \subseteq A'$, it is easy to verify, that the bounds computed according to (PRC1) are at least as precise as the bounds computed according to (CR)(a).

5) (WCR) is a special case of (PRC1), because of (TK)(a) and (CO)(b).

6) (N) is a special case of (CRN), because of (TK)(a).

## B.3 Some extensions to the uncertain calculus

According to Def. 3.3 our probabilistic knowledge is defined with respect to conjunctive t-classes. This restriction is made in order to keep the search space small. Do we have to "pay" this advantage by some other disadvantages? Do our computations for precise bounds become more complex by this restriction to conjunctive t-classes? The following rules are useful to give an answer to this question. There are cases in which the bounds computed by these extensions are more precise than the bounds computed by the calculus, presented before. Let $A$, $C_\emptyset \in \mathcal{C}$ and $C_{\{i\}} \in \mathcal{C}$ for all $i \in [1:n]$ with $C_\emptyset = C_{\{1\}} \cup \cdots \cup C_{\{n\}}$. For all $\{i_1, \ldots, i_k\} \subseteq [1:n]$ with $k \geq 1$ let $C_{\{i_1,\ldots,i_k\}} := C_{\{i_1\}} \cap \cdots \cap C_{\{i_k\}}$. Let the mapping $\varphi : \mathbf{N} \longrightarrow \{1,2\}$ be defined by $\varphi(i) := 1$ for all odd numbers $i$ and $\varphi(i) := 2$ for all even numbers $i$. The following rules are valid.

**(GC)** Generalized Chaining:

$$\{ A \xrightarrow{x_{I,1},x_{I,2}} AC_I \,|\, I \subseteq [1:n],\, I \neq \emptyset \} \vdash A \xrightarrow{z_1,z_2} AC_\emptyset,$$

$$z_1 = \begin{cases} 1 & \text{if } A \subseteq AC_\emptyset \\ \max(0, \Sigma_{j\in[1:n]}\Sigma_{I\subseteq[1:n],|I|=j}\,(-1)^{j+1}x_{I,\,\varphi(j)}) & \text{otherwise} \end{cases}$$

$$z_2 = \begin{cases} 1 & \text{if } A \subseteq AC_\emptyset \\ \min(1, \Sigma_{j\in[1:n]}\Sigma_{I\subseteq[1:n],|I|=j}\,(-1)^{j+1}x_{I,\,\varphi(j+1)}) & \text{otherwise} \end{cases}$$

**(GCRN)** Generalized Conjunction Right with Negation:

$$\{ A \xrightarrow{x_{I,1},x_{I,2}} AC_I \,|\, I \subseteq [1:n],\, I \neq \{1\} \} \vdash A \xrightarrow{z_1,z_2} AC_{\{1\}},$$

$$z_1 = \max(0, y_1 - \Sigma_{j\in[1:n]}\Sigma_{I\subseteq[1:n],|I|=j,I\neq\{1\}}(-1)^{j+1}x_{I,\,\varphi(j+1)})$$

$$z_2 = \min(1, y_2 - \Sigma_{j\in[1:n]}\Sigma_{I\subseteq[1:n],|I|=j,I\neq\{1\}}(-1)^{j+1}x_{I,\,\varphi(j)})$$

**Example B.3**
Let us consider (GC) for the case $n = 3$:

$$\begin{aligned}
&\{ A \xrightarrow{x_{\{1\},1},x_{\{1\},2}} AC_{\{1\}},\ A \xrightarrow{x_{\{2\},1},x_{\{2\},2}} AC_{\{2\}},\ A \xrightarrow{x_{\{3\},1},x_{\{3\},2}} AC_{\{3\}},\\
&A \xrightarrow{x_{\{1,2\},1},x_{\{1,2\},2}} AC_{\{1,2\}},\ A \xrightarrow{x_{\{1,3\},1},x_{\{1,3\},2}} AC_{\{1,3\}},\\
&A \xrightarrow{x_{\{2,3\},1},x_{\{2,3\},2}} AC_{\{2,3\}},\ A \xrightarrow{x_{\{1,2,3\},1},x_{\{1,2,3\},2}} AC_{\{1,2,3\}} \}\\
&\vdash A \xrightarrow{z_1,z_2} AC_\emptyset,
\end{aligned}$$

$$z_1 = x_{\{1\},1} + x_{\{2\},1} + x_{\{3\},1} - x_{\{1,2\},2} - x_{\{1,3\},2} - x_{\{2,3\},2} + x_{\{1,2,3\},1}$$
$$z_2 = x_{\{1\},2} + x_{\{2\},2} + x_{\{3\},2} - x_{\{1,2\},1} - x_{\{1,3\},1} - x_{\{2,3\},1} + x_{\{1,2,3\},2} \qquad \square$$

We can extract the following specializations of (GC) and (GCRN):

(DGC) Disjoint Generalized Chaining:

(GC) with $C_{\{1\}} \parallel \cdots \parallel C_{\{n\}}$,

(DGCRN) Disjoint Generalized Conjunction Right with Negation:

(GCRN) with $C_{\{1\}} \parallel \cdots \parallel C_{\{n\}}$,

(GN) Generalized Negation:

(GCRN) with $C_\emptyset = \mathcal{O}$ and

(DGN) Disjoint Generalized Negation:

(GCRN) with $C_\emptyset = \mathcal{O}$ and $C_{\{1\}} \parallel \cdots \parallel C_{\{n\}}$.

**Theorem B.4**

*1) The rules (DGC) and (DGCRN) are not subsumed by the new calculus.*

*2) The rules (GC) and (GCRN) are not subsumed by the new calculus and the rules (DGC) and (DGCRN).*

**Proof:** 1) We prove the claim for (DGC). The proof for (DGCRN) is similar. For $A_\mathcal{T}, B_\mathcal{T} \in \mathcal{C}_\mathcal{T}$ let $A_\mathcal{T} \vee B_\mathcal{T}$ denote the supremum of $A_\mathcal{T}$ and $B_\mathcal{T}$ in $(\mathcal{C}_\mathcal{T}, \subseteq_\mathcal{T})$. Let $\mathcal{G} = \{A_i \mid i \in N\} \cup \{C\}$ be a set of positive t-class terms. Let the set of $n$ conjunctive t-class terms $\mathcal{M}_n = \{C_{\{i\}} \mid i \in [1:n]\}$ be recursively defined by:

$$\begin{array}{rcl}
\mathcal{M}_2 & = & \{CA_1, C\overline{A}_1\} \\
\mathcal{M}_3 & = & \{CA_1A_2, C\overline{A}_1A_3, C\overline{A}_2\overline{A}_3\} \\
\mathcal{M}_4 & = & \{CA_1A_2A_4, C\overline{A}_1A_3A_5, C\overline{A}_2\overline{A}_3A_6, C\overline{A}_4\overline{A}_5\overline{A}_6\} \\
\mathcal{M}_5 & = & \{CA_1A_2A_4A_7, C\overline{A}_1A_3A_5A_8, C\overline{A}_2\overline{A}_3A_6A_9, C\overline{A}_4\overline{A}_5\overline{A}_6A_{10}, \\
& & \;C\overline{A}_7\overline{A}_8\overline{A}_9\overline{A}_{10}\} \\
& \vdots &
\end{array}$$

We assume that we have pos$(D)$ for all minimal conjunctive t-class terms $D \in \mathcal{CC}$ for which there exists a $C_{\{i\}} \in \mathcal{M}_n$ with $\models D \subseteq C_{\{i\}}$. Furthermore we assume that we have $D = \emptyset$ for all minimal conjunctive t-class terms $D \in \mathcal{CC}$ with $\models D \subseteq C$ and $\not\models D \subseteq C_{\{i\}}$ for all $i \in [1:n]$. Then we get $C_{\{1\}} \| \ldots \| C_{\{n\}} = C$ and $(C_{\{i\}})_\mathcal{T} \vee (C_{\{j\}})_\mathcal{T} = C_\mathcal{T}$ for all $i, j \in [1:n]$ with $i < j$. Given the probabilistic knowledge-base

$$\{\mathcal{O} \xrightarrow{x_{\{i\},1}, x_{\{i\},2}} C_{\{i\}} \,|\, i \in [1:n]\},$$

we can derive the bounds $z_1 = \max(x_{\{1\},1}, \ldots, x_{\{n\},1})$ and $z_2 = 1$ for the uncertain rule $\mathcal{O} \xrightarrow{z_1, z_2} C$ by applying (TK)(a), (CO)(b), (PRC1) and (S). The bounds computed by (DGC) are much better, they are given by $z_1 = x_{\{1\},1} + \ldots + x_{\{n\},1}$ and $z_2 = x_{\{1\},2} + \ldots + x_{\{n\},2}$.

2) We prove the claim for (GC). The proof is similar to 1), we just have to modify the set of $n$ conjunctive t-classes $\mathcal{M}_n = \{C_{\{j\}} \mid j \in [1:n]\}$:

$$\mathcal{M}_n \quad = \quad \{CA_jA_{j+1} \mid j \in \{1,3,\ldots,2n-1\}\}$$

With the assumptions of 1) we get $C_{\{1\}} \cup \ldots \cup C_{\{n\}} = C$ and $(C_{\{i\}})_\mathcal{T} \vee (C_{\{j\}})_\mathcal{T} = C_\mathcal{T}$ for all $i, j \in [1:n]$ with $i < j$. Given the set of uncertain rules

$$\{\, \mathcal{O} \xrightarrow{x_{I,1},x_{I,2}} C_I \mid I \in \mathcal{P}([1:n]) \backslash \emptyset \,\},$$

we can derive the bounds $z_1 = \max(x_{\{1\},1}, \ldots, x_{\{1,\ldots,n\}1})$ and $z_2 = 1$ for the uncertain rule $\mathcal{O} \xrightarrow{z_1,z_2} C$ by applying (TK)(a), (CO)(b), (PRC1) and (S). The bounds computed by (GC) are much better (see B.3). Since we do not have any disjointness formula, we cannot apply the rules (DGC) and (DGCRN) either. The proof for (GCRN) is similar. $\square$

Note that (GC) and (GCRN) are special cases of a more general rule of the following kind

$$\begin{array}{l} j \in [0:n],\, \{\, A \xrightarrow{x_{I,1},x_{I,2}} AC_I \mid I \subseteq [1:n],\, I \neq [1:j] \,\} \\ \vdash A \xrightarrow{z_1,z_2} AC_{[1:j]}. \end{array}$$

Note that (PRC2) could also be generalized accordingly.

**Example B.5**
Let $\mathcal{B}^+ := \{C, A_1, A_2, A_3\}$. Let the probabilistic knowledge-base $\mathcal{P}$ be given by

$$\begin{array}{rcl} \mathcal{P} & := & \{\, CA_1\overline{A}_2A_3 = \emptyset,\, C\overline{A}_1A_2\overline{A}_3 = \emptyset,\, \mathcal{O} \xrightarrow{0.3} CA_1A_2, \\ & & \mathcal{O} \xrightarrow{0.3} C\overline{A}_1A_3,\, \mathcal{O} \xrightarrow{0.3} C\overline{A}_2\overline{A}_3 \,\}. \end{array}$$

We get $\mathcal{P} \underset{(DGC)}{\vdash} \mathcal{O} \xrightarrow{0.9} C$. Our calculus just permits us to conclude $\mathcal{O} \xrightarrow{0.3,1} C$. $\square$

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