Uncertain Reasoning in Concept Lattices

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Lehrstuhl für Informatik II, Universität Augsburg, Universitätsstr. 2,
86135 Augsburg, Germany, lukasiewicz@uni-augsburg.de

Abstract. This paper presents concept lattices as a natural representation of class hierarchies in object-oriented databases and frame based knowledge representations. We show how to extend concept lattices by uncertainty in the form of conditional probabilities. We illustrate that uncertain reasoning within the hierarchical structure of concept lattices can be performed efficiently and makes uncertain conclusions more precise.

1 Introduction

The aim of this paper is to integrate uncertainty into class hierarchies of object-oriented databases and frame based knowledge representations.

Extensional subclass relationships and disjointness statements are characteristic of class hierarchies. They can naturally be represented by *concept lattices* (see e.g. [14]). A concept is a pair consisting of a set of objects and a set of properties that all these objects share. The concept order is based on a coupled extensional and intensional order. For our purpose it is sufficient to concentrate just on the intensional part.

Uncertainty is integrated into concept lattices by a model based on conditional probabilities with intervals (see e.g. [10], [11], [12]). Surprisingly, uncertain reasoning in concept lattices has a lot of advantages. We show that the availability of conceptual knowledge yields an enormous search space reduction and that conceptual knowledge can be exploited for precise conclusions.

More details about the integration of uncertainty into object-oriented databases in the form of *uncertain constraints* are provided by [7] and [8].

2 Syntax and Semantics of Conceptual Knowledge

We start with the definition of a language for conceptual knowledge. We introduce *c-terms* as a means to refer to concepts. Classification among concepts can then be formulated by conceptual formulas.

Definition 2.1 (Syntax of c-terms) We consider an alphabet $\mathcal{A} := \{\emptyset, \mathcal{U}, B_1, \ldots, B_k\}$ of constants. The set $\mathcal{C}$ of all *c-terms* is the minimal set with $\mathcal{A} \subseteq \mathcal{C}$ and $C, D \in \mathcal{C} \Longrightarrow CD \in \mathcal{C}$.

Definition 2.2 (Semantics of c-terms) An *interpretation* $\mathcal{J} := (\mathcal{U}, J)$ of the set of c-terms $\mathcal{C}$ consists of a finite set of objects $\mathcal{U}$ and a mapping $J\colon \{B_1, \ldots, B_k\} \longrightarrow 2^{\mathcal{U}}$. J is extended to $\mathcal{C}$ by: $J(\boldsymbol{\mathcal{U}}) := \mathcal{U}$, $J(\boldsymbol{\emptyset}) := \emptyset$, $J(CD) := J(C) \cap J(D)$. In the sequel, we identify $\boldsymbol{\mathcal{U}}$ with $\mathcal{U}$ and $\boldsymbol{\emptyset}$ with $\emptyset$.

Definition 2.3 (Syntax of conceptual formulas) Let C, D, $D_1, \ldots, D_m \in \mathcal{C}$. The set of all conceptual formulas $\mathcal{CF}$ comprises *subclass formulas* $C \subseteq D$, *equality formulas* $C = D$ and *disjointness formulas* $D_1 \parallel \ldots \parallel D_m$.

Definition 2.4 (Semantics of conceptual formulas) An interpretation $\mathcal{J} = (\mathcal{U}, J)$ of $\mathcal{C}$ is extended to an interpretation of $\mathcal{CF}$ as follows:

1. $\mathcal{J} \models C \subseteq D$, iff $J(C) \subseteq J(D)$,
2. $\mathcal{J} \models C = D$, iff $\mathcal{J} \models C \subseteq D$ and $\mathcal{J} \models D \subseteq C$,
3. $\mathcal{J} \models D_1 \parallel \ldots \parallel D_m$, iff $\mathcal{J} \models D_i \cap D_j = \emptyset$ for all $i, j \in [1 : m]$ with $i < j$.

The notions of models, satisfiability and logical consequence are defined as usual.

Definition 2.5 (Conceptual knowledge-base) A *conceptual knowledge-base* consists of a set of conceptual formulas.

Example 2.6 The conceptual knowledge that penguins are birds, that birds are winged animals, that penguins do not fly and that all flying objects are winged can be expressed by the following conceptual knowledge-base over the alphabet $\mathcal{A} = \{\emptyset, \mathcal{U}, \mathsf{penguins}, \mathsf{birds}, \mathsf{animals}, \mathsf{flying}, \mathsf{winged}\}$:

$$\begin{aligned} \mathcal{T}_1 = \{&\mathsf{penguins} \subseteq \mathsf{birds},\ \mathsf{birds} \subseteq \mathsf{winged\ animals},\ \mathsf{penguins} \parallel \mathsf{flying}, \\ &\mathsf{flying} \subseteq \mathsf{winged}\}\ . \end{aligned}$$

3 Concept Lattices

A conceptual knowledge-base can completely be represented by a concept lattice.

Definition 3.1 (Concept lattice) Let $\mathcal{T}$ be conceptual knowledge-base. Let the equivalence relation $\sim_{\mathcal{T}}$ on $\mathcal{C}$ be defined by $C_1 \sim_{\mathcal{T}} C_2 :\Leftrightarrow \mathcal{T} \models C_1 = C_2$. Let $\mathcal{C}_{\mathcal{T}} := \mathcal{C}_{/\sim_{\mathcal{T}}}$ and $C_{\mathcal{T}} := [C]_{\sim_{\mathcal{T}}}$ for all $C \in \mathcal{C}$. The elements of $\mathcal{C}_{\mathcal{T}}$ are called *concepts*. $\emptyset_{\mathcal{T}}$ is called *empty concept*, $\mathcal{U}_{\mathcal{T}}$ is called *universal concept*. $(B_1)_{\mathcal{T}}, \ldots, (B_k)_{\mathcal{T}}$ are called *basic concepts*.

The partial order $\subseteq$ on $\mathcal{C}$ is canonically extended to a partial order $\subseteq_{\mathcal{T}}$ on $\mathcal{C}_{\mathcal{T}}$ by $A_{\mathcal{T}} \subseteq_{\mathcal{T}} B_{\mathcal{T}} :\Leftrightarrow (AB)_{\mathcal{T}} = A_{\mathcal{T}}$. The finite partially ordered set $(\mathcal{C}_{\mathcal{T}}, \subseteq_{\mathcal{T}})$ is a complete lattice. It is called the *concept lattice* for $\mathcal{T}$.

Example 3.2 Figure 1 shows the Hasse-diagram of $(\mathcal{C}_{\mathcal{T}_1}, \subseteq_{\mathcal{T}_1})$. We use the constants of the alphabet $\mathcal{A}$ as a label to the corresponding nodes in the Hasse-diagrams. This way we can get a representative for all non-labeled nodes: each node contains the conjunction of all constants attached to upper nodes.

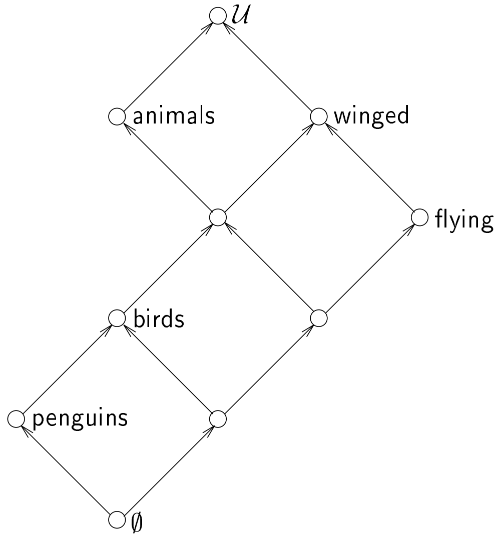

Fig. 1. The concept lattice for Ex. 2.6

4 Representation of Concept Lattices

Concepts can be internally represented by subsets of $\mathcal{A}$. The operation $\vee$ can be realized by intersection, the operation $\wedge$ can be performed by the use of a hull-operator on $2^{\mathcal{A}}$ based on a set of subclass formulas equivalent to the given conceptual knowledge-base. The time complexity of the hull-operator on $2^{\mathcal{A}}$ and hence of the operation $\wedge$ is linear in the number of subclass formulas ([1], [9]).

Definition 4.1 Let $\mathcal{A}(\cdot) : \mathcal{C} \to 2^{\mathcal{A}}$ be defined by $\mathcal{A}(C_1 C_2 \ldots C_l) := \{C_1, C_2, \ldots, C_l\}$ for all $C_1 C_2 \ldots C_l \in \mathcal{C}$ with $C_i \in \mathcal{A}$ for all $i \in [1:l]$, $l \geq 1$. Let $\mathcal{I} \subseteq \mathcal{CF}$ be a set of subclass formulas. Let the unary function $\mathcal{F}_{\mathcal{I}}$ on $2^{\mathcal{A}}$ be defined by:

$$\mathcal{F}_{\mathcal{I}}(A) := A \cup \bigcup \{\mathcal{A}(C) \mid P \subseteq C \in \mathcal{I},\, \mathcal{A}(P) \subseteq A\} \ .$$

Let the unary function $\mathcal{F}^{\star}_{\mathcal{I}}$ on $2^{\mathcal{A}}$ be defined by $\mathcal{F}^{\star}_{\mathcal{I}}(A) := \mathcal{F}^{n}_{\mathcal{I}}(A)$ with $n \in [0:k]$ such that $\mathcal{F}^{n}_{\mathcal{I}}(A) = \mathcal{F}^{n+1}_{\mathcal{I}}(A)$ and n minimal.

Theorem 4.2 *a) The partially ordered set $(\mathcal{F}^{\star}_{\mathcal{I}}(2^{\mathcal{A}}) \backslash \{\emptyset\}, \subseteq)$ is a complete lattice with $A \wedge B = \mathcal{F}^{\star}_{\mathcal{I}}(A \cup B)$ and $A \vee B = A \cap B$ for all $A, B \in \mathcal{F}^{\star}_{\mathcal{I}}(2^{\mathcal{A}}) \backslash \{\emptyset\}$.*
b) $(\mathcal{F}^{\star}_{\mathcal{I}}(2^{\mathcal{A}}) \backslash \{\emptyset\}, \subseteq)$ is anti-isomorphic to $(\mathcal{C}_{\mathcal{I}}, \subseteq_{\mathcal{I}})$.

Proof. For the proof of a) refer to [3] or [4], for the proof of b) refer to [8].

Example 4.3 An equivalent set of subclass formulas for the conceptual knowledge-base $\mathcal{T}_1$ is given by:

$$\mathcal{I} = \{\mathsf{penguins} \subseteq \mathsf{birds},\ \mathsf{birds} \subseteq \mathsf{winged\ animals},\ \mathsf{flying\ penguins} \subseteq \emptyset,$$
$$\mathsf{flying} \subseteq \mathsf{winged},\ \mathsf{winged} \subseteq \mathcal{U},\ \mathsf{animals} \subseteq \mathcal{U},\ \emptyset \subseteq \mathsf{flying\ penguins}\} \ .$$

5 Syntax and Semantics of Uncertain Knowledge

The notations for the probabilistic model are adapted from [5] and [11].

Definition 5.1 (Syntax of probabilistic formulas) Let A, B, $C \in \mathcal{C}$ and $x_1, x_2, y_1, y_2 \in [0,1]$ with $x_1 \leq x_2$ and $y_1 \leq y_2$. The set of all probabilistic formulas $\mathcal{PF}$ comprises *positive probability statements* pos(A), *uncertain rules* $A \xrightarrow{x_1,x_2} B$ and *correlation rules* $A \xleftrightarrow[y_1,y_2]{x_1,x_2} B$.

Definition 5.2 (Semantics of probabilistic formulas) An interpretation $\mathcal{J} = (\mathcal{U}, J, P)$ of $\mathcal{PF}$ consists of an interpretation $(\mathcal{U}, J)$ of $\mathcal{C}$ and a probability measure $P : \mathcal{S} \longrightarrow [0,1]$ for the measure space $(\mathcal{U}, \mathcal{S})$.[1]

1. $\mathcal{J} \models pos(A)$, iff $P(J(A)) > 0$,
2. $\mathcal{J} \models A \xrightarrow{x_1,x_2} B$, iff $P(J(A)) = 0$ or $x_1 \leq P(J(B)|J(A)) \leq x_2$,[2]
3. $\mathcal{J} \models A \xleftrightarrow[y_1,y_2]{x_1,x_2} B$, iff $\mathcal{J} \models A \xrightarrow{x_1,x_2} B$ and $\mathcal{J} \models B \xrightarrow{y_1,y_2} A$.

The notions of models, satisfiability and logical consequence are defined as usual. The uncertain rule $A \xrightarrow{z_1,z_2} B$ is a *precise* logical consequence of $\mathcal{R} \subseteq \mathcal{CF} \cup \mathcal{PF}$ (denoted $\mathcal{R} \models_{precise} A \xrightarrow{z_1,z_2} B$), iff z_1 is the greatest lower bound and z_2 is the least upper bound for $\mathrm{P}(B|A)$ that follows from $\mathcal{R}$ and the laws of probability.

Definition 5.3 (Probabilistic knowledge-base) A conceptual knowledge-base, extended by a non-empty set of probabilistic formulas, is called a *probabilistic knowledge-base*. Let V be an infinite set of variables. For $A, B \in \mathcal{C} \cup V$, $x_1, x_2 \in [0,1] \cup V$, the expression $?A \xrightarrow{x_1,x_2} B$ is called a *probabilistic rule query.*

Example 5.4 Gaining the uncertain knowledge that between 10 and 20 per cent of all animals are winged, that between 70 and 80 per cent of all winged animals fly, that between 40 and 50 per cent of all winged animals are flying birds, that between 50 and 60 per cent of all winged animals are birds, that between 10 and 20 per cent of all birds are penguins, that there are penguins and flying birds, we can add the set of uncertain rules

$$\mathcal{P} = \{\mathsf{animals} \xrightarrow{0.1,0.2} \mathsf{winged},\ \mathsf{winged\ animals} \xrightarrow{0.7,0.8} \mathsf{flying},$$
$$\mathsf{winged\ animals} \xrightarrow{0.4,0.5} \mathsf{flying\ birds},\ \mathsf{winged\ animals} \xrightarrow{0.5,0.6} \mathsf{birds},$$
$$\mathsf{birds} \xrightarrow{0.1,0.2} \mathsf{penguins},\ \mathrm{pos}(\mathsf{penguins}),\ \mathrm{pos}(\mathsf{flying\ birds})\}$$

[1] $\mathcal{S}$ denotes the smallest σ-algebra over $J(\mathcal{C})$.
[2] $P(J(B)|J(A))$ is the conditional probability of $J(B)$ under $J(A)$.

to our conceptual knowledge-base $\mathcal{T}_1$.

6 Uncertain Knowledge in Concept Lattices

Uncertain rules and correlation rules are extended from $\mathcal{C}$ to $\mathcal{C}_\mathcal{T}$ by:

$$A_\mathcal{T} \xrightarrow{x_1,x_2} (AB)_\mathcal{T} :\Leftrightarrow A \xrightarrow{x_1,x_2} B \ .$$

Positive probability statements are extended from $\mathcal{C}$ to $\mathcal{C}_\mathcal{T}$ by:

$$\mathrm{pos}(A_\mathcal{T}) :\Leftrightarrow \mathrm{pos}(A) \ .$$

Example 6.1 Figure 2 shows the Hasse-diagram of $(\mathcal{C}_{\mathcal{T}_1}, \subseteq_{\mathcal{T}_1})$ enriched by uncertain rules and positive probabilities. Uncertain rules are graphically represented by dashed, downward arcs and positive probabilities by black-filled nodes.

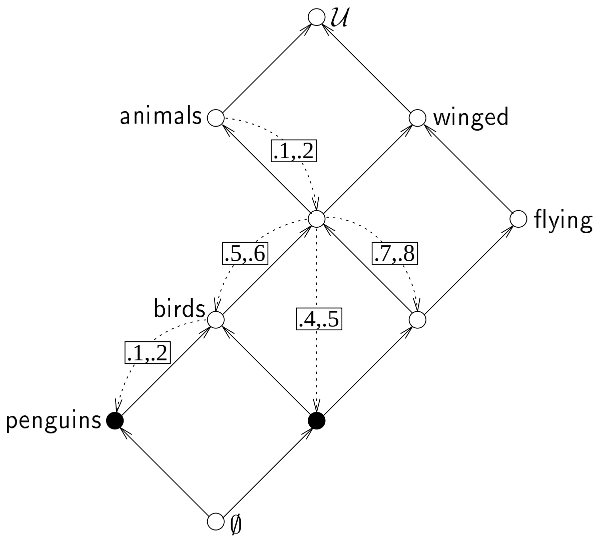

Fig. 2. Concept lattice with uncertain knowledge

7 Uncertain Deduction in Concept Lattices

Uncertain deduction is now performed on the complete lattice $(\mathcal{C}_\mathcal{T}, \subseteq_\mathcal{T})$, reducing the search space enormously.

Example 7.1 Table 7.1 gives a comparison of the number of elements in $\mathcal{C}_\mathcal{T}$, the number of uncertain and correlation rules over $\mathcal{C}_\mathcal{T}$ occuring w.r.t. $\mathcal{T} = \emptyset$ and $\mathcal{T} = \mathcal{T}_1$.

Table 1. Search space reduction by conceptual knowledge

	$\mathcal{T}=\emptyset$	$\mathcal{T}=\mathcal{T}_1$
number of elements in $\mathcal{C}_\mathcal{T}$	33	10
number of uncertain rules over $\mathcal{C}_\mathcal{T}$	276	47
number of correlation rules over $\mathcal{C}_\mathcal{T}$	1089	100

Conceptual knowledge can be exploited for increased precision of uncertain conclusions. This can be illustrated by referring to the chaining of correlations with the inference pattern $\{A \longleftrightarrow B, B \longleftrightarrow C\} \vdash A \longleftrightarrow C$. This syllogism is widely explored in the literature (see e.g. [2], [6], [11]), but all these considerations are based on the assumption that the involved events A, B and C are *elementary*. If A, B and C are implicitly or explicitly conceptually related, the inference rules presented in the literature for this pattern are not precise anymore. This is shown by the following theorem that provides the precise bounds under conceptual relationships between A, B and C. Note that conceptual relationships are directly encoded in the representation of concepts. Hence we get just one inference rule in which the conceptual relationships are evaluated on the representation of the concepts A, B and C.

Theorem 7.2 *Let* $\mathcal{T} \subseteq \mathcal{CF}$ *and* $\mathcal{P} = \{A \xleftrightarrow[v_1,v_2]{u_1,u_2} B, \mathrm{pos}(A), B \xleftrightarrow[y_1,y_2]{x_1,x_2} C\}$ *with* $u_1, v_1, x_1, y_1 > 0$, $u_1 = 1 \Leftrightarrow \mathcal{T} \models A \subseteq B$, $v_1 = 1 \Leftrightarrow \mathcal{T} \models B \subseteq A$, $x_1 = 1 \Leftrightarrow \mathcal{T} \models B \subseteq C$, $y_1 = 1 \Leftrightarrow \mathcal{T} \models C \subseteq B$, $\mathcal{T} \not\models A \subseteq C$, $\mathcal{T} \not\models AC = \emptyset$ *and* $\mathcal{T} \cup \mathcal{P}$ *satisfiable. It holds* $\mathcal{T} \cup \mathcal{P} \models_{precise} A \xrightarrow{z_1,z_2} C$ *with*

$$z_1 = \begin{cases} \frac{u_1}{y_2} & \textit{if } \mathcal{T} \models C \subseteq A,\ \mathcal{T} \models AB \subseteq C \\ \frac{u_1 x_1}{v_2 y_2} & \textit{if } \mathcal{T} \models C \subseteq A,\ \mathcal{T} \not\models AB \subseteq C \\ u_1 & \textit{if } \mathcal{T} \not\models C \subseteq A,\ \mathcal{T} \models AB \subseteq C \\ \frac{u_1 x_1}{v_2} & \textit{if } \mathcal{T} \not\models C \subseteq A,\ \mathcal{T} \not\models AB \subseteq C,\ \mathcal{T} \models BC \subseteq A \\ \max(0, u_1 - \frac{u_1}{v_1} + \frac{u_1 x_1}{v_1}) & \textit{otherwise} \end{cases}$$

$$z_2 = \begin{cases} \min(1-u_1, \frac{u_2(1-y_1)\min(x_2,1-v_1)}{v_1 y_1}, & \\ \quad \frac{(1-y_1)\min(x_2,1-v_1)}{y_1 v_1 + (1-y_1)\min(x_2,1-v_1)}) & \textit{if } \mathcal{T} \models ABC = \emptyset \\ \min(u_2, \frac{u_2 x_2}{v_1}) & \textit{if } \mathcal{T} \not\models ABC = \emptyset,\ \mathcal{T} \models AC \subseteq B \\ \min(1, \frac{u_2 x_2}{v_1 y_1}, 1 - u_1 + \frac{u_1 x_2}{v_1}, \frac{u_2}{y_1}, & \\ \quad \frac{x_2}{v_1 y_1 + (1-y_1) x_2}) & \textit{if } \mathcal{T} \not\models ABC = \emptyset,\ \mathcal{T} \not\models AC \subseteq B, \\ & \quad \mathcal{T} \models BC \subseteq A \\ \min(1, \frac{u_2 x_2}{v_1 y_1}, 1 - u_1 + \frac{u_1 x_2}{v_1}, & \\ \quad u_2 - \frac{u_2 x_2}{v_1} + \frac{u_2 x_2}{v_1 y_1}, \frac{x_2}{y_1 v_1 + (1-y_1) x_2}) & \textit{otherwise} \end{cases}$$

Proof. The proof of Theorem 7.2 is given in the appendix.

Example 7.3 Let $\mathcal{A} = \{\emptyset, \mathcal{O}, \mathsf{A}, \mathsf{B}, \mathsf{C}\}$,

$$\mathcal{T}_1 = \emptyset,\ \mathcal{P}_1 = \{\mathsf{A} \overset{0.4,0.4}{\underset{0.1,0.3}{\longleftrightarrow}} \mathsf{B},\ \mathrm{pos}(\mathsf{A}),\ \mathsf{B} \overset{0.3,1}{\underset{0.8,0.8}{\longleftrightarrow}} \mathsf{AC}\}\ ,$$
$$\mathcal{T}_2 = \{\mathsf{BC} \subseteq \mathsf{A}\},\ \mathcal{P}_2 = \{\mathsf{A} \overset{0.4,0.4}{\underset{0.1,0.3}{\longleftrightarrow}} \mathsf{B},\ \mathrm{pos}(\mathsf{A}),\ \mathsf{B} \overset{0.3,1}{\underset{0.8,0.8}{\longleftrightarrow}} \mathsf{C}\}\ .$$

We get $\mathcal{T}_1 \cup \mathcal{P}_1 \models_{precise} \mathsf{A} \xrightarrow{0.5,0.5} \mathsf{C}$ and $\mathcal{T}_2 \cup \mathcal{P}_2 \models_{precise} \mathsf{A} \xrightarrow{0.4,0.5} \mathsf{C}$. These results can be computed with Theorem 7.2, since it holds $\mathcal{T}_1 \models \mathsf{AC} \subseteq \mathsf{A}$ and $\mathcal{T}_2 \models \mathsf{BC} \subseteq \mathsf{A}$. In contrast, all the inference rules for the chaining of correlation, which do not incorporate implicit or explicit conceptual knowledge between the correlated events, just enable us to compute the lower bound 0 and the upper bound 1.

8 Related Work

In [6] Heinsohn integrates uncertainty into terminological languages. Our approach is similar, but much more tractable for the use in object-oriented databases and frame-based knowledge representations, since we do not consider disjunction and negation. Furthermore we show how to reduce the search space by conceptual knowledge and how to integrate conceptual knowledge into the deduction of uncertain knowledge for an increased precision. The problem of uncertain deduction can also be solved by linear optimization. In [13] von Rimscha shows how to use certain knowledge to eliminate some variables of a linear optimization problem.

9 Summary and Outlook

We extended concept lattices by uncertainty in the form of conditional probabilities. We illustrated that uncertain reasoning within the hierarchical structure of concept lattices can be performed efficiently and makes uncertain conclusions more precise. Concept lattices seem to be a promising tractable subclass of the general problem of uncertain reasoning with conditional probabilities. There is more work to be done to find out in which special cases we can draw globally precise conclusions. Another topic for future work is the integration of objects in uncertain reasoning within concept lattices.

10 Acknowledgements

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11 Appendix

Definition 11.1 a) Let $\mathcal{L}(\mathcal{A})$ denote the Lindenbaumalgebra over $\mathcal{A}$. Let $\mathcal{M}$ denote the set of all probability measures over $\mathcal{L}(\mathcal{A})$.

b) Let $\mathrm{P} \in \mathcal{M}$ be a probability measure and $\mathcal{T}$ be a conceptual knowledge-base. P *satisfies* $\mathcal{T}$ ($\mathrm{P} \models \mathcal{T}$), iff $\mathrm{P}(A) = 0$ for all $A \in \mathcal{L}(\mathcal{A})$ with $\mathcal{T} \models A = \emptyset$.

Lemma 11.2 *Let $\mathcal{T} \subseteq \mathcal{CF}$ and $\mathcal{P} = \{A \overset{u}{\underset{v}{\longleftrightarrow}} B, \mathrm{pos}(A), B \overset{x}{\underset{y}{\longleftrightarrow}} C\}$ with $u, v, x, y > 0$. The probabilistic knowledge-base $\mathcal{T} \cup \mathcal{P}$ is satisfiable, iff it holds*

$$
\begin{array}{ll}
\mathcal{T} \models ABC = \emptyset & \Longrightarrow x + v \leq 1 \\
\mathcal{T} \models A \subseteq B & \Longrightarrow u = 1 \\
\mathcal{T} \models B \subseteq A & \Longrightarrow v = 1 \\
\mathcal{T} \models B \subseteq C & \Longrightarrow x = 1 \\
\mathcal{T} \models C \subseteq B & \Longrightarrow y = 1 \\
\mathcal{T} \models A \subseteq C & \Longrightarrow v \leq x,\ vy(1-u) \leq ux(1-y) \\
\mathcal{T} \models C \subseteq A & \Longrightarrow x \leq v,\ ux(1-y) \leq vy(1-u) \\
\mathcal{T} \models AB \subseteq C & \Longrightarrow v \leq x \\
\mathcal{T} \models BC \subseteq A & \Longrightarrow x \leq v
\end{array}
$$

Proof: Let $u, v, x, y > 0$ with $u, v, x, y \leq 1$. For all probability measures $\mathrm{P} \in \mathcal{M}$ with $\mathrm{P}(A) > 0$ it holds

$$\frac{\mathrm{P}(AB)}{\mathrm{P}(A)} = u, \quad \frac{\mathrm{P}(AB)}{\mathrm{P}(B)} = v, \quad \frac{\mathrm{P}(BC)}{\mathrm{P}(B)} = x, \quad \frac{\mathrm{P}(BC)}{\mathrm{P}(C)} = y \ , \tag{1}$$

iff

$$\frac{\mathrm{P}(A)}{\mathrm{P}(A)} = 1, \quad \frac{\mathrm{P}(AB)}{\mathrm{P}(A)} = u, \quad \frac{\mathrm{P}(B)}{\mathrm{P}(A)} = \frac{u}{v}, \quad \frac{\mathrm{P}(BC)}{\mathrm{P}(A)} = \frac{ux}{v}, \quad \frac{\mathrm{P}(C)}{\mathrm{P}(A)} = \frac{ux}{vy} \ , \tag{2}$$

iff

$$\frac{\mathrm{P}(A\overline{B})}{\mathrm{P}(A)} = 1 - u, \quad \frac{\mathrm{P}(\overline{B}C)}{\mathrm{P}(A)} = \frac{ux}{vy} - \frac{ux}{v} \tag{3}$$

and

$$\frac{\mathrm{P}(AB)}{\mathrm{P}(A)} = u, \quad \frac{\mathrm{P}(BC)}{\mathrm{P}(A)} = \frac{ux}{v}, \quad \frac{\mathrm{P}(\overline{A}B)}{\mathrm{P}(A)} = \frac{u}{v} - u \ , \tag{4}$$

iff

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \cdot \left(\frac{\mathrm{P}(A\overline{B}\,\overline{C})}{\mathrm{P}(A)}, \frac{\mathrm{P}(\overline{A}\,\overline{B}C)}{\mathrm{P}(A)}, \frac{\mathrm{P}(A\overline{B}C)}{\mathrm{P}(A)} \right)^{\mathrm{T}} = \begin{pmatrix} 1-u \\ \frac{ux}{vy} - \frac{ux}{v} \end{pmatrix} \tag{5}$$

and

$$\begin{pmatrix} 0\,1\,0\,1 \\ 0\,0\,1\,1 \\ 1\,0\,1\,0 \end{pmatrix} \cdot \left(\frac{\mathrm{P}(\overline{A}B\overline{C})}{\mathrm{P}(A)}, \frac{\mathrm{P}(AB\overline{C})}{\mathrm{P}(A)}, \frac{\mathrm{P}(\overline{A}BC)}{\mathrm{P}(A)}, \frac{\mathrm{P}(ABC)}{\mathrm{P}(A)} \right)^{\mathrm{T}} = \begin{pmatrix} u \\ \frac{ux}{v} \\ \frac{u}{v} - u \end{pmatrix} \quad . \tag{6}$$

Hence there exists a probability measure $\mathrm{P} \in \mathcal{M}$ with $\mathrm{P}(A) > 0$, $\mathrm{P} \models \mathcal{T}$ and (1), iff the following two systems of equations are solvable:

$$\begin{aligned} &\begin{pmatrix} 1\,0\,1 \\ 0\,1\,1 \end{pmatrix} \cdot (s_1, s_2, s_3)^{\mathrm{T}} = \begin{pmatrix} 1-u \\ \frac{ux}{vy} - \frac{ux}{v} \end{pmatrix} \\ &s_1, s_2, s_3 \geq 0 \\ &s_1 = 0, \text{ if } \mathcal{T} \models A \subseteq B \text{ or } \mathcal{T} \models A \subseteq C \\ &s_2 = 0, \text{ if } \mathcal{T} \models C \subseteq B \text{ or } \mathcal{T} \models C \subseteq A \\ &s_3 = 0, \text{ if } \mathcal{T} \models AC \subseteq B \end{aligned} \tag{7}$$

and

$$\begin{aligned} &\begin{pmatrix} 0\,1\,0\,1 \\ 0\,0\,1\,1 \\ 1\,0\,1\,0 \end{pmatrix} \cdot (s_4, s_5, s_6, s_7)^{\mathrm{T}} = \begin{pmatrix} u \\ \frac{ux}{v} \\ \frac{u}{v} - u \end{pmatrix} \\ &s_4, s_5, s_6, s_7 \geq 0 \\ &s_4 = 0, \text{ if } \mathcal{T} \models B \subseteq A \text{ or } \mathcal{T} \models B \subseteq C \\ &s_5 = 0, \text{ if } \mathcal{T} \models AB \subseteq C \\ &s_6 = 0, \text{ if } \mathcal{T} \models BC \subseteq A \\ &s_7 = 0, \text{ if } \mathcal{T} \models ABC = \emptyset \end{aligned} \tag{8}$$

The probability measure $\mathrm{P} \in \mathcal{M}$ can be computed from a solution s_i $(i \in [1:7])$ by:

$$\mathrm{P}(A) = (\textstyle\sum_{i=1}^{7} s_i)^{-1}, \ \mathrm{P}(\overline{A}\,\overline{B}\,\overline{C}) = 0$$

$$\begin{aligned} (\mathrm{P}(A\overline{B}\,\overline{C}), \mathrm{P}(\overline{A}\,\overline{B}C), \mathrm{P}(A\overline{B}C)) &= (s_1, s_2, s_3) \cdot \mathrm{P}(A) \\ (\mathrm{P}(\overline{A}B\overline{C}), \mathrm{P}(AB\overline{C}), \mathrm{P}(\overline{A}BC), \mathrm{P}(ABC)) &= (s_4, s_5, s_6, s_7) \cdot \mathrm{P}(A) \end{aligned}$$

Now we can easily prove Lemma 11.2:
"$\Longrightarrow$": The equations in (7) are not solvable for:

$$\begin{aligned} &\mathcal{T} \models A \subseteq B \text{ and } u < 1 \\ &\mathcal{T} \models A \subseteq C \text{ and } vy(1-u) > ux(1-y) \\ &\mathcal{T} \models C \subseteq B \text{ and } y < 1 \\ &\mathcal{T} \models C \subseteq A \text{ and } ux(1-y) > vy(1-u) \end{aligned}$$

The equations in (8) are not solvable for:

$$\begin{aligned}
&\mathcal{T} \models B \subseteq A \text{ and } v < 1\\
&\mathcal{T} \models B \subseteq C \text{ and } x < 1\\
&\mathcal{T} \models AB \subseteq C \text{ and } v > x\\
&\mathcal{T} \models BC \subseteq A \text{ and } x > v\\
&\mathcal{T} \models ABC = \emptyset \text{ and } x + v > 1
\end{aligned}$$

"$\Longleftarrow$": We assume that the conditions stated in Lemma 11.2 hold. A solution (s_1, s_2, s_3) of (7) is given by:

$$\begin{cases}
(0,0,0) & \text{if } \mathcal{T} \models A \subseteq B,\ \mathcal{T} \models C \subseteq B\\
(0,0,1-u) & \text{if } \mathcal{T} \models A = C,\ \mathcal{T} \not\models AC \subseteq B\\
(0, \frac{x}{vy} - \frac{x}{v}, 0) & \text{if } \mathcal{T} \models A \subseteq B,\ \mathcal{T} \not\models C \subseteq B,\ \mathcal{T} \not\models C \subseteq A\\
(1-u,0,0) & \text{if } \mathcal{T} \models C \subseteq B,\ \mathcal{T} \not\models A \subseteq B,\ \mathcal{T} \not\models A \subseteq C\\
(0, \frac{ux}{vy} - \frac{ux}{v} + u - 1, 1-u) & \text{if } \mathcal{T} \models A \subseteq C,\ \mathcal{T} \not\models C \subseteq B,\ \mathcal{T} \not\models C \subseteq A,\\
& \quad \mathcal{T} \not\models AC \subseteq B\\
(1 - u + \frac{ux}{v} - \frac{ux}{vy}, 0, \frac{ux}{vy} - \frac{ux}{v}) & \text{if } \mathcal{T} \models C \subseteq A,\ \mathcal{T} \not\models A \subseteq B,\ \mathcal{T} \not\models A \subseteq C,\\
& \quad \mathcal{T} \not\models AC \subseteq B\\
(1-u, \frac{ux}{vy} - \frac{ux}{v}, 0) & \text{if } \mathcal{T} \models AC \subseteq B,\ \mathcal{T} \not\models C \subseteq B,\ \mathcal{T} \not\models C \subseteq A,\\
& \quad \mathcal{T} \not\models A \subseteq B,\ \mathcal{T} \not\models A \subseteq C\\
(1-u, \frac{ux}{vy} - \frac{ux}{v}, 0) & \text{if } \mathcal{T} \not\models AC \subseteq B,\ \mathcal{T} \not\models C \subseteq B,\ \mathcal{T} \not\models C \subseteq A,\\
& \quad \mathcal{T} \not\models A \subseteq B,\ \mathcal{T} \not\models A \subseteq C
\end{cases}$$

A solution (s_4, s_5, s_6, s_7) of (8) is given by:

$$\begin{cases}
(0,0,0,u) & \text{if } \mathcal{T} \models B \subseteq A,\ \mathcal{T} \models B \subseteq C,\ \mathcal{T} \not\models ABC = \emptyset\\
(0, u - ux, 0, ux) & \text{if } \mathcal{T} \models B \subseteq A,\ \mathcal{T} \not\models AB \subseteq C,\ \mathcal{T} \not\models ABC = \emptyset\\
(0, 0, \frac{u}{v} - u, u) & \text{if } \mathcal{T} \models B \subseteq C,\ \mathcal{T} \not\models BC \subseteq A,\ \mathcal{T} \not\models ABC = \emptyset\\
(\frac{u}{v} - u, 0, 0, u) & \text{if } \mathcal{T} \models AB \subseteq C,\ \mathcal{T} \models BC \subseteq A,\ \mathcal{T} \not\models B \subseteq C,\\
& \quad \mathcal{T} \not\models B \subseteq A,\ \mathcal{T} \not\models ABC = \emptyset\\
(\frac{u}{v} - \frac{ux}{v}, 0, \frac{ux}{v} - u, u) & \text{if } \mathcal{T} \models AB \subseteq C,\ \mathcal{T} \not\models BC \subseteq A,\ \mathcal{T} \not\models B \subseteq C,\\
& \quad \mathcal{T} \not\models ABC = \emptyset\\
(\frac{u}{v} - u, u - \frac{ux}{v}, 0, \frac{ux}{v}) & \text{if } \mathcal{T} \models BC \subseteq A,\ \mathcal{T} \not\models AB \subseteq C,\ \mathcal{T} \not\models B \subseteq A,\\
& \quad \mathcal{T} \not\models ABC = \emptyset\\
(\frac{u}{v} - u - \frac{ux}{v}, u, \frac{ux}{v}, 0) & \text{if } \mathcal{T} \models ABC = \emptyset,\ \mathcal{T} \not\models BC \subseteq A,\ \mathcal{T} \not\models AB \subseteq C\\
(\frac{u}{v} - \frac{ux}{v}, 0, \frac{ux}{v} - u, u) & \text{if } \mathcal{T} \not\models ABC = \emptyset,\ \mathcal{T} \not\models BC \subseteq A,\ \mathcal{T} \not\models AB \subseteq C,\\
& \quad v \leq x\\
(\frac{u}{v} - u, u - \frac{ux}{v}, 0, \frac{ux}{v}) & \text{if } \mathcal{T} \not\models ABC = \emptyset,\ \mathcal{T} \not\models BC \subseteq A,\ \mathcal{T} \not\models AB \subseteq C,\\
& \quad x \leq v
\end{cases}$$

Proof for Theorem 7.2: It holds:

$$\begin{aligned} z_1 = \min\{&\min\{\mathrm{P}(C|A) \mid \mathrm{P} \in \mathcal{M}, \mathrm{P}(A) > 0, \mathrm{P} \models \mathcal{T}, \mathrm{P}(B|A) = u, \mathrm{P}(A|B) = v, \\ &\mathrm{P}(C|B) = x, \mathrm{P}(C|B) = y\} \mid (u,v,x,y) \in [u_1,u_2] \times [v_1,v_2] \times [x_1,x_2] \\ &\times [y_1,y_2], \mathcal{T} \cup \{A \xleftrightarrow[v]{u} B,\ \mathrm{pos}(A),\ B \xleftrightarrow[y]{x} C\} \text{ satisfiable}\} \end{aligned}$$

$$\begin{aligned} z_2 = \max\{&\max\{\mathrm{P}(C|A) \mid \mathrm{P} \in \mathcal{M}, \mathrm{P}(A) > 0, \mathrm{P} \models \mathcal{T}, \mathrm{P}(B|A) = u, \mathrm{P}(A|B) = v, \\ &\mathrm{P}(C|B) = x, \mathrm{P}(C|B) = y\} \mid (u,v,x,y) \in [u_1,u_2] \times [v_1,v_2] \times [x_1,x_2] \\ &\times [y_1,y_2], \mathcal{T} \cup \{A \xleftrightarrow[v]{u} B,\ \mathrm{pos}(A),\ B \xleftrightarrow[y]{x} C\} \text{ satisfiable}\} \end{aligned}$$

According to the proof of Lemma 11.2, we can maximize/minimize $\mathrm{P}(C|A)$ under $\mathrm{P}(A) > 0$, $\mathrm{P} \models \mathcal{T}$ and (1) by maximizing/minimizing $\mathrm{P}(\overline{B}C|A)$ under $\mathrm{P}(A) > 0$, $\mathrm{P} \models \mathcal{T}$ and (5), and by maximizing/minimizing $\mathrm{P}(BC|A)$ under $\mathrm{P}(A) > 0$, $\mathrm{P} \models \mathcal{T}$ and (6). Or equivalently by maximizing/minimizing s_3 under (7) and by maximizing/minimizing s_7 under (8). We get the following precise lower bounds for $\mathrm{P}(\overline{B}C|A)$ (assuming $\mathcal{T} \not\models A \subseteq C$):

$$\begin{cases} \frac{ux}{vy} - \frac{ux}{v} & \text{if } \mathcal{T} \models C \subseteq A \\ 0 & \text{if } \mathcal{T} \not\models C \subseteq A \end{cases}$$

We get the following precise lower bounds for $\mathrm{P}(BC|A)$:

$$\begin{cases} 0 & \text{if } \mathcal{T} \models ABC = \emptyset,\ \mathcal{T} \not\models BC \subseteq A,\ \mathcal{T} \not\models AB \subseteq C \\ \frac{ux}{v} & \text{if } \mathcal{T} \not\models ABC = \emptyset,\ \mathcal{T} \models BC \subseteq A,\ \mathcal{T} \models AB \subseteq C \\ u & \text{if } \mathcal{T} \not\models ABC = \emptyset,\ \mathcal{T} \not\models BC \subseteq A,\ \mathcal{T} \models AB \subseteq C \\ \frac{ux}{v} & \text{if } \mathcal{T} \not\models ABC = \emptyset,\ \mathcal{T} \models BC \subseteq A,\ \mathcal{T} \not\models AB \subseteq C \\ \max(0, u - \frac{u}{v} + \frac{ux}{v}) & \text{if } \mathcal{T} \not\models ABC = \emptyset,\ \mathcal{T} \not\models BC \subseteq A,\ \mathcal{T} \not\models AB \subseteq C \end{cases}$$

Hence we get the following precise lower bounds for $\mathrm{P}(C|A)$:

$$\begin{cases} \frac{ux}{vy} & \text{if } \mathcal{T} \models C \subseteq A,\ \mathcal{T} \not\models ABC = \emptyset,\ \mathcal{T} \models BC \subseteq A, \\ & \quad \mathcal{T} \models AB \subseteq C \\ \frac{ux}{vy} & \text{if } \mathcal{T} \models C \subseteq A,\ \mathcal{T} \not\models ABC = \emptyset,\ \mathcal{T} \models BC \subseteq A, \\ & \quad \mathcal{T} \not\models AB \subseteq C \\ 0 & \text{if } \mathcal{T} \not\models C \subseteq A,\ \mathcal{T} \models ABC = \emptyset,\ \mathcal{T} \not\models BC \subseteq A, \\ & \quad \mathcal{T} \not\models AB \subseteq C \\ \frac{ux}{v} & \text{if } \mathcal{T} \not\models C \subseteq A,\ \mathcal{T} \not\models ABC = \emptyset,\ \mathcal{T} \models BC \subseteq A, \\ & \quad \mathcal{T} \models AB \subseteq C \\ u & \text{if } \mathcal{T} \not\models C \subseteq A,\ \mathcal{T} \not\models ABC = \emptyset,\ \mathcal{T} \not\models BC \subseteq A, \\ & \quad \mathcal{T} \models AB \subseteq C \\ \frac{ux}{v} & \text{if } \mathcal{T} \not\models C \subseteq A,\ \mathcal{T} \not\models ABC = \emptyset,\ \mathcal{T} \models BC \subseteq A, \\ & \quad \mathcal{T} \not\models AB \subseteq C \\ \max(0, u - \frac{u}{v} + \frac{ux}{v}) & \text{otherwise} \end{cases}$$

We get z_1 by nonlinear optimization under the constraints provided by Lemma 11.2 (assuming $u_1 = 1 \Leftrightarrow \mathcal{T} \models A \subseteq B$, $v_1 = 1 \Leftrightarrow \mathcal{T} \models B \subseteq A$, $x_1 = 1 \Leftrightarrow \mathcal{T} \models B \subseteq C$ and $y_1 = 1 \Leftrightarrow \mathcal{T} \models C \subseteq B$):

$$\begin{cases} \frac{u_1}{y_2} & \text{if } \mathcal{T} \models C \subseteq A,\, \mathcal{T} \not\models ABC = \emptyset,\, \mathcal{T} \models BC \subseteq A, \\ & \quad \mathcal{T} \models AB \subseteq C \\ \frac{u_1 x_1}{v_2 y_2} & \text{if } \mathcal{T} \models C \subseteq A,\, \mathcal{T} \not\models ABC = \emptyset,\, \mathcal{T} \models BC \subseteq A, \\ & \quad \mathcal{T} \not\models AB \subseteq C \\ 0 & \text{if } \mathcal{T} \not\models C \subseteq A,\, \mathcal{T} \models ABC = \emptyset,\, \mathcal{T} \not\models BC \subseteq A, \\ & \quad \mathcal{T} \not\models AB \subseteq C \\ u_1 & \text{if } \mathcal{T} \not\models C \subseteq A,\, \mathcal{T} \not\models ABC = \emptyset,\, \mathcal{T} \models BC \subseteq A, \\ & \quad \mathcal{T} \models AB \subseteq C \\ u_1 & \text{if } \mathcal{T} \not\models C \subseteq A,\, \mathcal{T} \not\models ABC = \emptyset,\, \mathcal{T} \not\models BC \subseteq A, \\ & \quad \mathcal{T} \models AB \subseteq C \\ \frac{u_1 x_1}{v_2} & \text{if } \mathcal{T} \not\models C \subseteq A,\, \mathcal{T} \not\models ABC = \emptyset,\, \mathcal{T} \models BC \subseteq A, \\ & \quad \mathcal{T} \not\models AB \subseteq C \\ \max(0, u_1 - \frac{u_1}{v_1} + \frac{u_1 x_1}{v_1}) & \text{otherwise} \end{cases}$$

The claim follows from the observation that the satisfiability of $\mathcal{T} \cup \mathcal{P}$ under $\mathcal{T} \models ABC = \emptyset$ entails $x_1 + v_1 \leq 1$.

We get the following precise upper bounds for $\mathrm{P}(\overline{B}C|A)$ (assuming $\mathcal{T} \not\models A \subseteq C$):

$$\begin{cases} \frac{ux}{vy} - \frac{ux}{v} & \text{if } \mathcal{T} \models C \subseteq A,\, \mathcal{T} \not\models AC \subseteq B \\ 0 & \text{if } \mathcal{T} \models C \subseteq A,\, \mathcal{T} \models AC \subseteq B \\ 0 & \text{if } \mathcal{T} \not\models C \subseteq A,\, \mathcal{T} \models AC \subseteq B \\ \min(1-u, \frac{ux}{vy} - \frac{ux}{v}) & \text{if } \mathcal{T} \not\models C \subseteq A,\, \mathcal{T} \not\models AC \subseteq B \end{cases}$$

We get the following precise upper bounds for $\mathrm{P}(BC|A)$:

$$\begin{cases} 0 & \text{if } \mathcal{T} \models ABC = \emptyset,\, \mathcal{T} \not\models BC \subseteq A,\, \mathcal{T} \not\models AB \subseteq C \\ \frac{ux}{v} & \text{if } \mathcal{T} \not\models ABC = \emptyset,\, \mathcal{T} \models BC \subseteq A,\, \mathcal{T} \models AB \subseteq C \\ u & \text{if } \mathcal{T} \not\models ABC = \emptyset,\, \mathcal{T} \not\models BC \subseteq A,\, \mathcal{T} \models AB \subseteq C \\ \frac{ux}{v} & \text{if } \mathcal{T} \not\models ABC = \emptyset,\, \mathcal{T} \models BC \subseteq A,\, \mathcal{T} \not\models AB \subseteq C \\ \min(u, \frac{ux}{v}) & \text{if } \mathcal{T} \not\models ABC = \emptyset,\, \mathcal{T} \not\models BC \subseteq A,\, \mathcal{T} \not\models AB \subseteq C \end{cases}$$

For determining the precise upper bounds for $\mathrm{P}(C|A)$, we must distinguish the following cases (assuming $\mathcal{T} \not\models AC = \emptyset$):

(a) $\mathcal{T} \models C \subseteq A$, $\mathcal{T} \not\models AC \subseteq B$, $\mathcal{T} \not\models ABC = \emptyset$, $\mathcal{T} \models BC \subseteq A$, $\mathcal{T} \models AB \subseteq C$

(b) $\mathcal{T} \models C \subseteq A$, $\mathcal{T} \not\models AC \subseteq B$, $\mathcal{T} \not\models ABC = \emptyset$, $\mathcal{T} \models BC \subseteq A$, $\mathcal{T} \not\models AB \subseteq C$

(c) $\mathcal{T} \models C \subseteq A$, $\mathcal{T} \models AC \subseteq B$, $\mathcal{T} \not\models ABC = \emptyset$, $\mathcal{T} \models BC \subseteq A$, $\mathcal{T} \models AB \subseteq C$

(d) $\mathcal{T} \models C \subseteq A$, $\mathcal{T} \models AC \subseteq B$, $\mathcal{T} \not\models ABC = \emptyset$, $\mathcal{T} \models BC \subseteq A$, $\mathcal{T} \not\models AB \subseteq C$

(e)	$\mathcal{T} \not\models C \subseteq A$, $\mathcal{T} \models AC \subseteq B$, $\mathcal{T} \not\models ABC = \emptyset$, $\mathcal{T} \models BC \subseteq A$, $\mathcal{T} \models AB \subseteq C$
(f)	$\mathcal{T} \not\models C \subseteq A$, $\mathcal{T} \models AC \subseteq B$, $\mathcal{T} \not\models ABC = \emptyset$, $\mathcal{T} \not\models BC \subseteq A$, $\mathcal{T} \models AB \subseteq C$
(g)	$\mathcal{T} \not\models C \subseteq A$, $\mathcal{T} \models AC \subseteq B$, $\mathcal{T} \not\models ABC = \emptyset$, $\mathcal{T} \models BC \subseteq A$, $\mathcal{T} \not\models AB \subseteq C$
(h)	$\mathcal{T} \not\models C \subseteq A$, $\mathcal{T} \models AC \subseteq B$, $\mathcal{T} \not\models ABC = \emptyset$, $\mathcal{T} \not\models BC \subseteq A$, $\mathcal{T} \not\models AB \subseteq C$
(i)	$\mathcal{T} \not\models C \subseteq A$, $\mathcal{T} \not\models AC \subseteq B$, $\mathcal{T} \models ABC = \emptyset$, $\mathcal{T} \not\models BC \subseteq A$, $\mathcal{T} \not\models AB \subseteq C$
(j)	$\mathcal{T} \not\models C \subseteq A$, $\mathcal{T} \not\models AC \subseteq B$, $\mathcal{T} \not\models ABC = \emptyset$, $\mathcal{T} \models BC \subseteq A$, $\mathcal{T} \models AB \subseteq C$
(k)	$\mathcal{T} \not\models C \subseteq A$, $\mathcal{T} \not\models AC \subseteq B$, $\mathcal{T} \not\models ABC = \emptyset$, $\mathcal{T} \not\models BC \subseteq A$, $\mathcal{T} \models AB \subseteq C$
(l)	$\mathcal{T} \not\models C \subseteq A$, $\mathcal{T} \not\models AC \subseteq B$, $\mathcal{T} \not\models ABC = \emptyset$, $\mathcal{T} \models BC \subseteq A$, $\mathcal{T} \not\models AB \subseteq C$
(m)	$\mathcal{T} \not\models C \subseteq A$, $\mathcal{T} \not\models AC \subseteq B$, $\mathcal{T} \not\models ABC = \emptyset$, $\mathcal{T} \not\models BC \subseteq A$, $\mathcal{T} \not\models AB \subseteq C$

We get the following precise upper bounds for $\mathrm{P}(C|A)$:

(a)	$\frac{ux}{vy}$	(h)	$\min(u, \frac{ux}{v})$
(b)	$\frac{ux}{vy}$	(i)	$\min(1-u, \frac{ux}{vy} - \frac{ux}{v})$
(c)	$\frac{ux}{v}$	(j)	$\min(1-u+\frac{ux}{v}, \frac{ux}{vy})$
(d)	$\frac{ux}{v}$	(k)	$\min(1, \frac{ux}{vy} - \frac{ux}{v} + u)$
(e)	$\frac{ux}{v}$	(l)	$\min(1-u+\frac{ux}{v}, \frac{ux}{vy})$
(f)	u	(m)	$\min(1, 1-u+\frac{ux}{v}, \frac{ux}{vy} - \frac{ux}{v} + u, \frac{ux}{vy})$
(g)	$\frac{ux}{v}$		

We get z_2 by nonlinear optimization under the constraints provided by Lemma 11.2 (assuming $u_1 = 1 \Leftrightarrow \mathcal{T} \models A \subseteq B$, $v_1 = 1 \Leftrightarrow \mathcal{T} \models B \subseteq A$, $x_1 = 1 \Leftrightarrow \mathcal{T} \models B \subseteq C$ and $y_1 = 1 \Leftrightarrow \mathcal{T} \models C \subseteq B$):

(a), (j)	$\min(1, \frac{u_2}{y_1})$
(b), (l)	$\min(1, \frac{u_2 x_2}{v_1 y_1}, 1-u_1+\frac{u_1 x_2}{v_1}, \frac{u_2}{y_1}, \frac{x_2}{v_1 y_1 + (1-y_1) x_2})$
(c), (e), (f)	u_2
(d), (g), (h)	$\min(u_2, \frac{u_2 x_2}{v_1})$
(i)	$\min(1-u_1, \frac{u_2(1-y_1)\min(x_2, 1-v_1)}{v_1 y_1}, \frac{(1-y_1)\min(x_2,1-v_1)}{y_1 v_1 + (1-y_1)\min(x_2,1-v_1)})$
(k)	$\min(1, u_2 - \frac{u_2 x_2}{v_1} + \frac{u_2 x_2}{v_1 y_1})$
(m)	$\min(1, \frac{u_2 x_2}{v_1 y_1}, 1-u_1+\frac{u_1 x_2}{v_1}, u_2 - \frac{u_2 x_2}{v_1} + \frac{u_2 x_2}{v_1 y_1}, \frac{x_2}{y_1 v_1 + (1-y_1) x_2})$

The claim follows from the observation that the satisfiability of $\mathcal{T} \cup \mathcal{P}$ under $\mathcal{T} \models AB \subseteq C$ entails $v_1 \leq x_2$.

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