A Fully Internalized Sequent Calculus for Hybrid Categorial Logics

Magdalena Ortiz

Institute of Information Systems, Vienna University of Technology.

Favoritenstraße 9-11, Vienna, Austria

magdalen@logic.at

Abstract. In this paper, a sequent calculus for a hybrid categorial type logic (HCTL) is obtained following Seligman's internalization strategy (Seligman 2001). With this strategy, a sequent calculus for a hybrid language can be developed starting from a first order sequent calculus. Seligman exemplifies his strategy developing a calculus for hybrid modal logics, but rises the question of whether the strategy works in general. We investigate this issue in the particular case of categorial type logics. As categorial type logics lack Boolean structure, a successful hybridization would indicate that the strategy is indeed rather general and does not depend on the availability of Boolean connectives. In this paper, we will see that this is the case, and moreover, since we can easily arrive to an intuitionistic version of the calculus, we will see that a classical base is not needed either.

1 Introduction

The basic categorial type language (CTL) is called NL and it was introduced by Lambek in (Lambek 1961) to model linguistic composition. It is a non-associative, non-commutative modal calculus of pure residuation containing just three operators: $\bullet$, $\rightarrow$ and $\leftarrow$. NL was extended by Moortgat to NL($\Diamond$) by the addition of a pair of residuated unary modalities $\vec{\Diamond}$ and $\overleftarrow{\Box}$ (Moortgat 1996).

NL($\Diamond$) is interesting as an example of a purely modal language: all its operators are modalities defined in terms of accessibility relations and it contains no Boolean structure. Despite its simplicity, it is considered to be suitable for reasoning about different linguistic phenomena and useful to describe complex relational structures (Moortgat 1996). But like all standard modal languages, NL($\Diamond$) lacks the means to directly refer to particular elements in a model. *Hybrid* modal languages overcome this shortcoming by providing a way to assign names to points and mechanisms to access them. In many cases, hybridization of a modal language does not only provide additional expressive power, but it also has a positive impact on its proof and model theory. Hybridization for the basic modal logic (e.g., the $\mathcal{H}(@, \downarrow)$ language) has been by now extensively investigated (Areces, Blackburn, and Marx 2001; HyLo 2006).

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In (2001) Seligman shows how, using a strategy called *internalization*, a sequent calculus for hybrid modal logics can be defined starting from a calculus for first order logic (FO). Seligman describes the internalization strategy by applying it to the basic modal logic, but mentions that it is unclear whether it will also work for other non-standard modal languages. In this sense, CTL is an specially interesting test case, as the absence of Boolean operators makes it hard to tell at first sight how far internalization would go.

In this paper, we apply Seligman's strategy to $\mathsf{NL}(\Diamond)$. The purpose of this paper is two-fold. First, we want to test Seligman's method in a fairly non-classical modal logic. Second, we aim to obtain a fully internalized sequent calculus for hybrid categorial type logics.

2 Hybrid Categorial Logic

We start by introducing the syntax and semantics of the hybrid categorial logic HCTL. Defining HCTL is simple: take the syntactic and semantic clauses of both $\mathsf{NL}(\Diamond)$ and $\mathcal{H}(@, \downarrow)$ and pool them together.

Formally, let $\mathcal{S}$ be a signature with variables $X = \{x, x_1, x_2, \dots\}$, propositional symbols $P = \{p, p_1, p_2, \dots\}$, a binary relational symbol $r_\Diamond$ and a ternary one $r_\bullet$. The atomic formulas of HCTL are the individual variables in X and the symbols in P. Complex formulas are defined as follows: if x is a variable in X, and φ, ψ are formulas, then $\overrightarrow{\Diamond}\varphi$, $\overleftarrow{\Box}\varphi$, $\mathsf{E}\varphi$, $x{:}\varphi$, $\downarrow_x\varphi$, $\varphi \bullet \psi$, $\varphi \leftarrow \psi$ and $\varphi \rightarrow \psi$ are formulas. Note that the Boolean operators $\neg, \wedge, \vee, \supset, \equiv$ do not occur in formulas of HCTL. Moreover, all connectives must be defined as primitive, and none of them can be expressed in terms of the others.

Like in any modal language, the semantics of HCTL is defined in terms of relational (Kripke) structures over the signature $\mathcal{S}$.

Definition 2.1. A model $\mathcal{M} = \langle \mathbf{M}, \cdot^{\mathcal{M}} \rangle$ for HCTL is a relational structure with a non-empty domain $\mathbf{M}$, and an interpretation function $\cdot^{\mathcal{M}}$ such that $(r_\Diamond)^{\mathcal{M}} \subseteq \mathbf{M}^2$, $(r_\bullet)^{\mathcal{M}} \subseteq \mathbf{M}^3$ and $p^{\mathcal{M}} \subseteq \mathbf{M}$ for each $p \in P$. Given a model $\mathcal{M}$, elements a, b, c in $\mathbf{M}$ and an assignment $g : X \to \mathbf{M}$ for the variables, the *satisfiability relation* is inductively defined as follows:

$\mathcal{M}, a, g \models x$	iff	$g(x) = a$ where $x \in X$
$\mathcal{M}, a, g \models p$	iff	$a \in p^{\mathcal{M}}$ where $p \in P$
$\mathcal{M}, a, g \models x{:}\varphi$	iff	$\mathcal{M}, g(x), g \models \varphi$
$\mathcal{M}, a, g \models \downarrow_x\varphi$	iff	$\mathcal{M}, a, g[x/a] \models \varphi$
$\mathcal{M}, a, g \models \mathsf{E}\varphi$	iff	$\exists b \in \mathbf{M}$, $\mathcal{M}, b, g \models \varphi$
$\mathcal{M}, a, g \models \overrightarrow{\Diamond}\varphi$	iff	$\exists b \in \mathbf{M}$, $\langle a, b \rangle \in (r_\Diamond)^{\mathcal{M}}$ and $\mathcal{M}, b, g \models \varphi$
$\mathcal{M}, b, g \models \overleftarrow{\Box}\varphi$	iff	$\forall a \in \mathbf{M}$, if $\langle a, b \rangle \in (r_\Diamond)^{\mathcal{M}}$ then $\mathcal{M}, a, g \models \varphi$
$\mathcal{M}, a, g \models \varphi \bullet \psi$	iff	$\exists b, c \in \mathbf{M}$, $\langle a, b, c \rangle \in (r_\bullet)^{\mathcal{M}}$, $\mathcal{M}, b, g \models \varphi$ and $\mathcal{M}, c, g \models \psi$
$\mathcal{M}, b, g \models \varphi \leftarrow \psi$	iff	$\forall a, c \in \mathbf{M}$, if $\langle a, b, c \rangle \in (r_\bullet)^{\mathcal{M}}$ and $\mathcal{M}, c, g \models \psi$ then $\mathcal{M}, a, g \models \varphi$
$\mathcal{M}, c, g \models \varphi \rightarrow \psi$	iff	$\forall a, b \in \mathbf{M}$, if $\langle a, b, c \rangle \in (r_\bullet)^{\mathcal{M}}$ and $\mathcal{M}, b, g \models \varphi$ then $\mathcal{M}, a, g \models \psi$

Note that the connectives $\overleftarrow{\Box}$ and $\overrightarrow{\Diamond}$ form a residuated pair (they are not duals as the $\Diamond$ and $\Box$ in the basic modal logic). Analogously, $\bullet$, $\leftarrow$ and $\rightarrow$ form a residuated triple.

$\Gamma \vdash \Delta$ is a *sequent* of HCTL if Γ and Δ are lists of HCTL formulas. $\mathcal{M}$ is a *model* of such a sequent if every $\mathcal{M}, a, g$ satisfying all formulas in Γ satisfies some formula in Δ. If every model is a model of $\Gamma \vdash \Delta$, then $\Gamma \vdash \Delta$ is a *valid* sequent.

3 Proving HCTL theorems in FO

As can be seen from the definitions in the previous section, every HCTL model $\mathcal{M}$ is a first order model. Moreover, the semantics of HCTL can be defined in terms of FO, and hence we can use first order machinery to prove valid sequents of HCTL. Following Seligman, we will now cast HCTL formulas as 'contextualized' first order formulas and express their semantics in FO.

We introduce the first order language obtained by extending the signature $\mathcal{S}$ with a new $n{+}1$-ary predicate symbol p_φ for each HCTL formula φ with n-free variables. Intuitively, the formula $p_\varphi x$ means that the point at which x is interpreted has the property of satisfying the formula φ, i.e., that φ is true at point $g(x)$, for g some assignment, according to the semantics of HCTL. Since the semantic conditions in Definition 2.1 can be easily expressed in FO, we can use this extended language to give an FO characterization of HCTL. For example, consider the formula $\overrightarrow{\Diamond}\varphi$, for φ without free variables. As defined above, this formula will be true at some point a of the model $\mathcal{M}$ iff $\mathcal{M}$ satisfies the formula $\exists y(r_\Diamond xy \wedge p_\varphi y)$, where x is a variable interpreted as a.

In order to make things more readable, we will write $x{:}\varphi$ instead of the formula $p_\varphi x$. The free variables of $x{:}\varphi$ are x and the free variables of φ. Note that in the previous section we have already introduced : as a connective of HCTL. Now we are introducing : with a different meaning, not as a logical connective (neither of FO nor of HCTL), but as a 'metasymbol', i.e., as a way to abbreviate certain fist order formulas. The reason for this double meaning will be made clear in Section 4, when we take the internalization step.

The semantics of HCTL can be expressed in FO through the following set Θ of formulas:

(θ_1)	$\forall z \forall x(z{:}x \equiv x = z)$	(θ_6)	$\forall z(z{:}\overrightarrow{\Diamond}\varphi \equiv \exists x(r_\Diamond zx \wedge x{:}\varphi))$
(θ_2)	$\forall z(z{:}p \equiv pz)$	(θ_7)	$\forall z(z{:}\overleftarrow{\Box}\varphi \equiv \forall x(r_\Diamond xz \supset x{:}\varphi))$
(θ_3)	$\forall z \forall x(z{:}(x{:}\varphi) \equiv x{:}\varphi)$	(θ_8)	$\forall z(z{:}\varphi \bullet \psi \equiv \exists x \exists y(r_\bullet zxy \wedge x{:}\varphi \wedge y{:}\psi))$
(θ_4)	$\forall z(z{:}{\downarrow_i}\varphi \equiv z{:}\varphi[z/i])$	(θ_9)	$\forall z(z{:}\varphi \leftarrow \psi \equiv \forall x \forall y(r_\bullet xzy \wedge y{:}\psi \supset x{:}\varphi))$
(θ_5)	$\forall z(z{:}\mathsf{E}\varphi \equiv \exists x(x{:}\varphi))$	(θ_{10})	$\forall z(z{:}\varphi \rightarrow \psi \equiv \forall x \forall y(r_\bullet xyz \wedge y{:}\varphi \supset x{:}\psi))$

Note that each θ_i is simply the formalization of a semantic conditions in Definition 2.1. The second occurrence of the symbol : in θ_3 corresponds to the connective in the syntax of HCTL, while : stands for the 'metasymbol' in FO in all remaining occurrences in Θ.

The set of sentences Θ will be our background theory. We say that a sequent $\Gamma \vdash \Delta$ (where Γ and Δ are now lists of FO formulas) is Θ-valid if every (first order) model of Θ is a model of $\Gamma \vdash \Delta$. For any list Γ of formulas of HCTL, $u{:}\Gamma$, denotes the list of (first order) formulas $u{:}\varphi$ for each $\varphi \in \Gamma$. Since Θ captures the semantics of HCTL, the following lemma holds:

$$\frac{}{\varphi, \Gamma \vdash \Delta, \varphi} \text{[Ax]} \qquad \frac{\Gamma \vdash \Delta}{\Gamma' \vdash \Delta'} \text{[S]}^{1}$$

^{1} if the lists Γ and Δ contain the same *set* of formulas as Γ′ and Δ′ resp.

Structural rules

$$\frac{\Gamma \vdash \Delta, \varphi}{\neg\varphi, \Gamma \vdash \Delta} [\neg\text{L}] \qquad \frac{\varphi\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg\varphi} [\neg\text{R}]$$

$$\frac{\varphi, \Gamma \vdash \Delta \quad \psi, \Gamma \vdash \Delta}{\varphi \vee \psi, \Gamma \vdash \Delta} [\vee\text{L}] \qquad \frac{\Gamma \vdash \Delta, \varphi, \psi}{\Gamma \vdash \Delta, \varphi \vee \psi} [\vee\text{R}]$$

$$\frac{\varphi[u/x], \Gamma \vdash \Delta}{\exists x\varphi, \Gamma \vdash \Delta} [\exists\text{L}]\ u \text{ new} \qquad \frac{\Gamma \vdash \Delta, \varphi[u/x]}{\Gamma \vdash \Delta, \exists x\varphi} [\exists\text{R}]$$

$$\frac{u{:}v, \Gamma[w/u] \vdash \Delta[w/u]}{u{:}v, \Gamma[w/v] \vdash \Delta[w/v]} [=\text{L}_1] \qquad \frac{u{:}v, \Gamma[w/v] \vdash \Delta[w/v]}{u{:}v, \Gamma[w/u] \vdash \Delta[w/u]} [=\text{L}_2] \qquad \frac{}{\Gamma \vdash \Delta, u = u} [=\text{R}]$$

Logical Rules

Figure 1.1: Sequent Calculus **S** for FO

Lemma 3.1. Let $\Gamma \vdash \Delta$ be a sequent of HCTL and u an arbitrary variable not occurring in $\Gamma \vdash \Delta$. The sequent $\Gamma \vdash \Delta$ is valid iff $u{:}\Gamma \vdash u{:}\Delta$ is Θ-valid.

Clearly, we can use the proof machinery of FO to prove valid HCTL sequents. For example, any standard sound and complete Gentzen sequent calculus for FO with equality will do. We pick (arbitrarily) the sequent calculus introduced in (Seligman 2001), given in Figure 1.1, and call it **S**.

S has two structural rules, [Ax] and [S], plus the standard (context sharing) rules for the logical connectives and the *Barwise* rules for equality. The connectives $\wedge$, $\supset$, $\equiv$ and $\forall$ are expressed as usual, and their respective rules can be derived.

Finally, the standard cut rule

$$\frac{\Gamma \vdash \Delta, \varphi \quad \varphi, \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'} [Cut]$$

is admissible in **S**, ensuring the subformula property[1]. This makes **S** a modular calculus: if we take any fragment F of FO closed under subformulas, the calculus obtained as a restriction of **S** to the operators mentioned in F will be sound and complete for validity of F sequents. As we will see in Section 4, choosing a modular calculus will be crucial for our aims.

[1] I.e., if the sequent $\Gamma \vdash \Delta$ is a theorem of **S** then it has a proof in which every formula as a subformula in Γ or Δ.

Let's state clearly how we would use **S** to prove Θ-validity. Since predicate logic is compact and **S** is sound and complete, it follows that a sequent $\Gamma \vdash \Delta$ of our first order language is Θ-valid iff there are formulas $\theta_1, \ldots, \theta_n$ in Θ such that $\theta_1, \ldots, \theta_n, \Gamma \vdash \Delta$ is a theorem of **S**. Thus we easily obtain a calculus for Θ-validity by adding the sentences of Θ as axioms to **S**, i.e., for each sentence θ_i in Θ we add to **S** the Θ-axiom:

$$\frac{}{\Gamma \vdash \Delta, \theta_i}\ [\mathrm{A}\theta_i].$$

Let us call such calculus **S**+Θ-Axioms. Then the following holds:

Theorem 3.1. A sequent of FO is Θ-valid iff it can be proved in **S**+Θ-Axioms+[Cut].

Proof. As the new rules are Θ-valid and the rules of **S** preserve validity and Θ-validity, all theorems of **S**+Θ-Axioms+[Cut] are Θ-valid. For the converse, observe that since $\vdash \theta_i$ is a Θ-axiom for each θ_i, a proof of $\theta_1, \ldots, \theta_n, \Gamma \vdash \Delta$ in **S** can be transformed into a proof of $\Gamma \vdash \Delta$ in **S**+Θ-Axioms+[Cut] with n applications of the cut rule. □

Note the use of the cut rule in the proof of Theorem 3.1. Indeed, the cut rule cannot be directly eliminated from **S**+Θ-Axioms+[Cut] and the subformula property is lost. In particular, the problematic cuts are those involving Θ-axioms. This is a standard problem when adding axioms to systems where cut was eliminated.

Lemma 3.2. A proof of **S**+Θ-Axioms+[Cut] can be transformed into a proof in which all cuts have the form

$$\frac{\dfrac{}{\Gamma \vdash \Delta, \theta}\ [\mathrm{A}\theta_i] \qquad \begin{array}{c}\pi \\ \vdots \\ \theta, \Gamma' \vdash \Delta'\end{array}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\ [\mathrm{Cut}]$$

where θ is a formula of Θ and the principal formula of the last rule of π.

4 Regaining cut elimination

By inspecting Lemma 3.2, we can realize that instead of the Θ-axioms, which generate unwanted cuts, we can use rules that do the same job. Each Θ-axiom is required to prove some particular kind of formula, and in each case we can define a rule that has the same effect. As an example, we show how we can prove $u{:}\overleftarrow{\Box}\varphi$ on the left of a sequent using the Θ-axiom θ_7.

$$\dfrac{\dfrac{}{\vdash \theta_7}\ [\mathrm{A}\theta_7] \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{u{:}\overleftarrow{\Box}\varphi, \Gamma \vdash \Delta, r_\diamond vu \qquad v{:}\varphi, u{:}\overleftarrow{\Box}\varphi, \Gamma \vdash \Delta}{r_\diamond vu \supset v{:}\varphi, u{:}\overleftarrow{\Box}\varphi, \Gamma \vdash \Delta}\ [\supset \mathrm{L}]}{u{:}\overleftarrow{\Box}\varphi, r_\diamond vu \supset v{:}\varphi, u{:}\overleftarrow{\Box}\varphi, \Gamma \vdash \Delta}\ [\mathrm{S}]}{u{:}\overleftarrow{\Box}\varphi, \forall x(r_\diamond xu \supset x{:}\varphi), u{:}\overleftarrow{\Box}\varphi, \Gamma \vdash \Delta}\ [\forall\mathrm{L}] \qquad \dfrac{}{u{:}\overleftarrow{\Box}\varphi, \Gamma \vdash \Delta, u{:}\overleftarrow{\Box}\varphi, \forall x(r_\diamond xu \supset x{:}\varphi)}\ [\mathrm{Ax}]}{u{:}\overleftarrow{\Box}\varphi \equiv \forall x(r_\diamond xu \supset x{:}\varphi), u{:}\overleftarrow{\Box}\varphi, \Gamma \vdash \Delta}\ [\equiv \mathrm{L}]}{\forall z(z{:}\overleftarrow{\Box}\varphi \equiv \forall x(r_\diamond xz \supset x{:}\varphi)), u{:}\overleftarrow{\Box}\varphi, \Gamma \vdash \Delta}\ [\forall\mathrm{L}]}{u{:}\overleftarrow{\Box}\varphi, \Gamma \vdash \Delta}\ [\mathrm{Cut}]$$

In this way, from a sequent of the form $\Gamma \vdash \Delta, r_\diamond vu$ and a sequent of the form $v{:}\varphi, \Gamma \vdash \Delta$, we can prove a sequent of the form $u{:}\overleftarrow{\Box}\varphi, \Gamma \vdash \Delta$. However, we could obtain the same result by using the following rules:

$$\frac{\Gamma \vdash \Delta, r_\diamond uv}{\Gamma \vdash \Delta, u{:}\overrightarrow{\diamond}v}\ [r_\diamond \mathrm{R}] \qquad\qquad \frac{\Gamma \vdash \Delta, v{:}\overrightarrow{\diamond}u \quad v{:}\varphi, \Gamma \vdash \Delta}{u{:}\overleftarrow{\Box}\varphi, \Gamma \vdash \Delta}\ [:\overleftarrow{\Box}\mathrm{L}].$$

Rule $[r_\diamond \mathrm{R}]$ does not have the subformula property, but we can improve it. It is easy to verify that when proving Θ-validity of a sequent with our new rules, the application of rule $[r_\diamond \mathrm{R}]$ commutes with all other rules and thus it can be pushed up to the leaves of the proof-tree. At the leaves, its application can be replaced by axioms. So, instead of the rule $[r_\diamond \mathrm{R}]$ we can use the axiom $[\mathrm{Ax}_\diamond \mathrm{R}]$:

$$\cfrac{\cfrac{}{r_\diamond uv, \Gamma \vdash \Delta, r_\diamond uv}\ [\mathrm{Ax}]}{r_\diamond uv, \Gamma \vdash \Delta, u{:}\overrightarrow{\diamond}v}\ [r_\diamond \mathrm{R}] \quad \rightsquigarrow \quad \cfrac{}{r_\diamond uv, \Gamma \vdash \Delta, u{:}\overrightarrow{\diamond}v}\ [\mathrm{Ax}_\diamond \mathrm{R}].$$

We can apply this method systematically and obtain a set of rules that replace all Θ-axioms, given in Figure 1.2. They are divided into two groups, which we call **IA** (Interface Axioms) and **LLR** (Labeled Logical Rules).

Since the rules in **LLR**+**IA** do the same work as Θ-axioms, we can replace **S**+Θ-Axioms+[Cut] by **S**+**LLR**+**IA**+[Cut].

Lemma 4.1. A sequent of FO is a theorem of **S**+Θ-Axioms+[Cut] iff it is theorem of **S**+**LLR**+**IA**+[Cut].

Moreover, [Cut] can now be eliminated.

Lemma 4.2. [Cut] is admissible in **S**+**LLR**+**IA**.

Proof. We have to prove two things:

1) Cuts can be pushed up though the new rules when the cut formula is not principal. For most of the rules this is trivial, since they do not alter any of the non-principal formulas. Only the $[:\mathrm{L}_1]$ and $[:\mathrm{L}_2]$ are different, but for them the proof is exactly as for the similar rules for handling equality in **S**.

2) Cut rank can be decreased when the cut formula is the main formula of one of the new rules. The proof of this part must be done rule by rule, but it is also quite straightforward. As an example, we show how cut rank is decreased when the cut formula is the main formula of the $[:\overleftarrow{\Box}\mathrm{R}]$ and $[:\overleftarrow{\Box}\mathrm{L}]$ rules. The proof:

$$\cfrac{\cfrac{\begin{matrix}\pi_1\\ \vdots\\ v{:}\overrightarrow{\diamond}u, \Gamma, \vdash \Delta, v{:}\varphi\end{matrix}}{\Gamma, \vdash \Delta, u{:}\overleftarrow{\Box}\varphi}\ [:\overleftarrow{\Box}\mathrm{R}] \quad \cfrac{\begin{matrix}\pi_2{}'\\ \vdots\\ \Gamma' \vdash \Delta', w{:}\overrightarrow{\diamond}u\end{matrix} \quad \begin{matrix}\pi_2{}''\\ \vdots\\ w{:}\varphi, \Gamma', \vdash \Delta'\end{matrix}}{u{:}\overleftarrow{\Box}\varphi, \Gamma' \vdash \Delta'}\ [:\overleftarrow{\Box}\mathrm{L}]}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\ [\mathrm{Cut}]$$

$$\frac{u{:}v, \Gamma[w/u] \vdash \Delta[w/u]}{u{:}v, \Gamma[w/v] \vdash \Delta[w/v]}\ [{:}\mathrm{L}_1] \qquad \frac{u{:}v, \Gamma[w/v] \vdash \Delta[w/v]}{u{:}v, \Gamma[w/u] \vdash \Delta[w/u]}\ [{:}\mathrm{L}_2] \qquad \frac{}{\Gamma \vdash \Delta, u{:}u}\ [{:}\mathrm{R}]$$

$$\frac{u{:}\overrightarrow{\Diamond} v, v{:}\varphi, \Gamma \vdash \Delta}{u{:}\overrightarrow{\Diamond}\varphi, \Gamma \vdash \Delta}\ [{:}\overrightarrow{\Diamond}\mathrm{L}]\ v \text{ new} \qquad \frac{\Gamma \vdash \Delta, u{:}\overrightarrow{\Diamond} v \quad \Gamma \vdash \Delta, v{:}\varphi}{\Gamma \vdash \Delta, u{:}\overrightarrow{\Diamond}\varphi}\ [{:}\overrightarrow{\Diamond}\mathrm{R}]$$

$$\frac{\Gamma \vdash \Delta, v{:}\overrightarrow{\Diamond} u \quad v{:}\varphi, \Gamma \vdash \Delta}{u{:}\overleftarrow{\Box}\varphi, \Gamma \vdash \Delta}\ [{:}\overleftarrow{\Box}\mathrm{L}] \qquad \frac{v{:}\overrightarrow{\Diamond} u, \Gamma \vdash \Delta, v{:}\varphi}{\Gamma \vdash \Delta, u{:}\overleftarrow{\Box}\varphi}\ [{:}\overleftarrow{\Box}\mathrm{R}]\ v \text{ new}$$

$$\frac{u{:}v \bullet w, v{:}\varphi, w{:}\psi, \Gamma \vdash \Delta}{u{:}\varphi \bullet \psi, \Gamma \vdash \Delta}\ [{:}\bullet\mathrm{L}]\ v, w \text{ new} \qquad \frac{\Gamma \vdash \Delta, u{:}v \bullet w \quad \Gamma \vdash \Delta, v{:}\varphi \quad \Gamma \vdash \Delta, w{:}\psi}{\Gamma \vdash \Delta, u{:}\varphi \bullet \psi}\ [{:}\bullet\mathrm{R}]$$

$$\frac{v{:}\varphi, \Gamma \vdash \Delta \quad \Gamma \vdash \Delta, v{:}u \bullet w \quad \Gamma \vdash \Delta, w{:}\psi}{u{:}\varphi \leftarrow \psi, \Gamma \vdash \Delta}\ [{:}\leftarrow\mathrm{L}] \qquad \frac{v{:}u \bullet w, w{:}\psi, \Gamma \vdash \Delta, v{:}\varphi}{\Gamma \vdash \Delta, u{:}\varphi \leftarrow \psi}\ [{:}\leftarrow\mathrm{R}]\ v, w \text{ new}$$

$$\frac{v{:}\psi, \Gamma \vdash \Delta \quad \Gamma \vdash \Delta, v{:}w \bullet u \quad \Gamma \vdash \Delta, w{:}\varphi}{u{:}\varphi \rightarrow \psi, \Gamma \vdash \Delta}\ [{:}\rightarrow\mathrm{L}] \qquad \frac{v{:}w \bullet u, w{:}\varphi, \Gamma \vdash \Delta, v{:}\psi}{\Gamma \vdash \Delta, u{:}\varphi \rightarrow \psi}\ [{:}\rightarrow\mathrm{R}]\ v, w \text{ new}$$

$$\frac{v{:}\varphi, \Gamma \vdash \Delta}{u{:}v{:}\varphi, \Gamma \vdash \Delta}\ [{:}{:}\mathrm{L}] \qquad \frac{\Gamma \vdash \Delta, v{:}\varphi}{\Gamma \vdash \Delta, u{:}v{:}\varphi}\ [{:}{:}\mathrm{R}]$$

$$\frac{u{:}\varphi[u/v], \Gamma \vdash \Delta}{u{:}\downarrow_v\varphi, \mathit{\Gamma} \vdash \mathit{\Delta}}\ [{:}\downarrow\mathrm{L}] \qquad \frac{\Gamma \vdash \Delta, u{:}\varphi[u/v]}{\Gamma \vdash \Delta, u{:}\downarrow_v\varphi}\ [{:}\downarrow\mathrm{R}]$$

$$\frac{v{:}\varphi, \Gamma \vdash \Delta}{u{:}\mathsf{E}\varphi, \Gamma \vdash \Delta}\ [{:}\mathsf{E}\mathrm{L}]\ v \text{ new} \qquad \frac{\Gamma \vdash \Delta, v{:}\varphi}{\Gamma \vdash \Delta, u{:}\mathsf{E}\varphi}\ [{:}\mathsf{E}\mathrm{R}]$$

Labeled logical rules LLR

$$\frac{}{u{:}p, \Gamma \vdash \Delta, pu}\ [\mathrm{Ax}_p\mathrm{L}] \qquad \frac{}{pu, \Gamma \vdash \Delta, u{:}p}\ [\mathrm{Ax}_p\mathrm{R}]$$

$$\frac{}{u{:}\overrightarrow{\Diamond} v, \Gamma \vdash \Delta, r_\Diamond uv}\ [\mathrm{Ax}_\Diamond\mathrm{L}] \qquad \frac{}{r_\Diamond uv, \Gamma \vdash \Delta, u{:}\overrightarrow{\Diamond} v}\ [\mathrm{Ax}_\Diamond\mathrm{R}]$$

$$\frac{}{u{:}v \bullet w, \Gamma \vdash \Delta, r_\bullet uvw}\ [\mathrm{Ax}_\bullet\mathrm{L}] \qquad \frac{}{r_\bullet uvw, \Gamma \vdash \Delta, u{:}v \bullet w}\ [\mathrm{Ax}_\bullet\mathrm{R}]$$

Interface axioms IA

Figure 1.2: Labeled sequent calculus for HCTL

becomes

$$
\dfrac{\begin{array}{c}\pi_2{}'\\ \vdots\\ \Gamma' \vdash \Delta', w{:}\overrightarrow{\Diamond}u\end{array} \qquad \dfrac{\begin{array}{c}\pi_1{}^*[w/v]\\ \vdots\\ w{:}\overrightarrow{\Diamond}u, \Gamma \vdash \Delta, w{:}\varphi\end{array} \quad \begin{array}{c}\pi_2{}''\\ \vdots\\ w{:}\varphi, \Gamma' \vdash \Delta'\end{array}}{w{:}\overrightarrow{\Diamond}u, \Gamma, \Gamma' \vdash \Delta, \Delta'}\,[\text{Cut}]}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,[\text{Cut}]
$$

where $\pi_1{}^*[w/v]$ is the result of renaming the occurrences of v by w in π_1. Note that such replacement leaves the contexts Γ and Δ unchanged since w does not occur in them. $\Box$

S+**LLR**+**IA** brings us very close to a real hybrid calculus for validity in HCTL. From the previous results, we know that since **S**+**LLR**+**IA** is a calculus for Θ-validity, $\Gamma \vdash \Delta$ is valid in HCTL iff $u{:}\Gamma \vdash u{:}\Delta$ is a theorem of **S**+**LLR**+**IA**. But we can go a step further. The **LLR** rules that we derived in our search for cut elimination, are actually rules for the logical operators in HCTL. Since the calculus **S**+**LLR**+**IA** has the subformula property, the proof of $u{:}\Gamma \vdash u{:}\Delta$ in **S**+**LLR**+**IA** will not require the rules for the logical operators of **S** or the interface axioms **IA**. We only need to keep the structural rules of **S** and the rules of **LLR** and we will still have a sound and complete calculus for HCTL. Let's call such a calculus **HS**, i.e., **HS** has as rules the structural rules of **S** and the rules of **LLR**.

Even more, up to now, we have been using a FO calculus to prove HCTL sequents 'indirectly', and the symbol : in the rules of **S** was a metasymbol of FO. Now everything is in place for the internalization flip:

> *From now on, consider that the symbol* : *in each rule of* **HS** *is the* HCTL *connective, and think of the rules as treating directly* HCTL *sequents.*

Note that the connective : allows us to easily capture in the formal language of HCTL the labels that we added to formulas in the metalanguage of FO. Without this operator, it is unclear how this would be achieved. This is precisely what makes hybrid logics so suitable for the study of internalization.

Theorem 4.1. A sequent $\Gamma \vdash \Delta$ of HCTL is valid iff $u{:}\Gamma \vdash u{:}\Delta$ is a theorem of **HS**.

5 A fully internalized sequent calculus for HCTL

HS is cut-free and enjoys the subformula property (with a suitable definition of subformula). It is also internalized, since no metalogical operators are required and only formulas of HCTL occur in proofs. However, it still has a minor drawback. It only covers a fragment of the language, namely we can prove only sequents where all formulas are of the form $u{:}\varphi$. Such a calculus is called *labeled* or :*-driven*. In practice this is not a big problem, since we know that a sequent $\Gamma \vdash \Delta$ of HCTL is valid iff the sequent $u{:}\Gamma \vdash u{:}\Delta$ is valid, where u is any arbitrary variable not occurring in $\Gamma \vdash \Delta$. But it would be nice if we could deal directly with arbitrary sequents of HCTL. This is easily achieved using *nominals*, the individual variables occurring as formulas in HCTL.

Recall that a nominal is true only when it is evaluated at the point it denotes, so a sequent $u, \Gamma \vdash \Delta$ will only be valid if it is evaluated at the point denoted by u and the sequent $\Gamma \vdash \Delta$ is true at this point. Thus when a single nominal occurs on the left hand side of a sequent, it anchors all non :-prefixed formulas to the same element, and hence they do not need to explicitly share a prefix. In Figure 1.3 we give the so called *nominal rules* **NR**, which control the interaction between :-prefixed formulas and nominals. We will use the context sharing conditions given by the nominals to do some cleaning up. We will rewrite our rules as follows:

$$\frac{u{:}\Gamma_1, \Gamma \vdash \Delta, u{:}\Delta_1}{u{:}\Gamma_2, \Gamma' \vdash \Delta', u{:}\Delta_2}\,[\text{:rule}] \quad \rightsquigarrow \quad \frac{u, \Gamma_1, \Gamma \vdash \Delta, \Delta_1}{u, \Gamma_2, \Gamma' \vdash \Delta', \Delta_2}\,[\text{rule}].$$

If additionally u does not occur in the rest of the rule, we can delete it[2]. This yields the *nominal-based hybrid rules* **HR**, given in Figure 1.3. Note that some of the rules of **LLR** become redundant when the :-prefixes are removed, so we do not include them in **HR**. Let's call **NHS** to the calculus shown in Figure 1.3 comprising the structural rules of **S**, the nominal rules **NR**, and the nominal-based hybrid rules **HR**.

Now we have finally achieved our goal: **NHS** is a sound and complete, cut free, fully internalized calculus for HCTL.

Theorem 5.1. A sequent of HCTL is valid iff it is a theorem of **NHS**.

Proof. First we prove that $u{:}\Gamma \vdash u{:}\Delta$ is a theorem of **HS** iff $\Gamma \vdash \Delta$ can be proved using **HS** and the nominal rules **NR**. For the only if direction, if we can prove $u{:}\Gamma \vdash u{:}\Delta$ in **HS**, then by weakening we get $u, u{:}\Gamma \vdash u{:}\Delta$ and then, by applying repeatedly the [$\wedge$:L] and [$\wedge$:R] rules, we get $u, \Gamma \vdash \Delta$. Finally, using [name] we get a proof of $\Gamma \vdash \Delta$. For the other direction, since all the rules of **NR** are sound, all theorems that can be proved with **HS** and **NR** are valid in HCTL. In a second step, it must be verified that any proof of **NHS** can be translated into a proof using the rules in **HS** and **NR**, and vice versa. The translation is straightforward: any occurrence of a rule [:rule] in a proof of **HS** can be replaced by its equivalent rule [rule] and the other way around. In both cases, some transformations on the sequent must be done using the nominal and structural rules, but the resulting proof is isomorphic to the original one. □

6 Examples

As an example, we give two short derivations in the calculus **NHS**. The first one corresponds to a sequent with both hybrid connectives and CTL formulas, and the second one to a sequent containing only CTL connectives. On the left column we prove the sequent $\vec{\Diamond}\varphi \vdash \mathsf{E}\varphi$. If $\vec{\Diamond}\varphi$ is true at some point a in $\mathcal{M}$, then there is some b in $\mathcal{M}$ (connected to a

[2]In this case the point of evaluation is the shared element of all formulas that are not :-prefixed.

by the $r_\Diamond$ relation) where φ holds, thus $\mathsf{E}\varphi$ holds in a as well. On the right we prove that $\varphi \vdash \overleftarrow{\Box}\overrightarrow{\Diamond}\varphi$. Since $\overleftarrow{\Box}$ and $\overrightarrow{\Diamond}$ form a residuated pair, $\overleftarrow{\Box}\overrightarrow{\Diamond}\varphi$ holds in any world where φ holds. Note that, although we are proving a theorem of CTL, hybrid operators are used during the proof.

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{u{:}\varphi, \overrightarrow{\Diamond}u \vdash u{:}\varphi}\,[\mathrm{Ax}]}{\overrightarrow{\Diamond}u, u{:}\varphi \vdash u{:}\varphi}\,[\mathrm{S}]}{\overrightarrow{\Diamond}u, u{:}\varphi \vdash \mathsf{E}\varphi}\,[\mathsf{ER}]}{\overrightarrow{\Diamond}\varphi \vdash \mathsf{E}\varphi}\,[\overrightarrow{\Diamond}\mathrm{L}]
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\overrightarrow{\Diamond}u, u, \varphi \vdash \overrightarrow{\Diamond}u}\,[\mathrm{Ax}] \quad \dfrac{\dfrac{\dfrac{\dfrac{}{u{:}\varphi, u, \overrightarrow{\Diamond}u \vdash u{:}\varphi}\,[\mathrm{Ax}]}{u, u{:}\varphi, \overrightarrow{\Diamond}u \vdash u{:}\varphi}\,[\mathrm{S}]}{u, \varphi, \overrightarrow{\Diamond}u \vdash u{:}\varphi}\,[\wedge{:}\mathrm{L}]}{\overrightarrow{\Diamond}u, u, \varphi \vdash u{:}\varphi}\,[\mathrm{S}]}{\overrightarrow{\Diamond}u, u, \varphi \vdash \overrightarrow{\Diamond}\varphi}\,[\overrightarrow{\Diamond}\mathrm{R}]}{u, \varphi \vdash u{:}\overleftarrow{\Box}\overrightarrow{\Diamond}\varphi}\,[\overleftarrow{\Box}\mathrm{R}]}{u, \varphi \vdash \overleftarrow{\Box}\overrightarrow{\Diamond}\varphi}\,[\vee{:}\mathrm{R}]}{\varphi \vdash \overleftarrow{\Box}\overrightarrow{\Diamond}\varphi}\,[\mathrm{name}]
$$

7 Conclusions and future work

Despite the peculiarities of CTL, the internalization strategy works fine and it yields an elegant, fully internalized calculus for HCTL. Note that the rules governing the modal operators in **NHS** contain also hybrid operators. This is a common feature of calculi for hybrid modal logics. We can see this as evidence that, in the absence of the hybrid machinery, it might be difficult to devise a fully internalized calculus. We also want to point out that the calculus **NHS** is given for sequents of the form $\Gamma \vdash \Delta$ where both Γ and Δ are lists of formulas. An intuitionistic version of the calculus (which is more natural in the context of categorial logics) can be obtained in a straightforward way, by restricting the right-hand-side of sequents to contain at most one formula.

It remains as future work to analyze in more detail the calculus we just obtained, and to compare it with other calculi. For example, in (Areces and Bernardi 2003) CTL has already been extended with hybrid operators in order to model some binding phenomena in natural language. The calculus given there is a labeled calculus. It would be interesting to compare it with our calculus **HS**, and even to explore the nominal technique to eliminate :-labels on it. Another issue that must be explored is a comparison of the calculus **NHS** with Moortgat's calculus $\mathsf{NL}(\Diamond)$. It is clear that we do not require any structural connectives, due to the expressiveness of the hybrid operators. Another relevant difference is that in our calculus sequents have more than one formula on the left-hand-side. This multiformula sequents imply a restricted form of conjunction, which is usually not present in CTL. Additionally, hybrid operators allow to simulate by themselves and to some extent, the behavior of Boolean operators. For example, if the formula $(w{:}p) \bullet (w{:}q)$ holds at some point of a model $\mathcal{M}$, then there is some point in $\mathcal{M}$ (namely, the point named w), where the conjunction of p and q holds. Similar formulas can be built that simulate other Boolean constructors at named worlds. Clearly HCTL is more expressive than ordinary CTL, but it remains as an open question to find out how much.

$$\frac{}{\varphi, \Gamma \vdash \Delta, \varphi}\ [\text{Ax}] \qquad \frac{\Gamma \vdash \Delta}{\Gamma' \vdash \Delta'}\ [\text{S}]^{1}$$

[1] if the lists Γ and Δ contain the same *set* of formulas as Γ' and Δ' resp.

Structural rules

$$\frac{u, \varphi, \Gamma \vdash \Delta}{u, u{:}\varphi, \Gamma \vdash \Delta}\ [\vee{:}\text{L}] \qquad \frac{u, \Gamma \vdash \Delta, u{:}\varphi}{u, \Gamma \vdash \Delta, \varphi}\ [\vee{:}\text{R}]$$

$$\frac{u, u{:}\varphi, \Gamma \vdash \Delta}{u, \varphi, \Gamma \vdash \Delta}\ [\wedge{:}\text{L}] \qquad \frac{u, \Gamma \vdash \Delta, \varphi}{u, \Gamma \vdash \Delta, u{:}\varphi}\ [\wedge{:}\text{R}]$$

$$\frac{u, \Gamma \vdash \Delta}{\Gamma \vdash \Delta}\ [\text{name}]^{1} \quad \frac{u, \Gamma \vdash \Delta}{\Gamma \vdash \Delta}\ [\text{term}]^{2} \quad \frac{\Gamma \vdash \Delta}{u, \Gamma \vdash \Delta}\ [\text{term}^{-}]^{2}$$

[1] if u does not occur in Γ,Δ [2] if all formulas in Γ,Δ are :-prefixed

Nominal rules NR

$$\frac{u, v, \Gamma[w/u] \vdash \Delta[w/u]}{u, v, \Gamma[w/v] \vdash \Delta[w/v]}\ [N\text{L}]$$

$$\frac{\overrightarrow{\Diamond} v, v{:}\varphi, \Gamma \vdash \Delta}{\overrightarrow{\Diamond}\varphi, \Gamma \vdash \Delta}\ [\overrightarrow{\Diamond}\text{L}]\ \ v \text{ new} \qquad \frac{\Gamma \vdash \Delta, \overrightarrow{\Diamond} v \quad \Gamma \vdash \Delta, v{:}\varphi}{\Gamma \vdash \Delta, \overrightarrow{\Diamond}\varphi}\ [\overrightarrow{\Diamond}\text{R}]$$

$$\frac{\Gamma \vdash \Delta, \overrightarrow{\Diamond} u \quad \varphi, \Gamma \vdash \Delta}{u{:}\overleftarrow{\Box}\varphi, \Gamma \vdash \Delta}\ [\overleftarrow{\Box}\text{L}] \qquad \frac{\overrightarrow{\Diamond} u, \Gamma \vdash \Delta, \varphi}{\Gamma \vdash \Delta, u{:}\overleftarrow{\Box}\varphi}\ [\overleftarrow{\Box}\text{R}]$$

$$\frac{v \bullet w, v{:}\varphi, w{:}\psi, \Gamma \vdash \Delta}{\varphi \bullet \psi, \Gamma \vdash \Delta}\ [\bullet\text{L}]\ \ v, w \text{ new} \qquad \frac{\Gamma \vdash \Delta, v \bullet w \quad \Gamma \vdash \Delta, v{:}\varphi \quad \Gamma \vdash \Delta, w{:}\psi}{\Gamma \vdash \Delta, \varphi \bullet \psi}\ [\bullet\text{R}]$$

$$\frac{\varphi, \Gamma \vdash \Delta \quad \Gamma \vdash \Delta, u \bullet w \quad \Gamma \vdash \Delta, w{:}\psi}{u{:}\varphi \leftarrow \psi, \Gamma \vdash \Delta}\ [\leftarrow \text{L}] \qquad \frac{u \bullet w, w{:}\psi, \Gamma \vdash \Delta, \varphi}{\Gamma \vdash \Delta, u{:}\varphi \leftarrow \psi}\ [\leftarrow \text{R}]\ \ w \text{ new}$$

$$\frac{\psi, \Gamma \vdash \Delta \quad \Gamma \vdash \Delta, w \bullet u \quad \Gamma \vdash \Delta, w{:}\varphi}{u{:}\varphi \rightarrow \psi, \Gamma \vdash \Delta}\ [\rightarrow \text{L}] \qquad \frac{w \bullet u, w{:}\varphi, \Gamma \vdash \Delta, \psi}{\Gamma \vdash \Delta, u{:}\varphi \rightarrow \psi}\ [\rightarrow \text{R}]\ \ w \text{ new}$$

$$\frac{u, \varphi[u/i], \Gamma \vdash \Delta}{u, \downarrow_i\varphi, \mathit{\Gamma} \vdash \mathit{\Delta}}\ [\downarrow\text{L}] \qquad \frac{u, \Gamma \vdash \Delta, \varphi[u/i]}{u, \Gamma \vdash \Delta, \downarrow_i\varphi}\ [\downarrow\text{R}]$$

$$\frac{v{:}\varphi, \Gamma \vdash \Delta}{\mathsf{E}\varphi, \Gamma \vdash \Delta}\ [\mathsf{EL}]\ \ v \text{ new} \qquad \frac{\Gamma \vdash \Delta, v{:}\varphi}{\Gamma \vdash \Delta, \mathsf{E}\varphi}\ [\mathsf{ER}]$$

Nominal-based hybrid rules HR

Figure 1.3: Internalized sequent calculus **NHS** for HCTL

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