Gentzen-Like Methods in Quantum Logic*

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Abstract. Quantum logic generally refers to the logical structure characterized by the class of orthomodular lattices. It originated from certain postulates advocated in the Hilbert-space formalism of modern quantum mechanics. In this paper, we investigate issues related to the proof theory of *minimal quantum logic*, i.e., quantum logic where the axiom of modularity is not stipulated. Based on a sequent-type calculus introduced by Nishimura, we will show that a modification of this system will result in a much more concise system. Moreover, a corresponding tableau system for minimal quantum logic will be proposed. An example from quantum mechanics will be used to illustrate how the principle of modularity can be encoded within our framework.

1 Introduction

Quantum logic has been introduced in the early 1930s by John von Neumann in his famous treatise on the mathematical foundations of quantum mechanics [15]. In that work, he proposed to regard projection operators over a given Hilbert space to *represent certain propositions of a corresponding quantum-mechanical system*. Later on, in a joint paper with Garrett Birkhoff, this "logic of projection operators" has been given a more elaborated treatment, characterizing its *algebraic* nature [4]. As it turned out, the resulting logical structure can be described in terms of *non-distributive, orthomodular lattices*, significantly different from classical Boolean algebra.

Although controversial right from the beginning, in the following decades a huge number of scientific investigations have been devoted to quantum logic. However, most of these research efforts were either conducted by physicists focusing purely on the *physical meaning* of quantum logics, or by mathematicians interested mainly in its algebraic properties—but there is only a minority of papers treating quantum logics in a suitable *proof-theoretical* manner, with, say, reference to a concrete axiomatic method.

In this paper, we will discuss quantum logic from the perspective of sequent-type methods. More specifically, we will present a sequent-type calculus, LMQ, for *minimal quantum logic* which turns out to be more concise than an earlier

* The authors would like to thank Markus Moschner for pointing out some of the relevant bibliographic references.

system proposed by Nishimura [17]. Although Takano [22] noted that one rule of Nishimura's calculus is superfluous, instead of eliminating this rule, a slight modification of it actually yields a large number of other rules redundant.

Based on the system LMQ, we will also present a tableau calculus for minimal quantum logic. To the best of our knowledge, this will constitute the first tableau-based account for any kind of quantum logic. The tableau system will be constructed in a style resembling the approach taken by Fitting in [9] for various modal logics—to wit, by exploiting the technique of *branch-modification rules*. Furthermore, a formalization of the quantum-mechanical principle known as *Lüders' Rule* will be used to illustrate how the property of *modularity* can be handled within the framework of minimal quantum logic. The paper is organized as follows. In Section 2, we will review the basic facts about quantum logic. In Section 3, we present our calculi and show that LMQ allows for significantly fewer rules than Nishimura's system. Section 4 contains the application of LMQ to Lüders' Rule, and Section 5 concludes with some general remarks.

2 Basics of Quantum Logic

From a strictly formal point of view, quantum logic can be characterized as the logical structure emanating from the *algebra of orthomodular lattices*. In this sense, quantum logic has a logical status similar to other non-classical logics like, say, intuitionistic logic or minimal logic, independent of its physical heritage.

Let us recall that an *orthomodular lattice* is a Boolean lattice in which the distributive laws are no longer stipulated, and in which the modular law holds.[2] If the modular law is also dispensed, one speaks of an *ortholattice*. The logic associated with ortholattices is referred to as *minimal quantum logic*, or simply *orthologic*. This variant of quantum logic has been introduced mainly for mathematical reasons, because it behaves more natural in certain respects than modular quantum logic.

In this section, we will review the basic facts about minimal quantum logic. In particular, we will introduce the sequent calculus GMQL, due to Nishimura [17]. This system traces back to earlier accounts given by Goldblatt [11], Nishimura [16], Cutland and Gibbins [5], Tamura [23], and others. For a thorough introduction to the subject of quantum logic, we refer the reader to several sources information: [6] provides a survey of quantum logic with emphasis on its logical structure; [13] and [19] discuss the logic of orthomodular structures; and [21] is a recent textbook on quantum logic with respect to physics. A sequent-type account of orthologic independent from the series of papers mentioned above—but related to a system described in [13]—is given in [2].

We will use a propositional language with the unary operator $'$ ("negation"), and the two binary operators $\wedge$ ("conjunction") and $\vee$ ("disjunction") as primitive logical connectives. We use $P, Q, R, \dots$ to denote atomic formulae and $A, B, C, \dots$ to denote composite formulae. The composite formulae are built

[2] A lattice $(L, \sqcup, \sqcap)$ is *modular* iff for all $x, y, z \in L$, $z \sqsubseteq x$ implies $x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup z$ (where, for any two elements $a, b \in L$, the relation $a \sqsubseteq b$ holds iff $a \sqcap b = a$).

from the atomic formulae according to the usual formation rules. Uppercase Greek letters Γ, Δ, $\Sigma, \ldots$ will be used to denote finite sets of formulae. Given a set Γ of formulae, then Γ' denotes the set $\{A' \mid A \in \Gamma\}$. By a *sequent* we understand an ordered pair $\Gamma \vdash \Delta$ of finite sets Γ, Δ. We tacitly employ the usual practice to write sequents like $\Gamma \cup \Delta \vdash \Sigma$ or $\Gamma \cup \{A\} \vdash \Sigma$ in the form $\Gamma, \Delta \vdash \Sigma$ and $\Gamma, A \vdash \Sigma$, respectively.

A formula B is called a *quasi-subformula* of a formula A iff it is either (i) a subformula of A, (ii) the negation of a subformula of A, or (iii) of the form $(C \circ D)''$, where $\circ \in \{\wedge, \vee\}$ and $C \circ D$ is a subformula of A.

We say that an inference rule R is *admissible* in a calculus $\mathcal{C}$ iff adjoining R to $\mathcal{C}$ does not change the class of provable formulae.

The sequent calculus GMQL for minimal quantum logic consists of axioms of the form $A \vdash A$ (where A is some formula), and the inference rules depicted in Figure 1.

Semantically, it is convenient to adopt the method introduced by Goldblatt in [11].

By an O-frame, $\mathcal{F}$, we understand a pair $\langle X, \perp \rangle$, where X is a nonempty set (the *carrier* of $\mathcal{F}$) and $\perp$ is an *orthogonality* relation on X, i.e., $\perp$ is an irreflexive and symmetric binary relation on X. Given a set $Y \subseteq X$, the *orthocomplement* of Y is the set

$$Y^{\perp} := \{x \in X \mid x \perp y, \text{ for all } y \in Y\}.$$

Observe that $Y \subseteq Y^{\perp\perp}$, for any set $Y \subseteq X$. If the converse also holds, i.e., if $Y^{\perp\perp} \subseteq Y$, then Y is said to be *closed*. Hence, Y is closed iff $Y = Y^{\perp\perp}$.

An O-model, $\mathcal{M}$, is a triple $\langle X, \perp, D \rangle$, where $\langle X, \perp \rangle$ is an O-frame and D is a mapping which assigns to each propositional variable p a closed subset $D(p)$ of X.

For any formula A, the relation $\|A\|$ is recursively defined as follows:

1. $\|p\| = D(p)$, for any propositional variable p;
2. $\|A'\| = \|A\|^{\perp}$;
3. $\|A \wedge B\| = \|A\| \cap \|B\|$;
4. $\|A \vee B\| = (\|A\|^{\perp} \cap \|B\|^{\perp})^{\perp}$

For an $x \in X$ and a formula A, we write $V(A; x) = 1$ if $x \in \|A\|$, and $V(A; x) = 0$ otherwise. Informally, $V(A; x) = 1$ may be read as "A is true at x in the O-model $\mathcal{M}$". Furthermore, for a sequent $\Gamma \vdash \Delta$, we set $V(\Gamma \vdash \Delta; x) = 0$ iff $x \in \bigcap_{A \in \Gamma} \|A\|$ and $x \notin (\bigcap_{B \in \Delta} \|B\|^{\perp})^{\perp}$, and $V(\Gamma \vdash \Delta; x) = 1$ otherwise.

A sequent $\Gamma \vdash \Delta$ is *falsifiable* iff there is some O-model $\langle X, \perp, D \rangle$ and some $x \in X$ such that $V(\Gamma \vdash \Delta; x) = 0$.[3] A sequent which is not falsifiable is said to be *valid*.

Proposition 1 [17]. *A sequent is provable in* GMQL *iff it is valid.*

[3] Nishimura [17] uses the somewhat misleading term "realizable" instead of "falsifiable".

$$\frac{\Gamma \vdash \Delta}{\Pi, \Gamma \vdash \Delta, \Sigma}\ ext \qquad \frac{\Gamma \vdash \Delta}{\Delta' \vdash \Gamma'}\ ({}' \vdash {}')$$

$$\frac{A, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta}\ (\wedge \vdash)_1 \qquad \frac{B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta}\ (\wedge \vdash)_2$$

$$\frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, A \vee B}\ (\vdash \vee)_1 \qquad \frac{\Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \vee B}\ (\vdash \vee)_2$$

$$\frac{A \vdash \Delta \qquad B \vdash \Delta}{A \vee B \vdash \Delta}\ (\vee \vdash) \qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ (\vdash \wedge)$$

$$\frac{\Gamma \vdash \Delta}{\Delta', \Gamma \vdash}\ ({}' \vdash) \qquad \frac{\Gamma \vdash \Delta}{\vdash \Delta, \Gamma'}\ (\vdash {}')$$

$$\frac{A, \Gamma \vdash \Delta}{A'', \Gamma \vdash \Delta}\ ({}'' \vdash) \qquad \frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, A''}\ (\vdash {}'')$$

$$\frac{A', \Gamma \vdash \Delta}{(A \vee B)', \Gamma \vdash \Delta}\ (\vee' \vdash)_1 \qquad \frac{B', \Gamma \vdash \Delta}{(A \vee B)', \Gamma \vdash \Delta}\ (\vee' \vdash)_2$$

$$\frac{\Gamma \vdash \Delta, A'}{\Gamma \vdash \Delta, (A \wedge B)'}\ (\vdash \wedge')_1 \qquad \frac{\Gamma \vdash \Delta, B'}{\Gamma \vdash \Delta, (A \wedge B)'}\ (\vdash \wedge')_2$$

$$\frac{A' \vdash \Delta \qquad B' \vdash \Delta}{(A \wedge B)' \vdash \Delta}\ (\wedge' \vdash) \qquad \frac{\Gamma \vdash A' \qquad \Gamma \vdash B'}{\Gamma \vdash (A \vee B)'}\ (\vdash \vee')$$

$$\frac{\Gamma \vdash A' \qquad \Gamma \vdash B'}{A \vee B, \Gamma \vdash}\ (\vee \vdash {}') \qquad \frac{A' \vdash \Delta \qquad B' \vdash \Delta}{\vdash \Delta, A \wedge B}\ ({}' \vdash \wedge)$$

Fig. 1. The inference rules of GMQL.

REMARK: We mentioned earlier that quantum logics are described in terms of ortholattices. Indeed, according to some standard results from abstract algebra, closed subsets of an O-frame form an ortholattice under the partial ordering of set inclusion (cf. Birkhoff [3] for more details).

An important role in the context of sequent-type calculi plays the well-known *cut-rule*. However, as pointed out by Cutland and Gibbins [5] (with reference to an unpublished paper by M. Dummett [7]), the inclusion of an unrestricted cut rule in a system like GMQL would imply a collapse to classical logic. So, in order to remain in the realm of quantum logic, only restricted forms of cut can be considered. In fact, following Cutland and Gibbins [5], Nishimura [17] considers the following two cut rules:

$$\frac{\Gamma \vdash \Delta_1, A \qquad A \vdash \Delta_2}{\Gamma \vdash \Delta_1, \Delta_2}\ cut_1 \qquad \frac{\Gamma_1 \vdash A \qquad A, \Gamma_2 \vdash \Delta}{\Gamma_1, \Gamma_2 \vdash \Delta}\ cut_2$$

The next result states that these forms of cut are indeed appropriate for GMQL.

Proposition 2 [17, 18]. *If the sequents* $\Gamma_1 \vdash \Delta_1, A$ *and* $A, \Gamma_2 \vdash \Delta_2$ *are provable in* GMQL*, where* $\Delta_1 = \emptyset$ *or* $\Gamma_2 = \emptyset$*, then the sequent* $\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2$ *is provable in* GMQL*. In other words,* cut_1 *and* cut_2 *are admissible in* GMQL*.*

As remarked by Nishimura in [17], if we remove the rules

$$(\vee' \vdash)_i,\ (\vdash \wedge')_i,\ (\wedge' \vdash),\ (\vdash \vee'),\ (\vee \vdash '),\ ('\vdash \wedge) \tag{1}$$

from GMQL (for $i = 1, 2$), but admit cut_1 and cut_2, then the system of Cutland and Gibbins [5] would be obtained. However, in the next section we will present a calculus in which the rules (1) are still superfluous, *but without the need of any cut-rule.*

3 New Calculi for Minimal Quantum Logic

3.1 The Sequent Calculus LMQ

Recently, Takano [22] identified the rule $(' \vdash ')$ to be superfluous in GMQL. However, as we are going to show, an even more compact calculus for minimal quantum logic exists, if we replace $(' \vdash ')$ by a rule which is in effect the converse of $(' \vdash ')$. Thus, the new rule, called *alternation rule*, has the following form:

$$\frac{\Gamma' \vdash \Delta'}{\Delta \vdash \Gamma}\ alt$$

We will see that this rule allows the elimination of the essentially "symmetric" rules in GMQL, involving negation and the binary connectives $\wedge$ and $\vee$, respectively. The new calculus is called LMQ and its inference rules are depicted in Figure 2.

Some words about the rule *alt* are in order. One might be puzzled about the fact that this rule is actually an *elimination* rule, contrasting the usual paradigm in sequent-type calculi of employing *introduction* rules (the notable exception being of course the cut-rule). However, elimination rules are a handy device if one is interested in comparing proofs in, e.g., intuitionistic sequent-type calculi like LJ with proofs in classical sequent-type calculi like LK. More exactly, one can show that a classically valid formula can be proven in LJ, together with the cut-rule, if one allows "non-logical axioms" of the form $\neg\neg A \vdash A$. Now, the point is that the same result holds if the cut-rule is replaced by double-negation elimination rules like the ones given in Lemma 5 below. (For more details on non-logical axioms, see [24].)

We commence with our result on the redundancy of a large number of GMQL-rules by using the new rule *alt*. Afterwards, soundness and completeness of the calculus LMQ will be established. We conclude this subsection with some remarks how LMQ can be used for a decision procedure of minimal quantum logic.

$$\frac{\Gamma \vdash \Delta}{\Pi, \Gamma \vdash \Delta, \Sigma}\ ext \qquad \frac{\Gamma' \vdash \Delta'}{\Delta \vdash \Gamma}\ alt$$

$$\frac{A, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta}\ (\wedge \vdash)_1 \qquad \frac{B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta}\ (\wedge \vdash)_2$$

$$\frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, A \vee B}\ (\vdash \vee)_1 \qquad \frac{\Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \vee B}\ (\vdash \vee)_2$$

$$\frac{A \vdash \Delta \qquad B \vdash \Delta}{A \vee B \vdash \Delta}\ (\vee \vdash) \qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ (\vdash \wedge)$$

$$\frac{\Gamma \vdash \Delta}{\Delta', \Gamma \vdash}\ ({}' \vdash) \qquad \frac{A, \Gamma \vdash \Delta}{A'', \Gamma \vdash \Delta}\ ({}'' \vdash) \qquad \frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, A''}\ (\vdash {}'')$$

Fig. 2. The inference rules of LMQ.

Theorem 3. *The* GMQL*-inferences* $(\vdash {}')$, $(\vee' \vdash)_i$, $(\vdash \wedge')_i$, $(\wedge' \vdash)$, $(\vdash \vee')$, $(\vee \vdash {}')$, *and* $({}' \vdash \wedge)$ *are admissible in* LMQ *(for* $i = 1, 2$*).*

Proof. We show the assertion of the theorem only for $(\vdash {}')$ and $(\wedge' \vdash)$; the remaining cases are similar. The relevant LMQ-deductions are given below.

Rule $(\vdash {}')$: Let Γ be partitioned into Γ_1, Γ_2', where Γ_2' contains all the formulae of Γ which are of the form A', for some formula A.

$$\dfrac{\dfrac{\dfrac{\Gamma_1, \Gamma_2' \vdash \Delta}{\Delta', \Gamma_1, \Gamma_2' \vdash}\ ({}' \vdash)}{\begin{array}{c}\vdots\ ({}'' \vdash)\\ \Delta', \Gamma_1'', \Gamma_2' \vdash\end{array}}}{\dfrac{\vdash \Delta, \Gamma_1', \Gamma_2}{\vdash \Delta, \Gamma_1', \Gamma_2''}\ (\vdash {}'')}\ alt$$

Rule $(\wedge' \vdash)$: Let Δ be similarly partitioned into Δ_1, Δ_2' as in the previous case.

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}A' \vdash \Delta_1, \Delta_2'\\ \vdots\ (\vdash {}'')\\ A' \vdash \Delta_1'', \Delta_2'\end{array}}{\Delta_1', \Delta_2 \vdash A}\ alt \quad \dfrac{\begin{array}{c}B' \vdash \Delta_1, \Delta_2'\\ \vdots\ (\vdash {}'')\\ B' \vdash \Delta_1'', \Delta_2'\end{array}}{\Delta_1', \Delta_2 \vdash B}\ alt}{\Delta_1', \Delta_2 \vdash A \wedge B}\ (\vdash \wedge)}{\Delta_1', \Delta_2'' \vdash (A \wedge B)''}\ ({}'' \vdash), (\vdash {}'')}{(A \wedge B)' \vdash \Delta_1, \Delta_2'}\ alt$$

■

Next, we turn to the soundness and completeness of LMQ. First, let us observe that the soundness of *alt* follows readily from Corollary 2.3(a) of Nishimura [17], which is as follows:

Proposition 4 [17]. *If a sequent $\Gamma, \Pi' \vdash \Delta, \Sigma'$ is provable in* GMQL*, then so is $\Delta', \Sigma \vdash \Gamma', \Pi$.*

Alternatively, the soundness of *alt* can also be shown by using the two lemmata given below.

Lemma 5. *The following inference rules for the elimination of double negation are admissible in* GMQL*.*

$$\frac{A'', \Gamma \vdash \Delta}{A, \Gamma \vdash \Delta}\ ({''}E\vdash) \qquad\qquad \frac{\Gamma \vdash \Delta, A''}{\Gamma \vdash \Delta, A}\ (\vdash{''}E)$$

Proof. The result follows from the derivations given below and the elimination of cut due to Proposition 2.

$$\frac{\dfrac{A \vdash A}{A \vdash A''}\ (\vdash{''}) \qquad A'', \Gamma \vdash \Delta}{A, \Gamma \vdash \Delta}\ cut_2 \qquad\qquad \frac{\Gamma \vdash \Delta, A'' \qquad \dfrac{A \vdash A}{A'' \vdash A}\ ({''}\vdash)}{\Gamma \vdash \Delta, A}\ cut_1$$

■

Lemma 6. *The inference rule* alt *is admissible in* GMQL*.*

Proof. By Lemma 5 and the following deduction:

$$\begin{array}{l} \dfrac{\Gamma' \vdash \Delta'}{\Delta'' \vdash \Gamma''}\ ({'}\vdash{'}) \\ \quad\vdots\ ({''}E\vdash) \\ \Delta \vdash \Gamma'' \\ \quad\vdots\ (\vdash{''}E) \\ \Delta \vdash \Gamma \end{array}$$

■

The last result immediately yields the soundness of *alt*. Since all other rules of LMQ are also rules of GMQL, by the soundness of the latter system we obtain the soundness of LMQ.

Theorem 7. *If a sequent is provable in* LMQ*, then it is valid.*

Conversely, the completeness of LMQ follows from Theorem 3 and the observation that $({'} \vdash {'})$ can be simulated by applications of $({''} \vdash)$, $(\vdash {''})$, and *alt*.

Theorem 8. *If a sequent is valid, then it is provable in* LMQ*.*

Inspection of the formulae occurring in the simulations in the proof of Theorem 3 yields the following corollary.

Corollary 9. LMQ *is complete without the cut rule. Moreover, for each valid sequent S, there is a proof in* LMQ *containing only quasi-subformulae of the formulae in S.*

This corollary provides a decision procedure for minimal quantum logic by Gentzen's finite top-down oriented construction of all sequent proofs which use possibly negated formulae from the formulae of the end sequent.

Furthermore, from the two cut rules given in Nishimura [17], one can be simulated by the other, together with rules from LMQ.

Theorem 10. *The inference rule cut_1 can be simulated by cut_2, alt, $('' \vdash)$, and $(\vdash '')$.*

Proof. The following derivation provides the simulation.

$$
\dfrac{\dfrac{\dfrac{\dfrac{A \vdash \Delta_2}{\vdots\ ('' \vdash), (\vdash '')}}{A'' \vdash \Delta_2''}}{\Delta_2' \vdash A'}\,alt \qquad \dfrac{\dfrac{\dfrac{\Gamma \vdash \Delta_1, A}{\vdots\ ('' \vdash), (\vdash '')}}{\Gamma'' \vdash \Delta_1'', A''}}{A', \Delta_1' \vdash \Gamma'}\,alt}{\dfrac{\Delta_1', \Delta_2' \vdash \Gamma'}{\Gamma \vdash \Delta_1, \Delta_2}\,alt}\,cut_2
$$

∎

Let us briefly compare the system LMQ with the calculus $^{\perp}\mathsf{O}$ for orthologic given in [2]. The main difference between these two systems is that in $^{\perp}\mathsf{O}$ negation is not a primitive concept, but defined in terms of linear recursion. Hence, $^{\perp}\mathsf{O}$ shifts the treatment of negation from the object level to the meta level. As a consequence, such a treatment *hides* certain inferences which should be taken into account if one is interested in proof-theoretical complexity estimations. In contrast, all inferences in LMQ are explicitly represented at the object level.

3.2 A Tableau System for Minimal Quantum Logic

By a simple rewriting technique, any sequent calculus can be transformed into a corresponding tableau calculus—basically by turning the sequent proofs upside-down and by "identifying" sequents with branches in the tableau. In the following, we apply such a procedure to the sequent calculus LMQ, thus obtaining a tableau calculus for minimal quantum logic in a rather straightforward manner.

As the first step of this procedure, we will introduce an intermediate sequent calculus LMQ_t. This system differs from LMQ by using generalized axioms of the form $\Gamma, A \vdash \Delta, A$, and the inference rules given in Figure 3.

It is obvious that this calculus is also sound and complete: completeness follows from the fact that any LMQ-proof is *a fortiori* an LMQ_t-proof, and soundness is an immediate consequence from the following two observations:

(i) Any axiom of LMQ_t is provable in LMQ (by invoking the rule *ext*).
(ii) The rules $(\vee \vdash)_t$, $(\vdash \wedge)_t$, and $(' \vdash)_t$ are admissible in LMQ (also by invoking the rule *ext*, and the corresponding LMQ-inference rules $(\vee \vdash)$, $(\vdash \wedge)$, and $(' \vdash)$, respectively).

$$\frac{\Gamma' \vdash \Delta'}{\Delta \vdash \Gamma}\ alt$$

$$\frac{A, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta}\ (\wedge\vdash)_1 \qquad \frac{B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta}\ (\wedge\vdash)_2$$

$$\frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, A \vee B}\ (\vdash\vee)_1 \qquad \frac{\Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \vee B}\ (\vdash\vee)_2$$

$$\frac{A \vdash \Delta \qquad B \vdash \Delta}{A \vee B, \Gamma \vdash \Delta}\ (\vee\vdash)_t \qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B, \Delta}\ (\vdash\wedge)_t$$

$$\frac{\Gamma \vdash \Delta}{\Delta', \Gamma \vdash \Sigma}\ ({}'\vdash)_t \qquad \frac{A, \Gamma \vdash \Delta}{A'', \Gamma \vdash \Delta}\ ({}''\vdash) \qquad \frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, A''}\ (\vdash {}'')$$

Fig. 3. The inference rules of LMQ_t.

It is common practice to specify tableau systems in terms of Smullyan's *uniform notation* [20], which allows an elegant formulation of the tableau rules and reduces the number of different cases. Therefore, let us fix some notation.

By a *signed formula* we understand an expression of the form $\mathsf{T}A$ or $\mathsf{F}A$, where A is a formula. We refer to the symbols T and F as the *signature* of the signed formulae $\mathsf{T}A$ and $\mathsf{F}A$, respectively. A signed formula is said to be of *positive signature* if its signature is T, and of *negative signature* otherwise.

A signed formula of the form $\mathsf{T}(A \wedge B)$, $\mathsf{F}(A \vee B)$, $\mathsf{T}A''$, or $\mathsf{F}A''$ is called an α*-formula*, and a signed formula of the form $\mathsf{F}(A \wedge B)$ or $\mathsf{T}(A \vee B)$ is called a β*-formula*. The *components*, α_i and β_i ($i = 1, 2$), of an α- or β-formula are given according to the following charts:

α	α_1	α_2
$\mathsf{T}(A \wedge B)$	$\mathsf{T}A$	$\mathsf{T}B$
$\mathsf{F}(A \vee B)$	$\mathsf{F}A$	$\mathsf{F}B$
$\mathsf{T}A''$	$\mathsf{T}A$	$\mathsf{T}A$
$\mathsf{F}A''$	$\mathsf{F}A$	$\mathsf{F}A$

β	β_1	β_2
$\mathsf{F}(A \wedge B)$	$\mathsf{F}A$	$\mathsf{F}B$
$\mathsf{T}(A \vee B)$	$\mathsf{T}A$	$\mathsf{T}B$

Given a set S of signed formulae, we define the following operations:

$$S^\flat := \{\mathsf{T}A \mid \mathsf{T}A \in S \text{ and } A \neq B' \text{ for any formula } B\} \cup \{\mathsf{F}A \mid \mathsf{T}A' \in S\};$$
$$S^\natural := \{\mathsf{T}A' \mid \mathsf{F}A \in S\} \cup \{\mathsf{F}A' \mid \mathsf{T}A \in S\};$$
$$S^\sharp_X := \begin{cases} \{\mathsf{F}A \mid \mathsf{F}A \in S\} & \text{if } X \text{ is a signed formula with positive signature;} \\ \{\mathsf{T}A \mid \mathsf{T}A \in S\} & \text{if } X \text{ is a signed formula with negative signature.} \end{cases}$$

Moreover, if $S = \{\mathsf{T}A_1, \ldots, \mathsf{T}A_n, \mathsf{F}B_1, \ldots, \mathsf{F}B_m\}$, then

$$|S| := A_1, \ldots, A_n \vdash B_1, \ldots, B_m.$$

With these notations at hand, the inference rules of LMQ_t read as follows:

$$\frac{|S^\natural|}{|S|} \qquad \frac{|S,\alpha_1|}{|S,\alpha|} \qquad \frac{|S,\alpha_2|}{|S,\alpha|} \qquad \frac{|S^\sharp_\beta,\beta_1| \qquad |S^\sharp_\beta,\beta_2|}{|S,\beta|} \qquad \frac{|S^\flat|}{|S|}$$

Now is it only a matter of adding a few further ingredients to obtain the desired tableau calculus.

Let $\mathcal{T}$ be a binary tree whose nodes are signed formulae. We define the following *branch modification rules*, corresponding to the operations $S^\flat$, $S^\natural$, and $S^\sharp_X$ given above, respectively:

Branch Modification Rule 1 ("BMR 1"): Given a branch b of $\mathcal{T}$, delete all signed formulae with negative signature, and replace signed formulae of the form $\mathsf{T}A'$ by $\mathsf{F}A$.

Branch Modification Rule 2 ("BMR 2"): Given a branch b of $\mathcal{T}$, replace all signed formulae $\mathsf{F}A$ by $\mathsf{T}A'$ and all $\mathsf{T}A$ by $\mathsf{F}A'$.

Branch Modification Rule 3[X] ("BMR 3[X]"): Given a branch b of $\mathcal{T}$, delete all signed formulae with positive signature providing the signed formula X has positive signature, and delete all signed formulae with negative signature providing X has negative signature.

Definition 11. Let $S = \{X_1, \ldots, X_n\}$ be a set of signed formulae. Then $\mathcal{T}$ is a *tableau for* S iff there exists a finite sequence $\mathcal{T}_1, \ldots, \mathcal{T}_m$ such that:

(1) $\mathcal{T}_1$ is a tree consisting of the single branch $X_1, \ldots, X_n$;
(2) $\mathcal{T}_m = \mathcal{T}$;
(3) for each $1 \leq i < m$, $\mathcal{T}_{i+1}$ is the result of applying one of the following rules to some branch b of $\mathcal{T}_i$:
   (i) Branch Modification Rule 1 (BMR 1);
   (ii) Branch Modification Rule 2 (BMR 2);
   (iii) Rule A: If α occurs on branch b, extend b by either α_1 or α_2.
   (iv) Rule B: If β occurs on branch b, apply Branch Modification Rule 3[β] to b and extend the resulting branch simultaneously by β_1 and β_2.

□

Schematically, applications of Rule A and Rule B can be written as follows:

$$\frac{\alpha}{\alpha_1} \qquad \frac{\alpha}{\alpha_2}$$

$$\frac{\beta}{\beta_1 \mid \beta_2} \text{ , provided that BMR 3}[\beta] \text{ has been applied}$$

Definition 12. Let $\mathcal{T}$ be a tableau for a set of signed formulae. A branch b is *closed* if $\mathsf{T}A$ and $\mathsf{F}A$ occur on b, for some formula A; otherwise b is *open*. We say that $\mathcal{T}$ is closed if all branches of $\mathcal{T}$ are closed; otherwise $\mathcal{T}$ is open. By a *tableau proof* of a formula A we understand a closed tableau for $\{\mathsf{F}A\}$. □

From the construction given above, it is clear that we have the following result:

Theorem 13. *A formula A has a tableau proof iff the sequent $\vdash A$ is provable in* LMQ.

4 An Application to Lüders' Rule

Thus far, we considered a very elementary form of quantum logic, namely *minimal quantum logic*. As already mentioned earlier, minimal quantum logic is algebraically characterized in terms of ortholattices. For most physical applications, however, the principle of modularity is required. In this section, we will describe how modularity can be included within our sequent-type formalism, by way of an example from quantum mechanics.

Consider the following sequent, taken from Gibbins [10]:

$$P \vdash Q \rightarrow (Q \wedge (P \vee Q')). \tag{2}$$

In quantum logic, there are several possibilities to define an implication. In our case, "$\rightarrow$" is defined by

$$(A \rightarrow B) := A' \vee (A \wedge B),$$

for any formulae A, B.

Informally, the sequent (2) encodes the so-called *Lüders' Rule* [14]. What does this rule mean? Let P correspond to the state of a quantum-mechanical system, and let Q correspond to the state of the system after a measurement has been performed. According to Lüders' Rule, the measurement has the effect that the state-vector of the system will be projected onto the state in the new subspace "most similar" to the original state-vector. Now, the state "most similar" to the original state lies in the subspace corresponding to Q, but also in the subspace spanned by P and Q'. In formal terms: given that P is true, if Q were true, then $Q \wedge (P \vee Q')$ would also be true.

As it turns out, the sequent (2) cannot be proved in LMQ, because modularity is needed in this particular instance. However, by using the technique of non-logical axioms—as already mentioned in Section 3.1—it is quite straightforward to include the principle of modularity in our present systems, and to obtain a proof of sequent (2).

Inspecting an alleged proof of sequent (2), one realizes that what is actually missing in order to obtain a proof are basically applications of modus ponens and (restricted forms) of cut. Indeed, given an axiomatization of orthologic which already contains the cut-rule, one can obtain an axiom system for quantum logic by adding a suitable rule enforcing applications of modus ponens. For instance, Goldblatt [11] obtains a system for quantum logic by adjoining to his axiomatization of orthologic (containing cut) axiom sequents of the form $A \wedge (A' \vee (A \wedge B)) \vdash B$. Similarly, Dalla Chiara [6] uses the inference rule

$$\frac{A \wedge (A \wedge (A \wedge B)')'}{B}$$

to obtain an axiomatization of quantum logic. Incidentally, at present it is an open question whether there is a proof system for quantum logic which does not require some form of cut, i.e., which is *cut-free*.

Summarizing our discussion, by adding non-logical axioms of the form $A \wedge (A \rightarrow B)) \vdash B$ and by allowing applications of restricted forms of cut (like, in our case, cut_1 and cut_2), we can effectively enforce modularity and thus obtain a calculus for quantum logic. Similarly, in tableau systems, the effect of non-logical axioms (encoding modularity) can be achieved by liberalizing the closure condition of a branch: in addition to the usual closure rule, call a branch closed if it contains signed formulae of the form $\mathsf{T}A \wedge (A' \vee (A \wedge B))$ and $\mathsf{F}B$. Of course, corresponding to the situation in sequent-type calculi, tableau systems for (modular) quantum logic must include some restricted forms of cut as well. A detailed mathematical proof of this informal discussion can be found in the full version of this paper [8].

As shown by Nishimura in [16], instead of non-logical axioms one can also use the following inference rule:

$$\frac{G' \vdash F' \quad F', G \vdash}{F' \vdash G'}\ OM$$

More precisely, one can always "expand" occurrences of non-logical axioms by suitable deductions using the rule *OM* and cut (a deduction of this kind—in the system LMQ, adjoined by *OM* and cut_1—is given in the appendix of this paper).

In the rest of this section, let us present our proof of Lüders' Rule by the technique outlined above. We start with the derivations ϑ and ξ, and ϱ. For convenience, we use F to denote the formula $(Q \wedge (Q \wedge (Q' \vee P)))$, hence Lüders' Rule has the form $P \vdash Q' \vee F$.

Derivation ϑ

$$\dfrac{\dfrac{\dfrac{Q' \vdash Q'}{Q' \vdash (Q' \vee F)}\,(\vdash \vee)_1}{Q' \vdash (Q' \vee F)''}\,(\vdash '')}{(Q' \vee F)' \vdash Q}\,alt$$

Derivation ξ

$$\dfrac{\dfrac{\dfrac{\dfrac{Q \vdash Q}{Q, P' \vdash Q}\,ext \quad \dfrac{P' \vdash P'}{Q, P' \vdash P'}\,ext}{Q, P' \vdash (Q \wedge P')}\,(\vdash \wedge)}{(Q \wedge P')' \vdash Q', P}\,('' \vdash), (\vdash ''), alt}{Q, (Q \wedge P')' \vdash Q' \vee P}\,(\vdash \vee)_1, (\vdash \vee)_2$$

Derivation ϱ

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{Q \vdash Q}{Q, (Q \wedge P')' \vdash Q}\; ext \qquad \dfrac{\dfrac{Q \vdash Q}{Q, (Q \wedge P')' \vdash Q}\; ext \qquad \overset{\xi}{Q, (Q \wedge P')' \vdash Q' \vee P}}{Q, (Q \wedge P')' \vdash Q \wedge (Q' \vee P)}\; (\vdash \wedge)}{Q, (Q \wedge P')' \vdash F}\; (\vdash \wedge)}{Q, (Q \wedge P')' \vdash (Q' \vee F)}\; (\vdash \vee)_2}{(Q' \vee F)' \vdash Q', (Q \wedge P')}\; ('' \vdash), (\vdash ''), alt}{(Q' \vee F)' \vdash Q' \vee (Q \wedge P')}\; (\vdash \vee)_1, (\vdash \vee)_2
$$

Derivation ψ

$$
\dfrac{\overset{\vartheta}{(Q' \vee F)' \vdash Q} \qquad \overset{\varrho}{(Q' \vee F)' \vdash Q' \vee (Q \wedge P')}}{(Q' \vee F)' \vdash Q \wedge (Q' \vee (Q \wedge P'))}\; (\vdash \wedge)
$$

Using these auxiliary derivations, the final proof of Lüders' Rule is given below. Notice that the rightmost sequent is a non-logical axiom.

$$
\dfrac{\dfrac{\overset{\psi}{(Q' \vee F)' \vdash Q \wedge (Q' \vee (Q \wedge P'))} \qquad Q \wedge (Q' \vee (Q \wedge P')) \vdash P'}{(Q' \vee F)' \vdash P'}\; cut_1}{P \vdash Q' \vee F}\; alt
$$

5 Conclusion and Discussion

We presented a sequent calculus for minimal quantum logic based on Nishimura's calculus GMQL. By modifying a rule which Takano identified to be superfluous, we were able to show that an even greater number of other rules becomes redundant. Moreover, we presented a corresponding tableau system for the new calculus as well.

Although quantum logics has its roots in physics, investigating its formal properties proved to be of interest also in other application domains. We like to mention the research initiated by Heelan showing that quantum logic can also be used as formalizations of *context-dependent logics* [12]. Also, quantum logic is a useful device if one is interested in the relation of different axiom systems weaker than classical logic (see, e.g., [1, 2]).

We already mentioned the open problem of whether there is a cut-free axiomatization of (modular) quantum logic, or even a (semi-)analytic proof procedure. Related to this problem is the question of the decidability of quantum logic, which remains unclear at the time of writing (recall that orthologic, i.e., minimal quantum logic, *is* in fact decidable). Also, proof-theoretical properties of extensions of the present calculi to the first-order case remain to be explored.

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A Appendix: A Derivation of the Non-Logical Axioms

Let G be an abbreviation for $A' \vee (A \wedge B)$. Below, a proof of the sequent $A \wedge G \vdash B$ is given.

DERIVATION ϕ_1

$$\dfrac{\dfrac{A \vdash A}{A \wedge B \vdash A}\,(\wedge\vdash)_1}{(A \wedge B)'' \vdash A}\,({''}\vdash)$$

DERIVATION ϕ_2

$$\dfrac{\dfrac{A \wedge B \vdash A \wedge B}{(A \wedge B)'' \vdash A \wedge B}\,({''}\vdash)}{(A \wedge B)'' \vdash G}\,(\vdash\vee)_2$$

DERIVATION ϕ_3

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A' \vdash A'}{A' \vdash A \wedge B, A'}\,ext \quad \dfrac{A \wedge B \vdash A \wedge B}{A \wedge B \vdash A \wedge B, A'}\,ext}{G \vdash A \wedge B, A'}\,(\vee\vdash)}{(A \wedge B)', A'', G \vdash}\,({'}\vdash)}{(A \wedge B)', A'', G'' \vdash}\,({''}\vdash)}{\vdash (A \wedge B), A', G'}\,alt}{\vdash (A \wedge B)'', A', G'}\,(\vdash {''})}{(A \wedge B)', A, G \vdash}\,alt}{(A \wedge B)', A \wedge G \vdash}\,(\wedge\vdash)_1, (\wedge\vdash)_2}{(A \wedge B)', (A \wedge G)'' \vdash}\,({''}\vdash)$$

DERIVATION ϕ_4

$$\dfrac{\dfrac{\dfrac{\dfrac{\phi_1}{(A \wedge B)'' \vdash A} \quad \dfrac{\phi_2}{(A \wedge B)'' \vdash G}}{(A \wedge B)'' \vdash A \wedge G}\,(\vdash\wedge)}{(A \wedge B)'' \vdash (A \wedge G)''}\,(\vdash {''}) \qquad \dfrac{\phi_3}{(A \wedge B)', (A \wedge G)'' \vdash}}{\dfrac{(A \wedge G)'' \vdash (A \wedge B)''}{A \wedge G \vdash A \wedge B}\,alt, alt}\,OM$$

DERIVATION ϕ_5

$$\dfrac{\dfrac{\phi_4}{A \wedge G \vdash A \wedge B} \quad \dfrac{B \vdash B}{A \wedge B \vdash B}\,(\wedge\vdash)_1}{A \wedge G \vdash B}\,cut_1$$

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